INFN
BOLOGNA
Istituto Nazionale di Fisica Nucleare
Sezione di Bologna

Towards $e^+e^- \to \gamma\gamma^*$ at two loops

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Theory and Phenomenology
of Fundamental Interactions
UNIVERSITY AND INFN · BOLOGNA

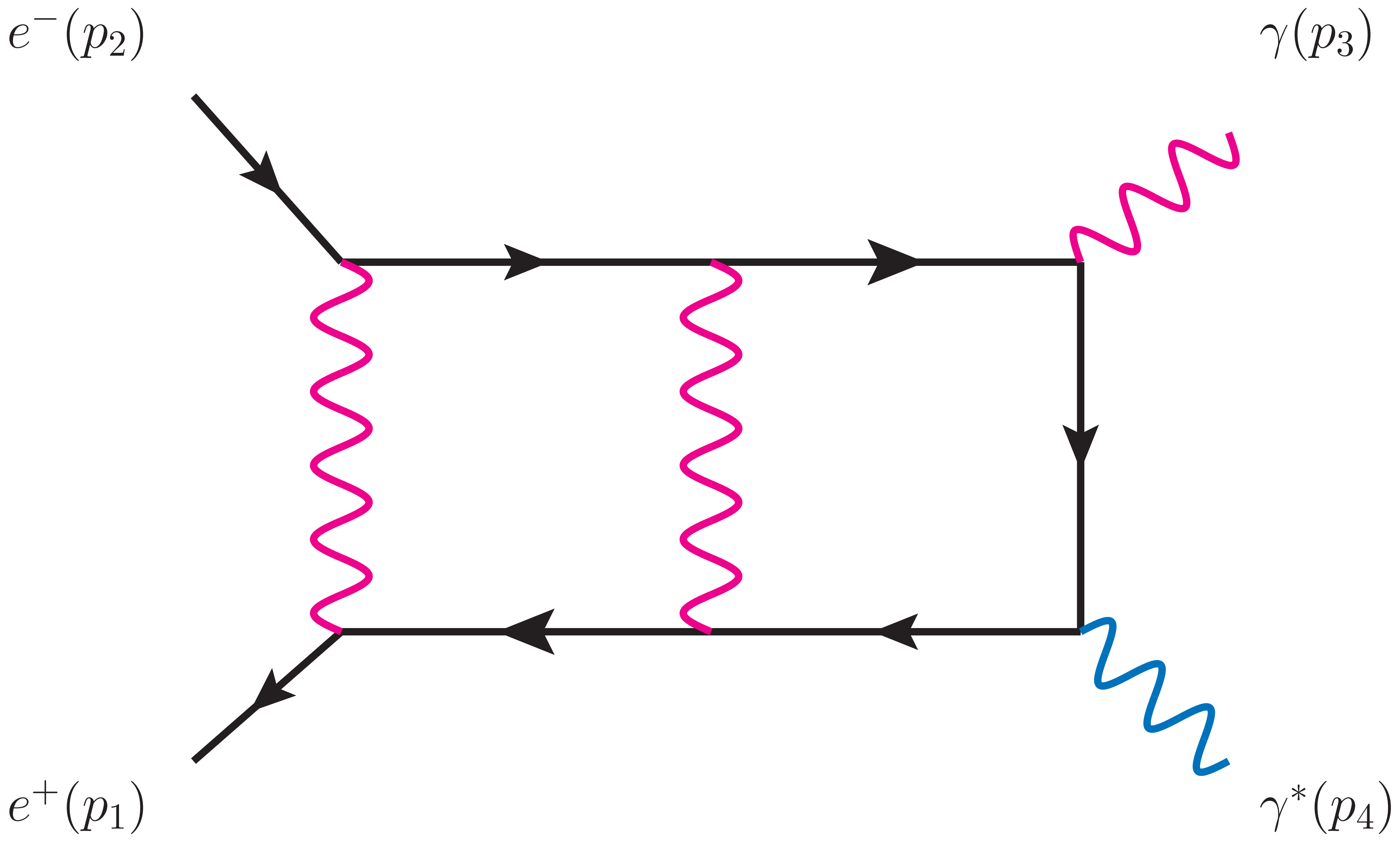
In collaboration with T. Dave, P. Petit Rosàs
and W. Torres Bobadilla

Workshop on Radiative Corrections and Monte Carlo Simulations, Torino, 4/6/2026

Setup and kinematics

$e^+(p_1) + e^-(p_2) + \gamma(p_3) + \gamma^*(p_4) \to 0$

2-loop corrections including loops of **massive electrons**



$$p_1^2 = p_2^2 = m^2, \quad p_3^2 = 0, \quad p_4^2 = q^2$$

4 invariants: $\vec{x} := \{s, t, m^2, q^2\}$

Dimensional regularisation: $d = 4 - 2\varepsilon$

Structure of the amplitude

We first decompose the amplitude into **12 independent tensor structures**, depending on the external polarisations

$$\mathscr{A} = \sum_{i=1}^{12} F^{(i)} T_i$$

See also Tom's talk!

