multi-loop methods and application: a status report

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thanks to my collaborators:

Becchetti, Brancaccio, Czakon, Hartanto, Peraro,
Poncelet, Ripani, Zoia

Radiative Corrections and Monte Carlo Simulations at Electron-Positron Colliders

Turin, 4th June 2026

FARE
RICERCA IN ITALIA
FRAMEWORK PER L'ATTRAZIONE E IL RAFFORZAMENTO
DELLE ECCELLENZE PER LA RICERCA IN ITALIA

Goal of this talk

Look ahead to see how we might reach the precision needs of current and future e+e- colliders
- low and high energy

Technology for underlying amplitude computations is the same independent of the collider

Different phase spaces can have a big effect on practical applications for precision phenomenology

Let's look at a few state-of-the art examples as motivation for discussion

other talks this week:
Rocco, Pozzoli, Dave

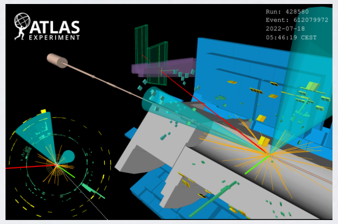
IBPs

DEQs

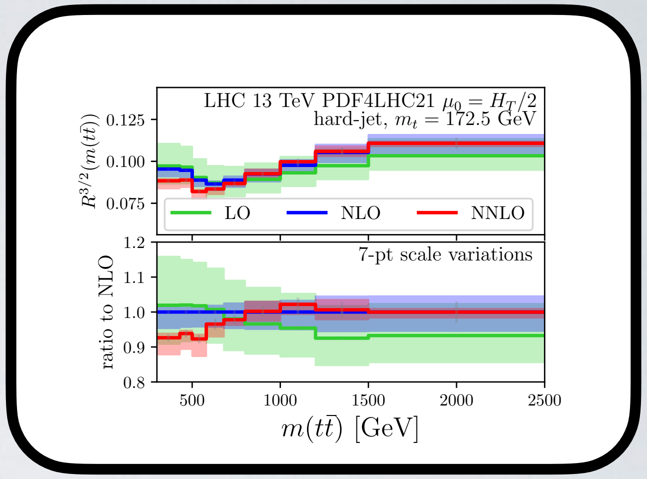
modular
artihmetic

$$\mathcal{A} = C.I$$

geometry/
mathematical
structure

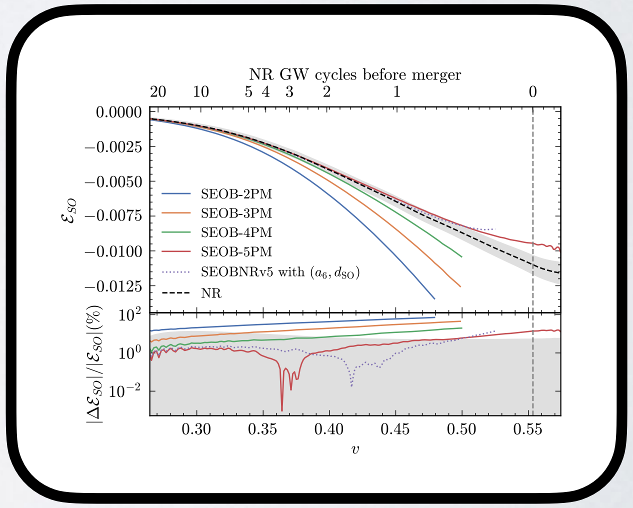


tt event

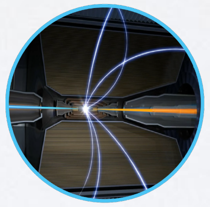


2511.11431

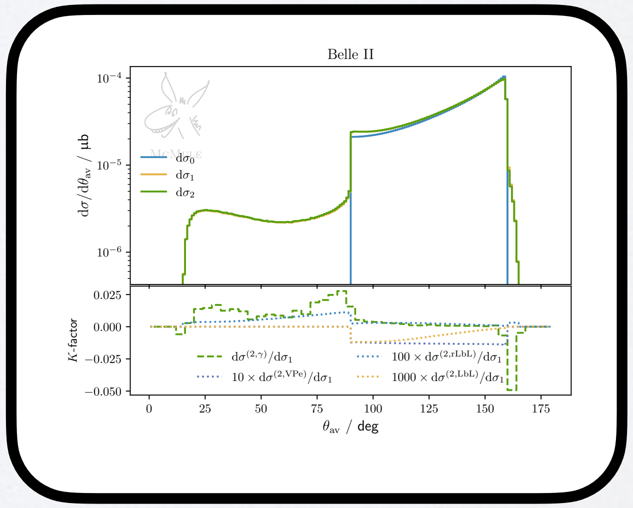
Ligo BH merger



2405.19181



Belle 2

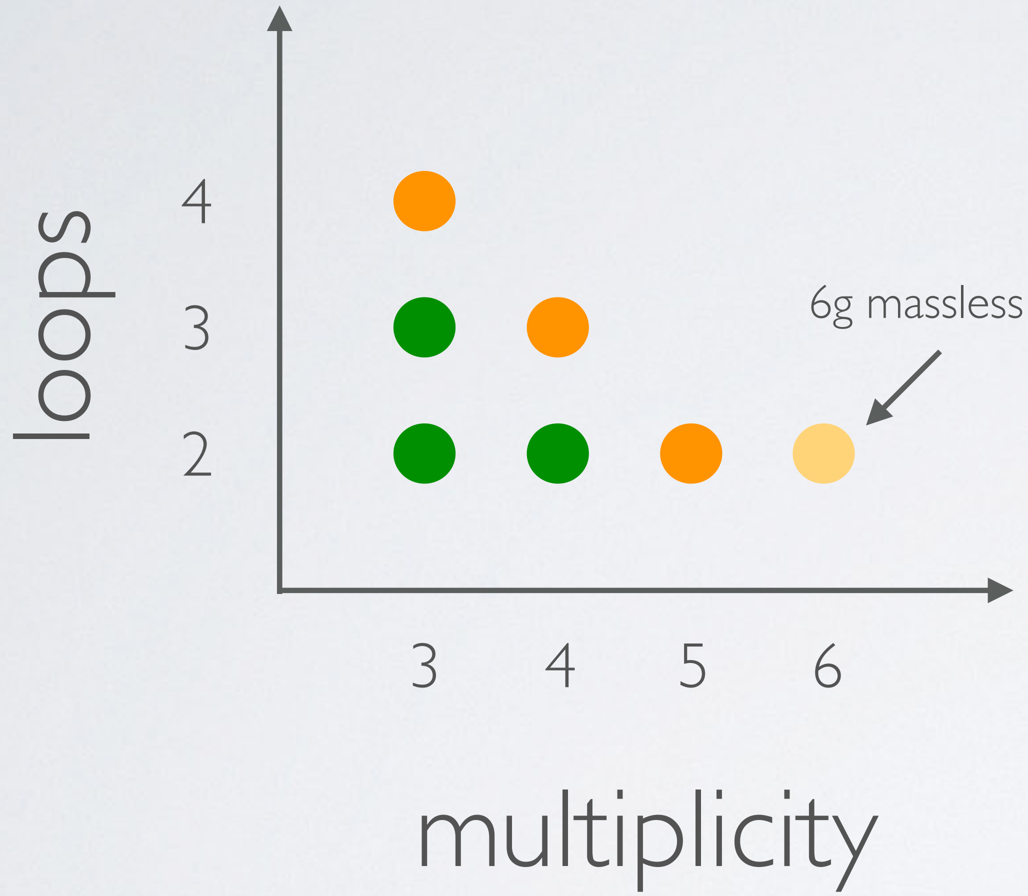


2512.22929

perturbative precision

one-loop/NLO fully automated!*

current frontier



6g massless e.g. Carrôlo et al. 2510.20565

better measure of complexity would be number of scales

internal massive propagators: complex underlying integral geometries lead to rapid growth in complexity (elliptic, K3, Calabi Yau)