The form factors $F^{(i)}$ only depend on the kinematics

We have a further decomposition as

$$F^{(i)} = \boxed{F^{(i,N_f)}} + \boxed{F^{(i,1)}}$$

42 Feynman diagrams

14 after Furry's theorem

60 Feynman diagrams

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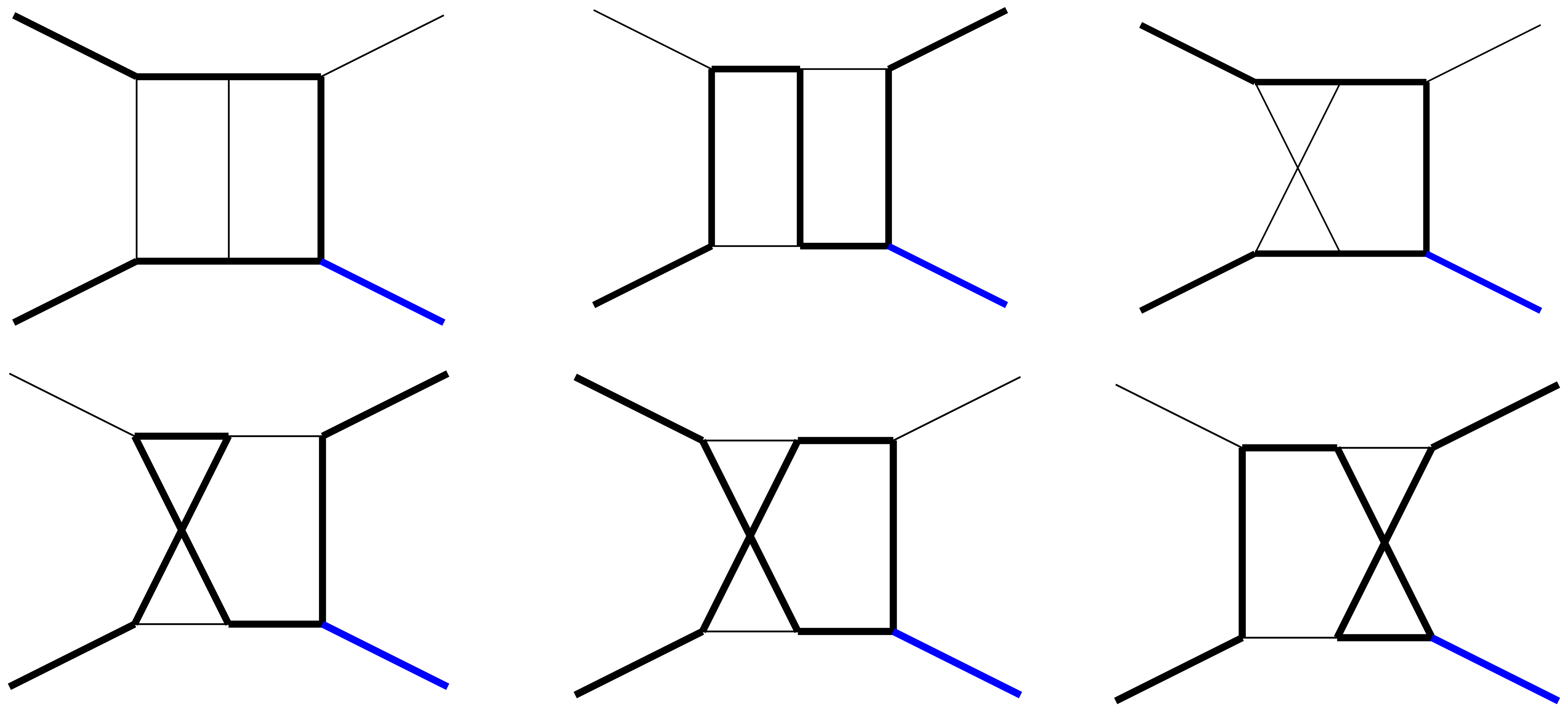
Amplitude expressed in terms of **13'844 scalar Feynman integrals**

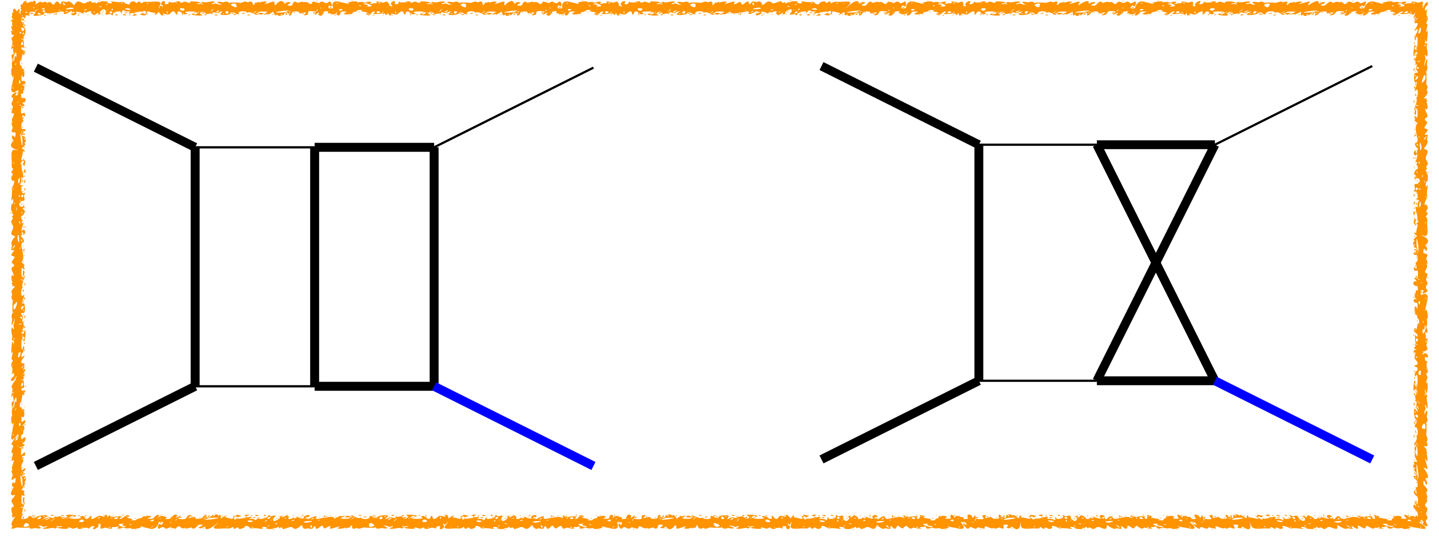
Computation of the Feynman integrals

Integral families

Eight genuine two-loop topologies up to permutations



Relevant for N_f contribution



$$I_{\vec{a}}(\vec{x}) = \int \mathrm{d}^d k_1 \mathrm{d}^d k_2 \frac{1}{D_1^{a_1} \cdots D_9^{a_9}}, \quad \vec{a} = (a_1, \ldots, a_9) \in \mathbb{Z}^9$$

$D_i = l_i^2 - m_i^2$: inverse propagators of the diagrams

$$D_1 = (k_1 - p_1)^2, \quad D_2 = k_1^2 - m^2$$
$$D_3 = (k_1 + p_2)^2, \quad D_4 = (k_2 - p_1)^2 - m^2$$
$$D_5 = (k_2 + p_2)^2 - m^2, \quad D_6 = (k_2 + p_{23})^2 - m^2$$
$$D_7 = (k_1 - k_2)^2 - m^2, \quad D_8 = (k_1 + p_{23})^2$$
$$D_9 = k_2^2$$

IBPs and reduction to master integrals

[Tkachov '81], [Chetyrkin, Tkachov '81]

Feynman integrals satisfy **linear relations**: integration by part identities (IBPs)

$$0 = \int \mathrm{d}^d k_1 \mathrm{d}^d k_2 \frac{\partial}{\partial k_l^\mu} \frac{v^\mu}{D_1^{a_1} \cdots D_9^{a_9}}, \qquad v^\mu \in \{k_j^\mu, p_j^\mu\}$$

$\Longrightarrow$ All integrals are linear combinations of a finite set of master integrals (MIs) $\vec{I}$

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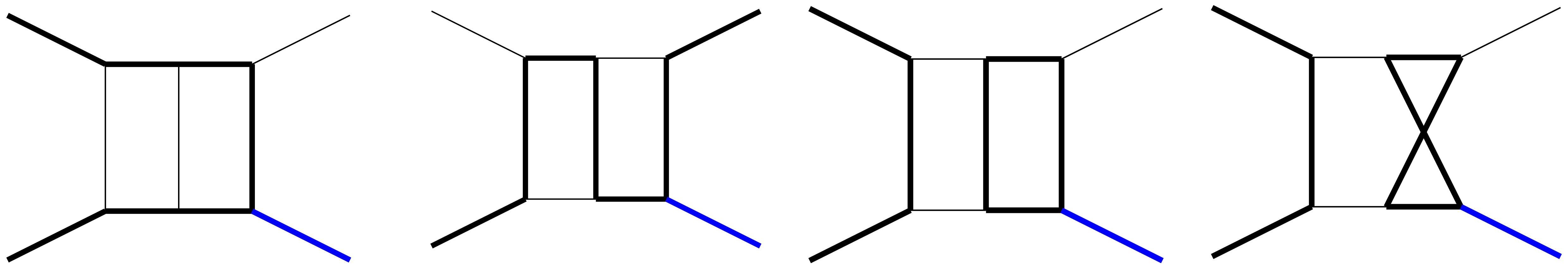
`LiteRed` [Lee 2012] and `FiniteFlow` [Peraro 2019]



PL1: 68 MIs PL2: 74 MIs PL3: 70 MIs NP5: 108 MIs

Master integrals satisfy differential equations

[Kotikov '91; Bern, Dixon, Kosower '94; Gehrmann, Remiddi 2000]

From the IBPs we derive

$$\forall \xi \in \vec{x}: \quad \partial_\xi \vec{I}(\vec{x};\varepsilon) = B_\xi(\vec{x};\varepsilon) \cdot \vec{I}(\vec{x};\varepsilon)$$

The basis is arbitrary: is there a natural choice?

[Henn 2014]: DEs in **canonical form** (no general algorithm)

Decouple $\vec{x}$ and ε and avoid spurious poles

$$\mathrm{d}\vec{I}(\vec{x};\varepsilon) = \varepsilon \; \mathrm{d}A(\vec{x}) \; \vec{I}(\vec{x};\varepsilon)$$

Simplest case: *dlog*-forms

$$\mathrm{d}A(\vec{x}) = \sum_i a_i \boxed{\mathrm{d}\log W_i(\vec{x})}$$

Letters

$$\mathrm{d}\log(z+c) = \frac{\mathrm{d}z}{z+c}$$

How do we construct a canonical basis?

Integrand analysis

[Arkani-Hamed, Bourjaily, Cachazo, Trnka 2012], [Henn, Mistlberger, Smirnov, Wasser, 2020]

Conjecture: integrals with a loop-integrand with **at most simple poles** and a **constant leading singularity** satisfy canonical DEs

$$I^{\varepsilon=0} \propto R(\vec{x}) \int \mathrm{d}\log(\ldots) \wedge \mathrm{d}\log(\ldots)$$

$$\boxed{\mathrm{d}\log(z+c) = \frac{\mathrm{d}z}{z+c}}$$

$\Longrightarrow \dfrac{1}{R(\vec{x})} I$ is a good MI

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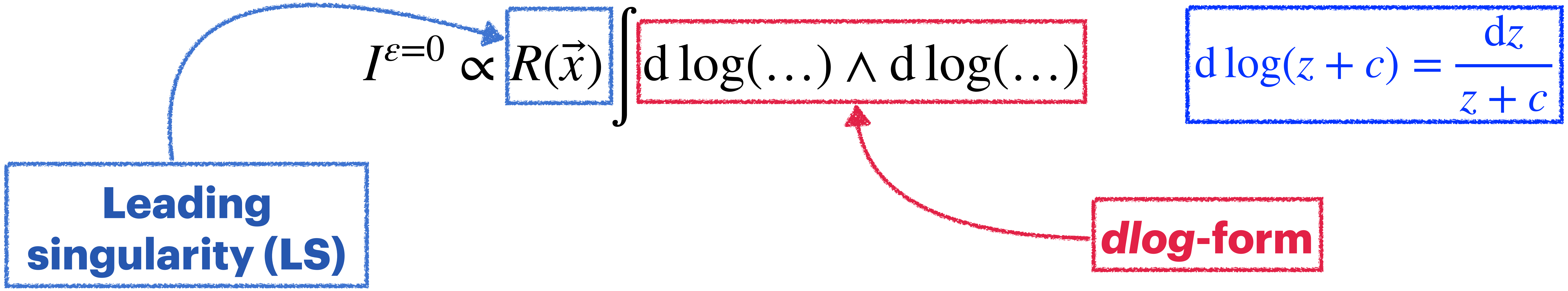

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Leading singularity (LS)