$$\mathcal{A} \to d\sigma$$

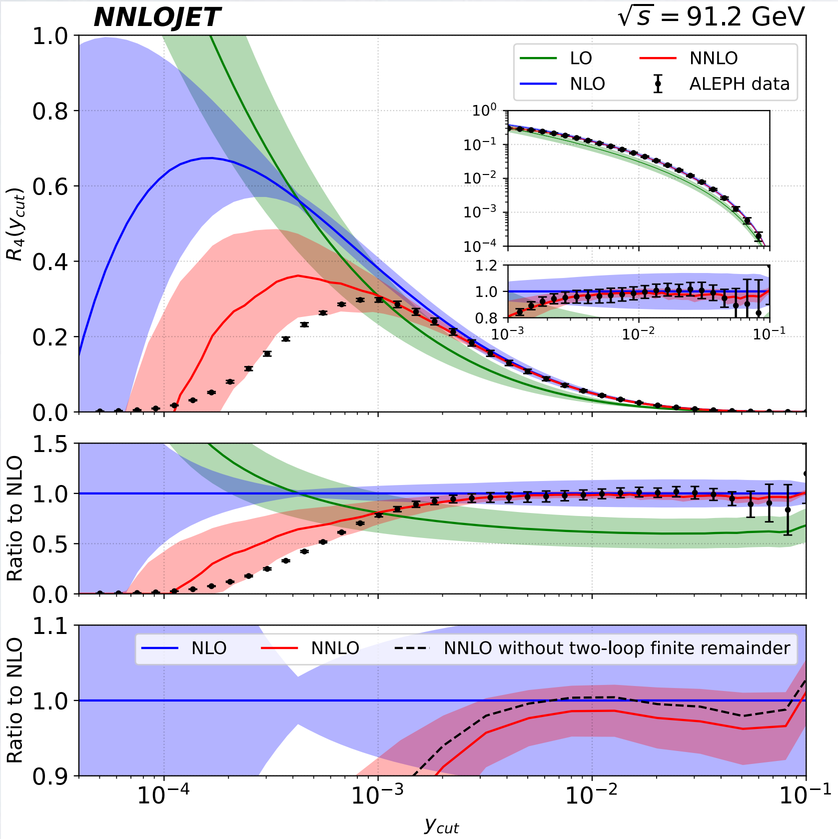
combining multi-loop multi-scale amplitudes with IR subtraction and challenging phase space

$e^+e^- \rightarrow$ jets

new results from antenna subtraction w/ NNLOJET

$e^+e^- \rightarrow$ 4 jets @ NNLO

Chen et al. 2602.18185



RV and RR small in this case but dominate computation time

compact two-loop leading colour amplitudes

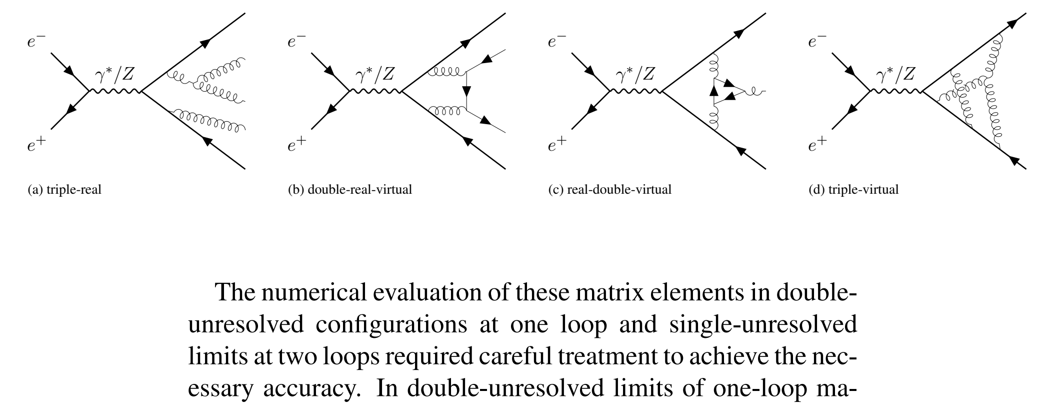
De Laurentis et al. 2503.10595

fast stable evaluation with (analytically continued) pentagon functions

Abreu et al. 2306.15431

$e^+e^- \rightarrow$ 2 jets @ N3LO

Chen et al. 2505.10618



(a) triple-real (b) double-real-virtual (c) real-double-virtual (d) triple-virtual

The numerical evaluation of these matrix elements in double-unresolved configurations at one loop and single-unresolved limits at two loops required careful treatment to achieve the necessary accuracy. In double-unresolved limits of one-loop ma-

example amplitude toolchain

tensor integral input

Feynman diagrams
Projectors [in 4D Tancredi, Peraro]
Unitarity cuts

tensor reduction to master integrals

integration-by-parts identities [Tkachov, Chetykin] [Laporta]
solve over finite fields e.g. FiniteFlow, FireFly, FIRE, Kira
optimized systems (Blade, NeatIBP etc.)

master integral expansions

identity function basis via differential equations for master integrals
canonical ‘ε’ -factorized form (Henn)

$$\begin{tikzcd} \mathcal{A} = C.I \arrow[d] \\ \mathcal{F} = c.f \end{tikzcd}$$

finite remainder

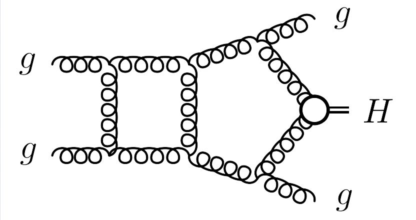
analytic reconstruction

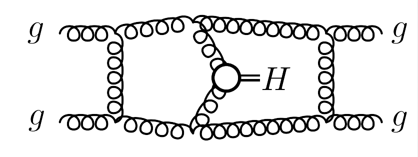
extract rational coefficients through multiple finite field evaluations
(in parallel) [Thiele interpolation]
optimize with linear relations, partial fractioning etc.