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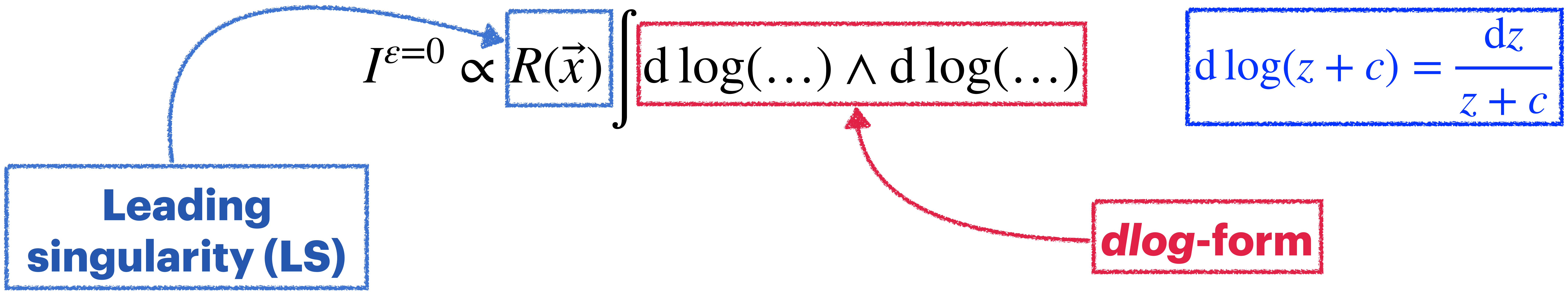

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***dlog*-form**

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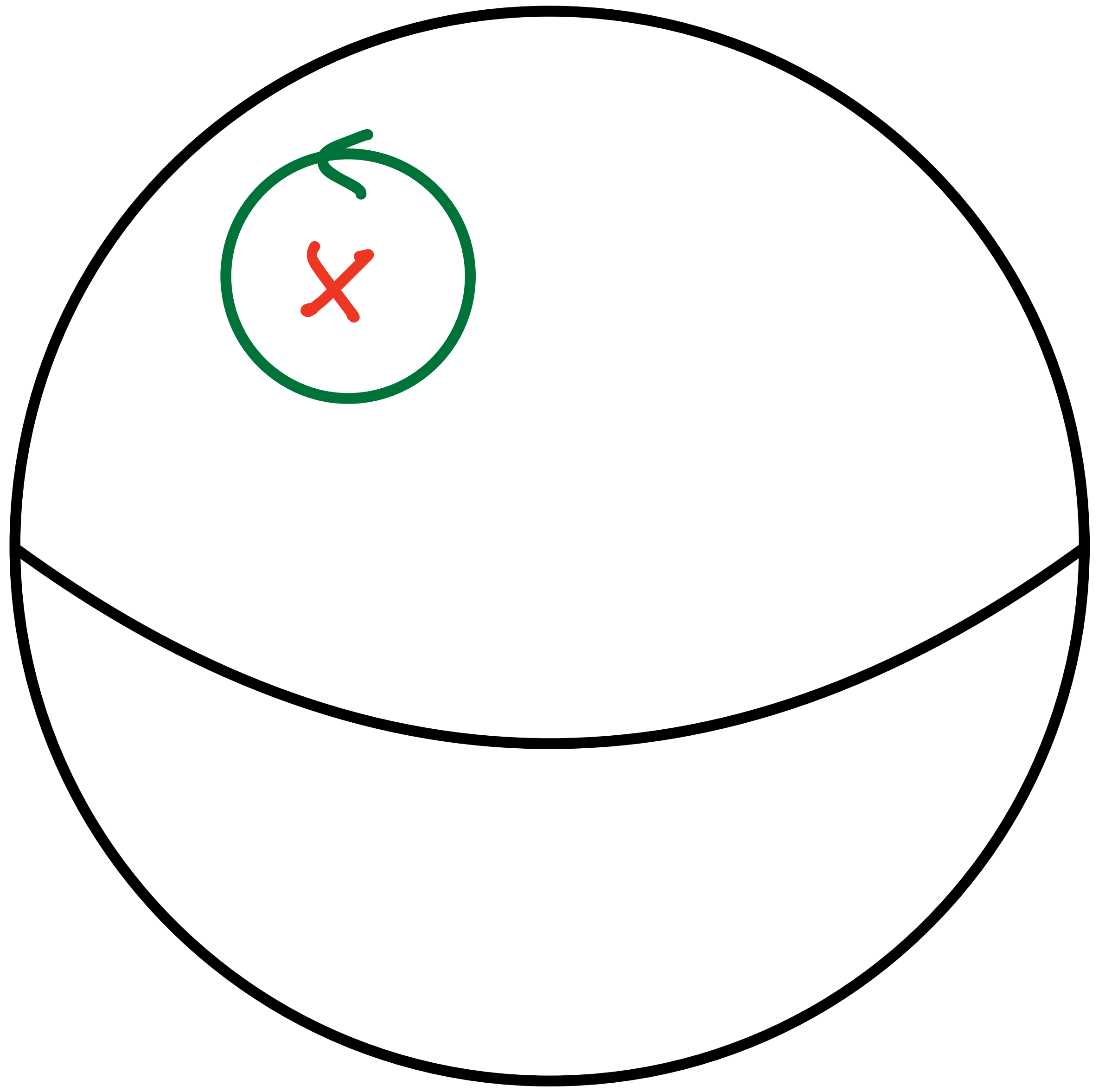
Commonly: **rational functions** and **simple square roots**

Connection to geometry

***Dlog*-integrals** are associated with the geometry of the (punctured) **Riemann sphere**



There is only one natural differential form:

$$\mathrm{d}\log(z+c) = \frac{\mathrm{d}z}{z+c}$$

Computing **leading singularities** $\sim$ taking **residues**

"Bad" integrals have poles of higher multiplicity

$$I \sim \int \frac{1}{z^2(z+c)}$$

Beyond the *dlog* case

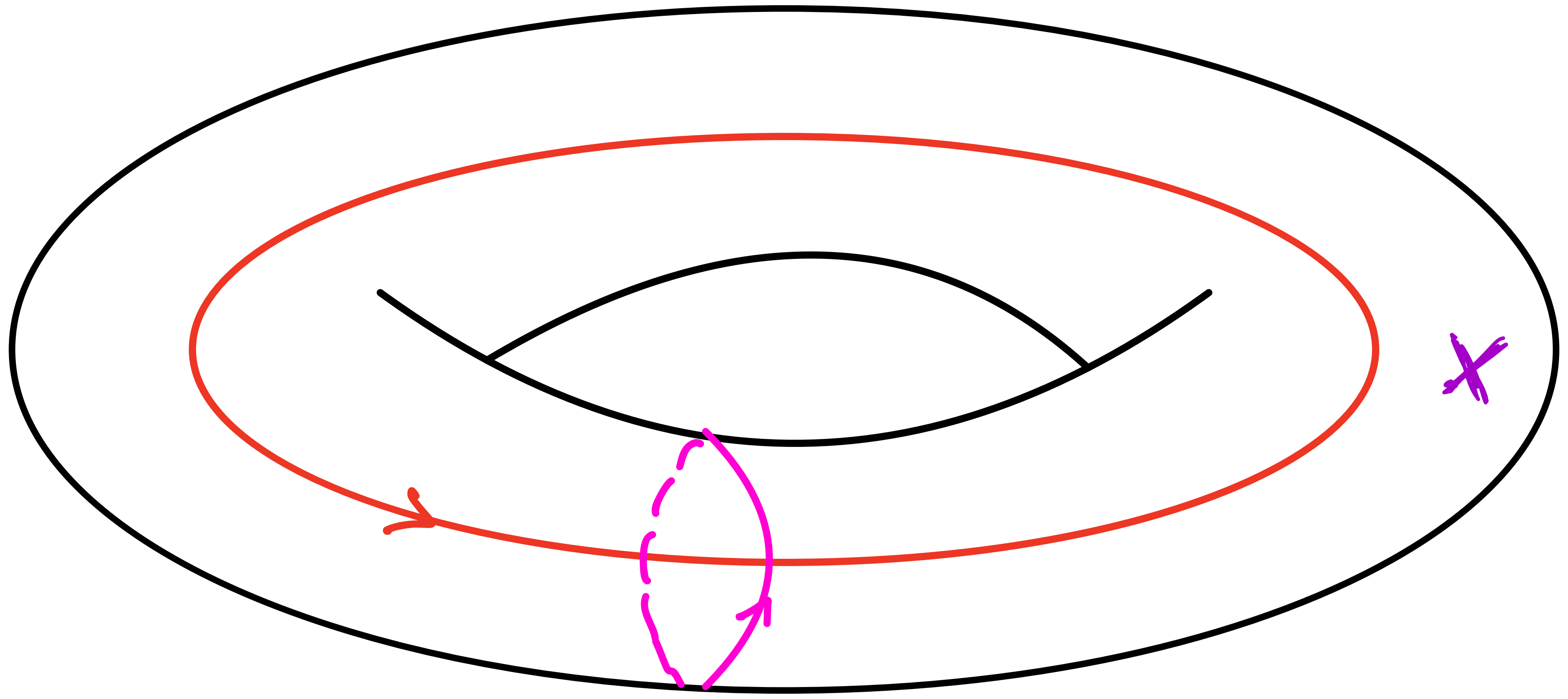
Starting at two-loops we find more complicated functions

Next-to-simplest example: **elliptic curves**

$$\mathscr{P}_4(z) = (z-a_1)(z-a_2)(z-a_3)(z-a_4)$$

The geometry is a **torus**: three natural differential forms

- First kind: holomorphic $\dfrac{\mathrm{d}z}{\sqrt{\mathscr{P}_4(z)}}$
- Second kind: double pole $\dfrac{z(z-c)\ \mathrm{d}z}{\sqrt{\mathscr{P}_4(z)}}$
- Third kind: single pole $\dfrac{z\ \mathrm{d}z}{\sqrt{\mathscr{P}_4(z)}}$



Elliptic integrals in $e^+e^- \to \gamma\gamma^*$

Sector: same $a_i > 0$

	PL1	**PL2**	**PL3**	**NP5**
MIs	68	74	70	108
Elliptic sectors (MIs)	2 (6)	7 (26)	6 (24)	5 (22) + hyper-elliptic top sector (7 MIs)

Master integrals for the elliptic case

Progress on general strategies to construct canonical integrals in recent years (e.g. [Görges et al. 2023], [ε collaboration 2025])

Still **substantially more complicated** than in the *dlog*-case, and it requires the introduction of **transcendental functions in the DEs**

$\Longrightarrow$ construct integrals satisfying DEs with a polynomial dependence on ε

Suitable for **evaluation numerically** or through **generalised series expansions**