Successful program at 2 loops: pp→3j, pp→3γ, pp→γγj,
pp→ γjj, pp→ W+2j, pp→ Wbb, pp→ Hbb, pp→ Wγγ

these cases have massless propagators…

pp → H+2 jets





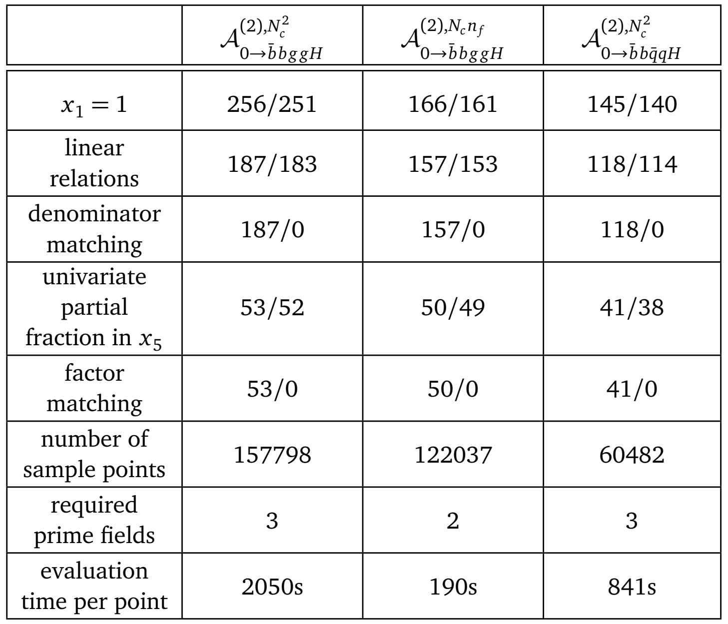
leading colour, heavy top quark limit

numerical unitarity method over finite fields (Caravel), IBPs with surface term decomposition (syzygy)

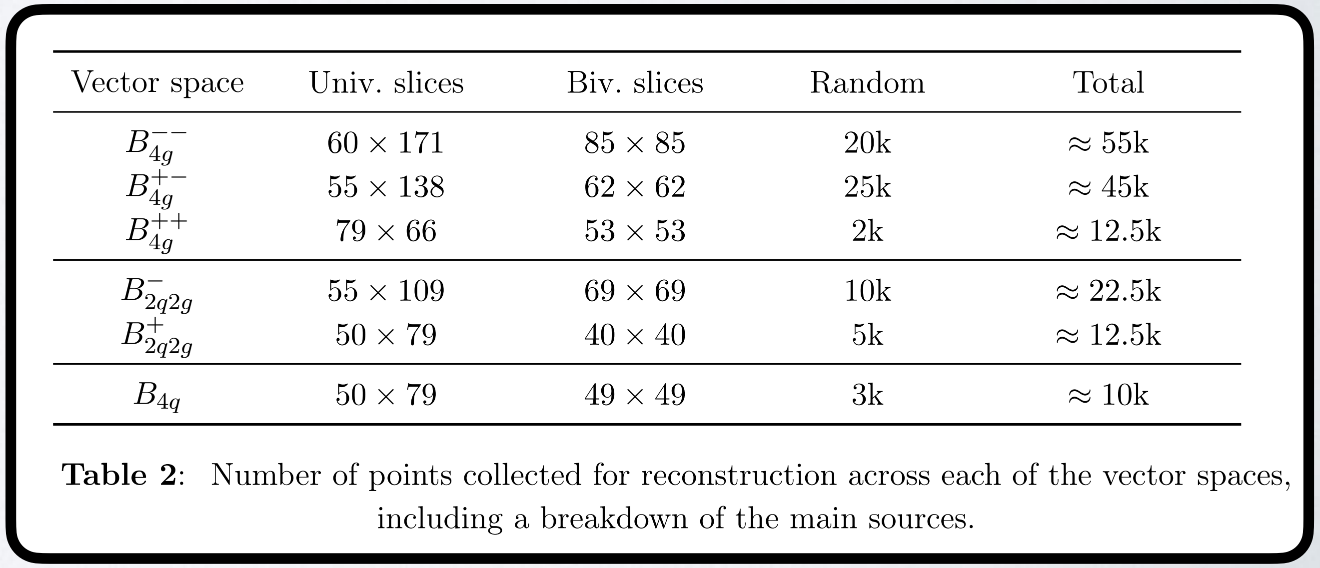
analytic reconstruction possible using optimized ansatz in terms of spinor helciity variables and multivariate partial fractioning