Simple elliptic sectors

Baikov representation:
$z_i = D_i$

Baikov representation of $I_{011111100}$ of PL1

$$\int \boxed{\frac{\mathrm{d}z_9}{(q^2 - s)\sqrt{\mathscr{P}_4(z_9)}}}$$

Differential of the first kind

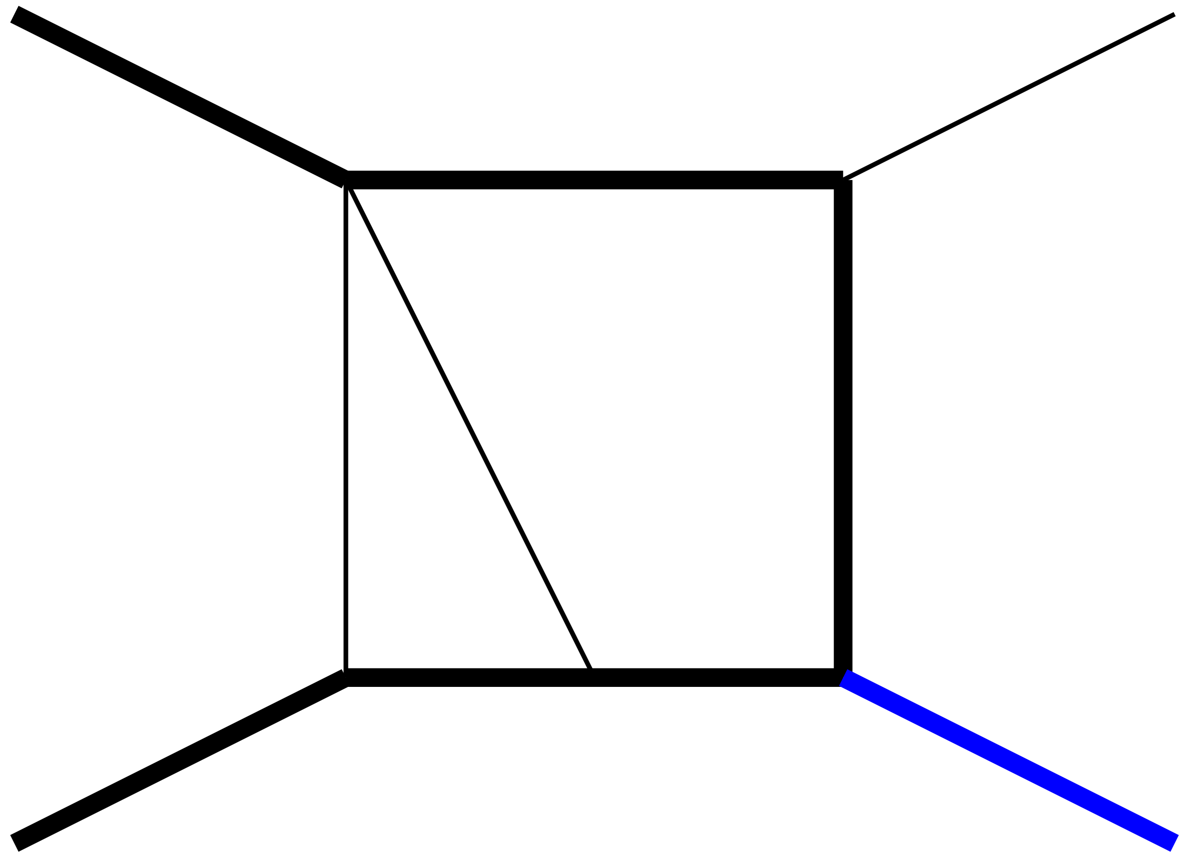
$$\implies \quad I_1 \propto (q^2 - s) I^{\mathrm{PL1}}_{011111100}$$

Second kind? Take a derivative [Görges et al. 2023]

$$I_2 \propto \partial_t \left((q^2 - s) I^{\mathrm{PL1}}_{011111100} \right)$$

Third kind: pole at infinity

$$I_3 \propto (q^2 - s) I^{\mathrm{PL1}}_{01111110-1}$$



3 MIs

Simple elliptic sectors

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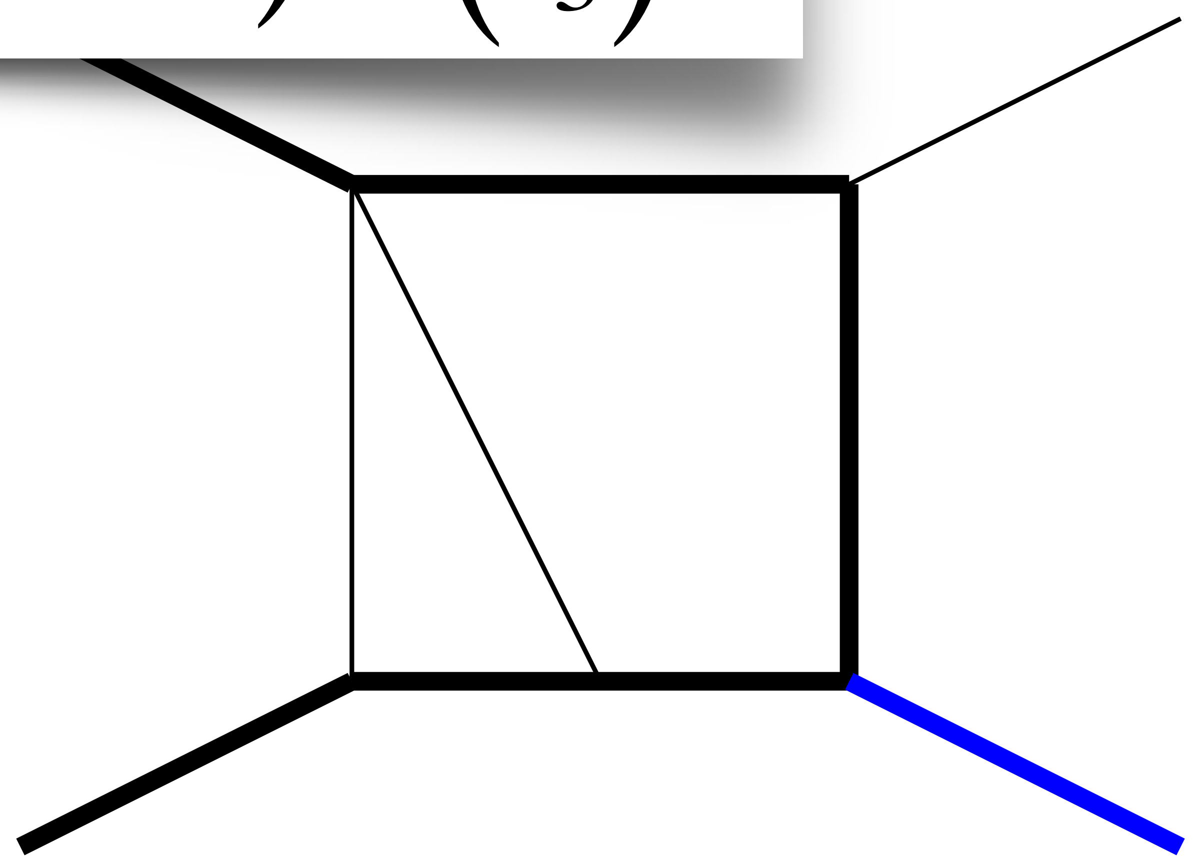
Baikov representat

$$\mathrm{d}\begin{pmatrix}I_1\\I_2\\I_3\end{pmatrix} = \begin{pmatrix} *+*\varepsilon & * & *\,\varepsilon \\ *+*\varepsilon+*\varepsilon^2 & *+*\varepsilon & *\,\varepsilon(*+*\varepsilon) \\ *+*\varepsilon & * & *\,\varepsilon \end{pmatrix}\cdot\begin{pmatrix}I_1\\I_2\\I_3\end{pmatrix},$$

$$\implies I_1 \propto (q^2-s)I^{\mathrm{PL1}}_{011111100}$$



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Four-point kite sectors

[Bargiela et al. 2025]: corner integral is in *dlog*-form

Integrals with **squared propagators** can be **elliptic**

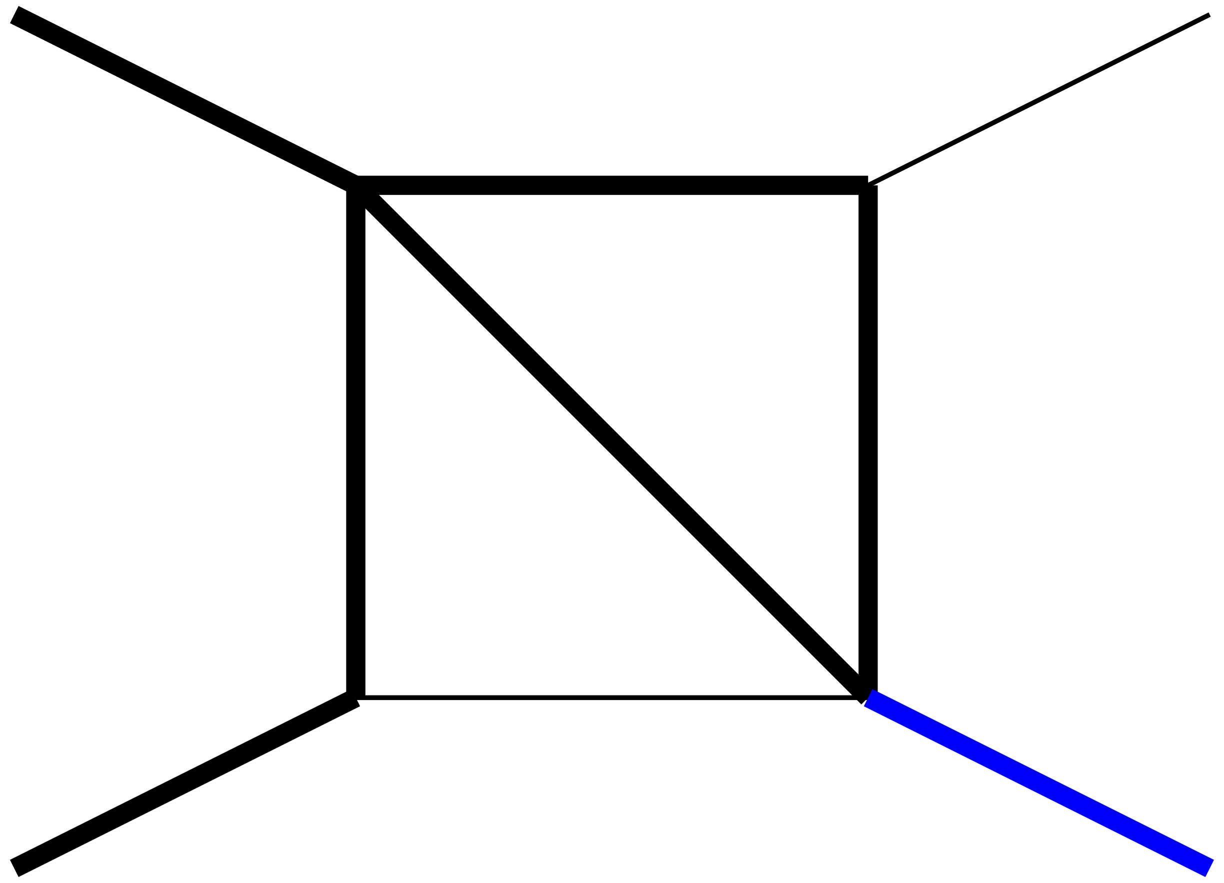
$\Longrightarrow$ identifying the differentials is more complicated

Solution: analyse **reducible super-sectors**

$$I^{\mathrm{PL3}}_{110111100} \propto \int \frac{\mathrm{d}z_9}{(q^2-s)\sqrt{\mathscr{P}_4(z_9)}}$$