De Laurentis et al. 2605.04009

see also pp → bbH Hartanto, Poncelet 2603.29480



	$\mathcal{A}^{(2),N_c^2}_{0\to\bar{b}bggH}$	$\mathcal{A}^{(2),N_c n_f}_{0\to\bar{b}bggH}$	$\mathcal{A}^{(2),N_c^2}_{0\to\bar{b}b\bar{q}qH}$
$x_1 = 1$	256/251	166/161	145/140
linear relations	187/183	157/153	118/114
denominator matching	187/0	157/0	118/0
univariate partial fraction in x_5	53/52	50/49	41/38
factor matching	53/0	50/0	41/0
number of sample points	157798	122037	60482
required prime fields	3	2	3
evaluation time per point	2050s	190s	841s



Vector space	Univ. slices	Biv. slices	Random	Total
B_{4g}^{--}	60×171	85×85	20k	$\approx$ 55k
B_{4g}^{+-}	55×138	62×62	25k	$\approx$ 45k
B_{4g}^{++}	79×66	53×53	2k	$\approx$ 12.5k
B_{2q2g}^{-}	55×109	69×69	10k	$\approx$ 22.5k
B_{2q2g}^{+}	50×79	40×40	5k	$\approx$ 12.5k
B_{4q}	50×79	49×49	3k	$\approx$ 10k

Table 2: Number of points collected for reconstruction across each of the vector spaces, including a breakdown of the main sources.

special functions with massive particles

massive internal propagators lead quickly to elliptic Feynman integrals (or worse)

what is a basis for the function space?
numerical evaluation not clear

even (nested) square roots can cause trouble for the numerical evaluations

$$dI = M(\epsilon, x).I$$ [Kotikov][Gehrmann, Remiddi]

$$dI = \epsilon M(x).I$$ [Henn]

$$M(x) = \sum_w \mathrm{dlog}(w)$$

leads to MPLs ⇒ good numerical performance

$$dI = \sum_{k=0}^{G} \epsilon^G M^{(G)}(x).I$$

can be acheived in general without elliptic/other objects

further rotation to ε factorised form can be done

numerical integration

when MPLs are not possible…

integrate along path from
boundary point x to y

generalized series expansions

e.g. Liu, Ma, Wang [1711.09572] Moriello [1907.13234]

automated codes

DiffExp

AMFlow

LINE

SeaSyde

Runge-Kutta methods (e.g.
ODEINT Boost Library)