$\Longrightarrow$ elliptic integrals like in the simple case

$I^{\mathrm{PL3}}_{110011100}$



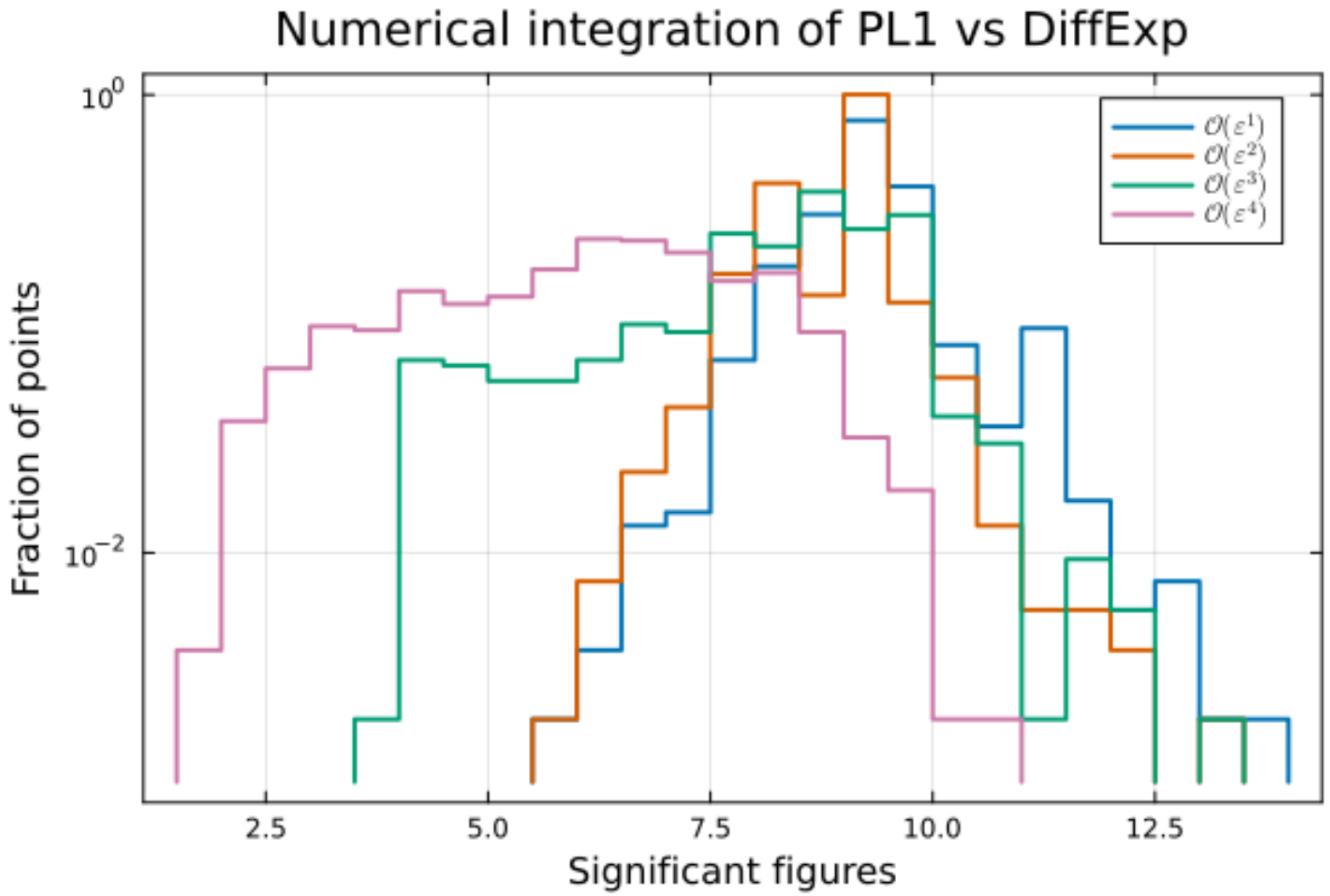
4 MIs

Numerical evaluation

Numerical integration of the DEs using the approach of [Petit Rosàs, Torres Bobadilla 2025], implemented in C++ and Julia

Performance: $\mathcal{O}(10-100\text{ms})$ per phase-space point

Check against `DiffExp` [Hidding 2020]: worst agreement 3 digits



Status of the calculation and outlook

- Feynman integrals:
  - Planar families PL1, PL2 and PL3 done [MP, W. Torres Bobadilla, to appear]
  - NP5: two sectors missing
  - Numerical integration of the DEs in $\mathcal{O}(10 - 100\mathrm{ms})$
- Amplitude calculation
  - N_f contribution generated and reduced to MIs [WIP with Tom, Pau and William]
  - N_f^0 contribution targeted in the near future

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Thank you!

Backup slides

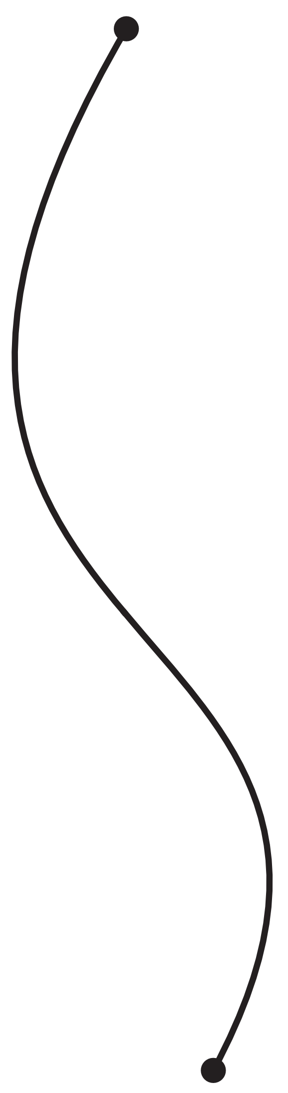
Evaluation through generalised series expansions

Parametrise a **linear path**

$$\gamma(\eta) = (1-\eta)\vec{x}_i + \eta\vec{x}_f$$

$\Longrightarrow$ DEs A_η in one variable

Boundary point $\vec{x}_i$:
values from `AMFlow`
[Liu, Ma 2022]



Target point $\vec{x}_f$

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The solution admits an **expansion**

$$I \backsim I_0 + \sum A_\eta \; \log(\eta-\tau)^{k_1}(\eta-\tau)^{k_2}$$

$\Longrightarrow$ **evolution of the solution** using `DiffExp` [Hidding 2020]

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$\mathcal{O}$(1 minute) for phase-space point for 16 digits

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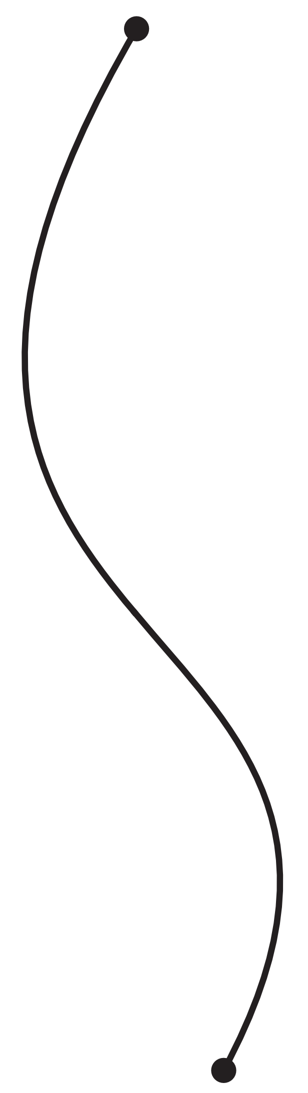

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