Bulirsch-Stoer shows
promising performance

Czakon, Niggetiedt 2001.03008

Czakon, Harlander, Klappert, Niggetiedt 2105.04436

Rosàs, Bobadilla 2507.12548

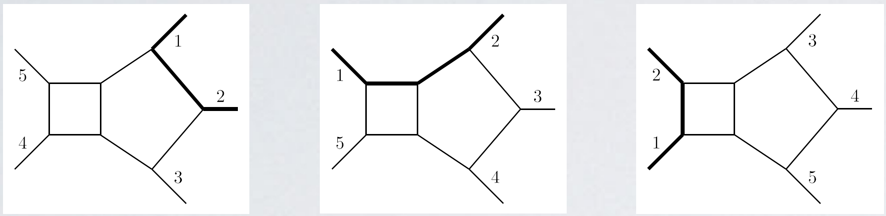
this week! → Baur, Duhr 2606.02744

SB, Becchetti, Brancaccio, Czakon,
Hartanto, Poncelet, Zoia

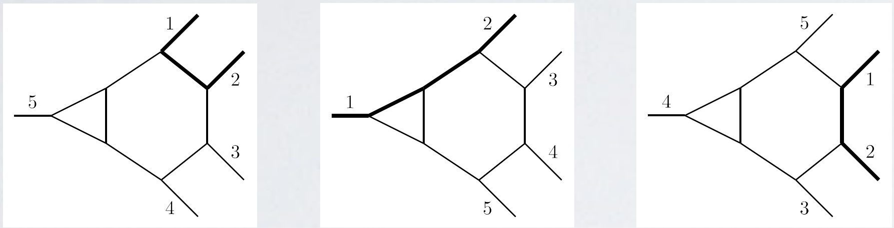
pp→ ttj

2404.12325 2412.13876

2511.11424 2511.11431

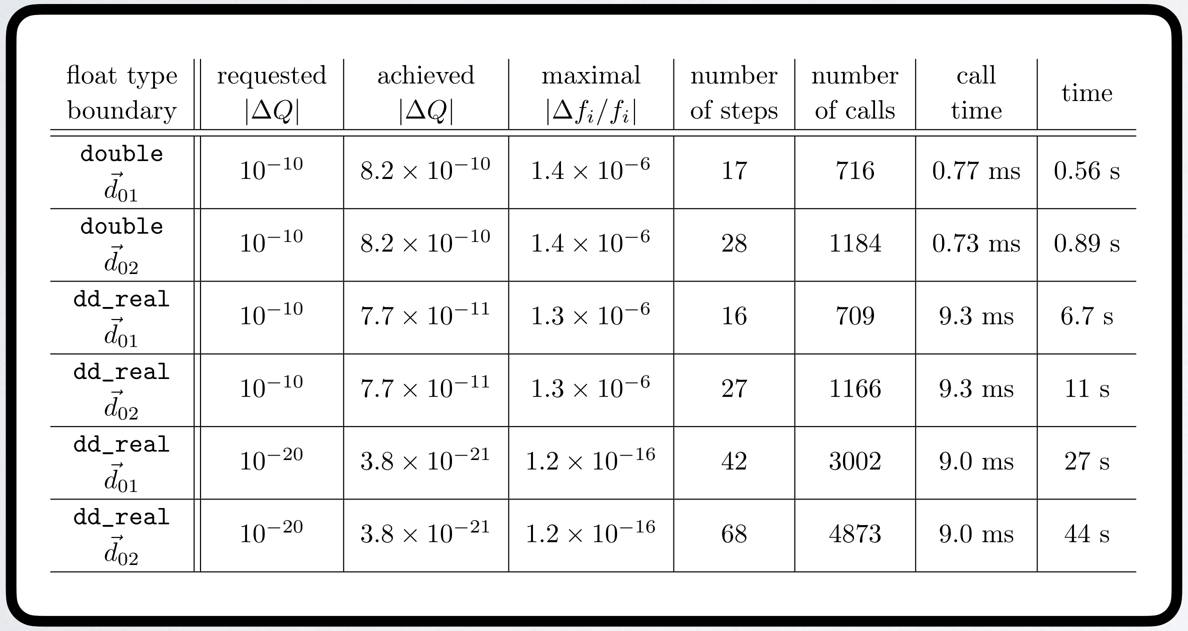


(a) Family PB$_A$. (b) Family PB$_B$. (c) Family PB$_C$.



(d) Family HT$_A$. (e) Family HT$_B$. (f) Family HT$_C$.

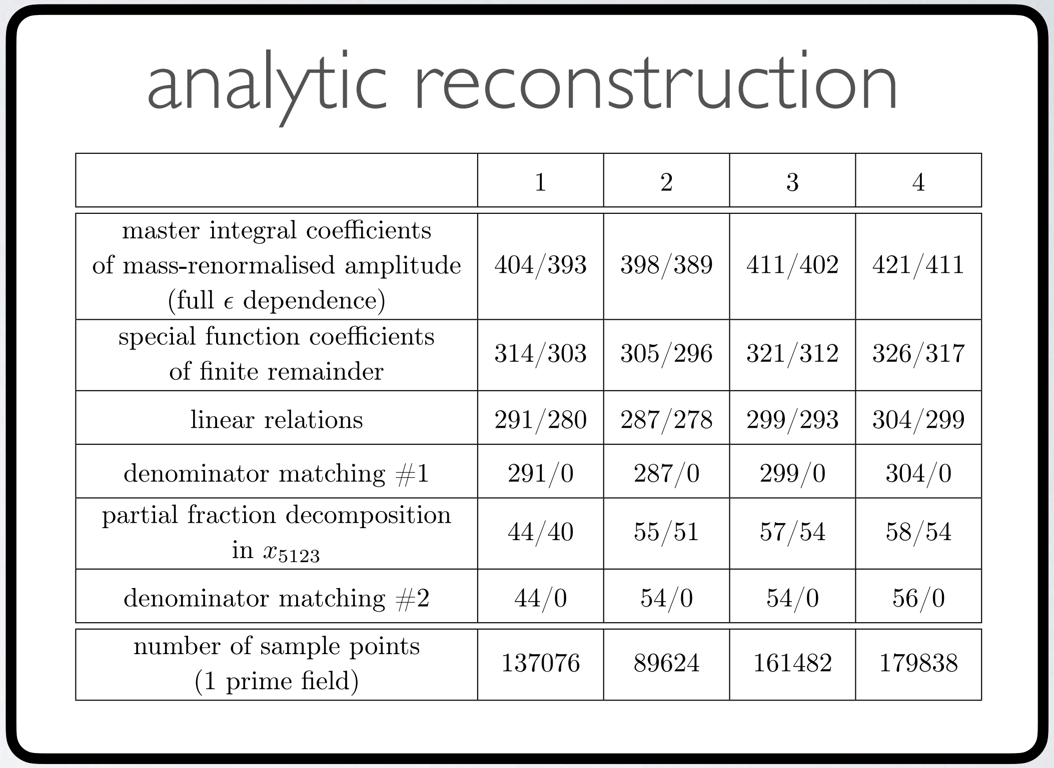
physical projectors, NeatIBP system



float type boundary	requested $\|\Delta Q\|$	achieved $\|\Delta Q\|$	maximal $\|\Delta f_i/f_i\|$	number of steps	number of calls	call time	time
`double` $\vec{d}_{01}$	10^{-10}	8.2×10^{-10}	1.4×10^{-6}	17	716	0.77 ms	0.56 s
`double` $\vec{d}_{02}$	10^{-10}	8.2×10^{-10}	1.4×10^{-6}	28	1184	0.73 ms	0.89 s
`dd_real` $\vec{d}_{01}$	10^{-10}	7.7×10^{-11}	1.3×10^{-6}	16	709	9.3 ms	6.7 s
`dd_real` $\vec{d}_{02}$	10^{-10}	7.7×10^{-11}	1.3×10^{-6}	27	1166	9.3 ms	11 s
`dd_real` $\vec{d}_{01}$	10^{-20}	3.8×10^{-21}	1.2×10^{-16}	42	3002	9.0 ms	27 s
`dd_real` $\vec{d}_{02}$	10^{-20}	3.8×10^{-21}	1.2×10^{-16}	68	4873	9.0 ms	44 s

dσ computed using STRIPPER IR subtraction

is leading colour enough?

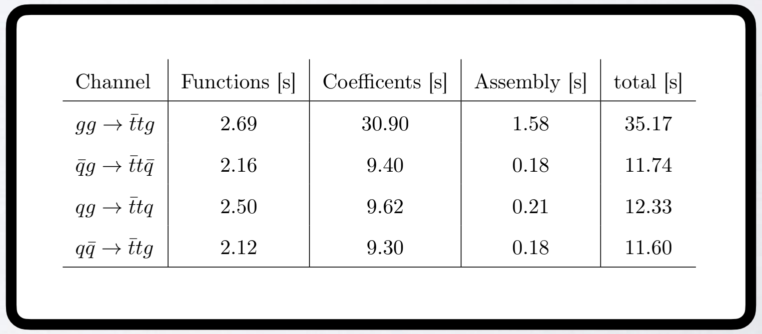


analytic reconstruction

	1	2	3	4
master integral coefficients of mass-renormalised amplitude (full ϵ dependence)	404/393	398/389	411/402	421/411
special function coefficients of finite remainder	314/303	305/296	321/312	326/317
linear relations	291/280	287/278	299/293	304/299
denominator matching #1	291/0	287/0	299/0	304/0
partial fraction decomposition in x_{5123}	44/40	55/51	57/54	58/54
denominator matching #2	44/0	54/0	54/0	56/0
number of sample points (1 prime field)	137076	89624	161482	179838

stable enough BUT rational coefficients
are extremely complicated

quadruple precision often neccesary for the
coefficients - basis functions much quicker



Channel	Functions [s]	Coefficents [s]	Assembly [s]	total [s]
$gg\to\bar{t}tg$	2.69	30.90	1.58	35.17
$\bar{q}g\to\bar{t}t\bar{q}$	2.16	9.40	0.18	11.74
$qg\to\bar{t}tq$	2.50	9.62	0.21	12.33
$q\bar{q}\to\bar{t}tg$	2.12	9.30	0.18	11.60

pp→ ttW

Becchetti, Canko, Chen, Chestnov,
Delto, Ditsch,Kallweit,Peraro, Pozzoli,
Savoini,Tancredi, Zoia, Grazzini

NNLO **total** cross sections using approximate 2-loop amplitudes

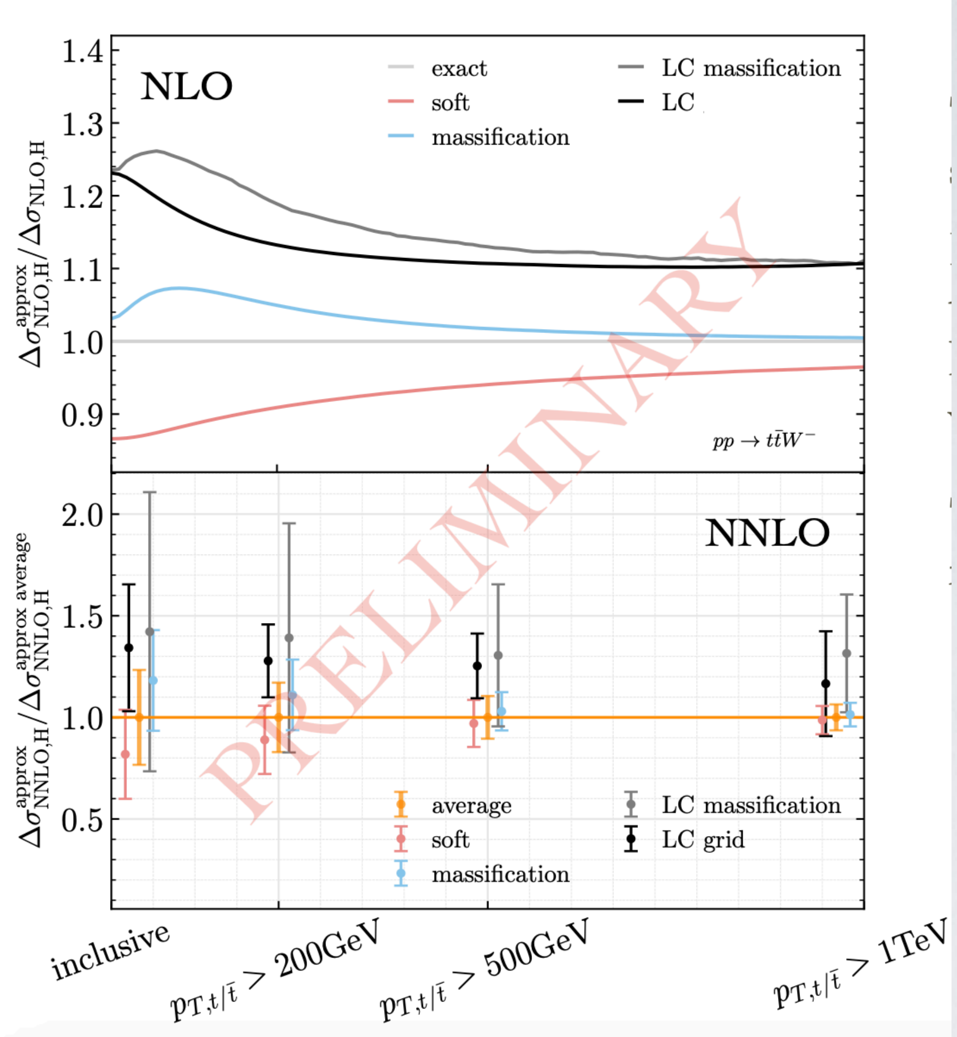
- soft boson limit
- massification [Penin][Moch, Mitov][Becher, Melnikov] [Engel,Gnendiger,Signer,Ulrich][Wang et a.]

NEW! Leading colour ampltudes

challenging amplitude computation
and numerical evaluation

analytic reconstruction not possible

200K points, interpolation
grid, integrals via AMFlow



Grazzini, Top XS WG, 29/05/26

outlook

mass effects particularly relevant for low energy QED computations

multi-loop computations with full mass dependence are developing fast

are we at the limit of analytic workflow with reconstruction over finite fields?

NLO solved? e.g. analytic 1L 2→4 for pp→γγ @ N3LO [Czakon et al. 2604.12613]

Simulations are non-trivial:

- can we share/interface to MC in a way useable to experiments?
- full mass effects come at considerable cost - CPU and human hours!
- fixed order will never be enough for all phase space region
- approximations can be much more efficient for event generation - can we guarentee reliable results when using them?