Phokhara @ NLO+

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Workshop on Radiative Corrections and Monte Carlo Simulations at Electron-Positron Colliders
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Università di Torino

In Pisa **[May 8, 2025]**, we discussed the ultimate plans of `Phokhara`

From Graziano's talk

Desiderable:

CODE	mmg	ppg	Comments (matrix element, FSC)
Phokhara	NNLO	NNLO	exponentiation, FxsQED, GVMD,FsQED

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Phokhara	NNLO	NNLO	exponentiation, FxsQED, GVMD,FsQED

Today's talk:
current status of `Phokhara`

Outline

- Fixed order NLO + soft photon resummation
- GVMD (NLO) and $F \times$ sQED (NNLO) within Phokhara

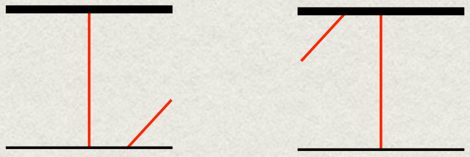
- Fixed order NNLO :: in the making —> first look at Initial State Content

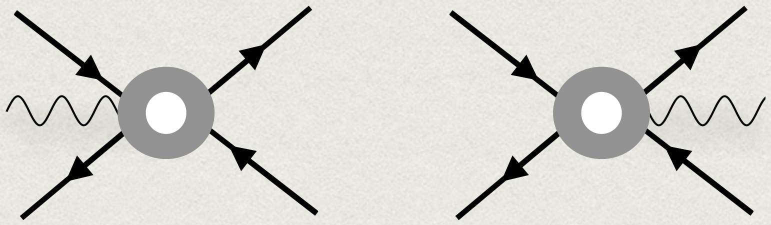
$e^+e^- \to F^+F^-\gamma$ @ NLO

☑ Phokhara is a Monte Carlo event generator for low energy e^+e^- colliders, **with +20 years of development** (https://looptreeduality.csic.es/phokhara/)

Anatomy @ LO

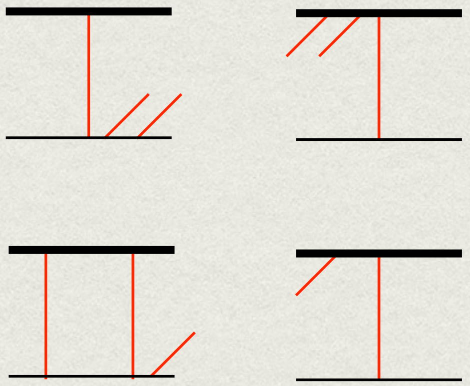
- Born matrix element tree-level & n-pt process





Anatomy @ NLO

- Real contribution tree-level (n+1)-particles
- Virtual Contribution one-loop (n+1)-particles



Use of F×sQED

☑ `Phokhara10` does great for radiative return processes @ NLO accuracy

☑ Several cross checks agains other codes: Radio MonteCarLow2

☑ Workhorse :: efficient evaluation of *multi-loop scattering amplitudes*

$e^+e^- \to F^+F^-\gamma$ @ NLO

- **`Phokhara10`** calculates interference $\mathscr{M}^{(1)} = 2\Re(A^{(1)}A^{(0)*})$ from polarised amplitudes — let's exploit amplitudes!
  - We construct efficient scattering amplitudes regardless of the loop order through a **four-dimensional tensor decomposition** -> immediately leads to independent **polarised/helicity amplitudes**

[Chen 2019; Peraro, Tancredi (2019)]

Tom's talk

- Analytic evaluation of polarised amplitudes
  $e^+(p_1)\, e^-(p_2) \to \pi^+(p_3)\, \gamma(p_4)\, \pi^-(p_5)$
  Understanding of Dirac structure in four dimensions.
  (same study carried out for $e^+e^- \to \mu^+\mu^-\gamma$)

$$\mathcal{A}^{(L)} = \sum_{i=1}^{8} \mathcal{F}_i^{(L)}\, \mathcal{T}_i$$

$$\begin{aligned}
\mathcal{T}_1 &= \bar{v}(p_2, m_e) \not{p}_3 u(p_1, m_e)(p_1 \cdot \varepsilon_4)\,,\\
\mathcal{T}_2 &= \bar{v}(p_2, m_e) \not{p}_3 u(p_1, m_e)(p_3 \cdot \varepsilon_4)\,,\\
\mathcal{T}_3 &= \bar{v}(p_2, m_e) \not{p}_5 u(p_1, m_e)(p_1 \cdot \varepsilon_4)\,,\\
\mathcal{T}_4 &= \bar{v}(p_2, m_e) \not{p}_5 u(p_1, m_e)(p_3 \cdot \varepsilon_4)\,,\\
\mathcal{T}_5 &= \bar{v}(p_2, m_e) u(p_1, m_e)(p_1 \cdot \varepsilon_4)\,,\\
\mathcal{T}_6 &= \bar{v}(p_2, m_e) u(p_1, m_e)(p_3 \cdot \varepsilon_4)\,,\\
\mathcal{T}_7 &= \bar{v}(p_2, m_e) \not{p}_3 \not{p}_5 u(p_1, m_e)(p_1 \cdot \varepsilon_4)\,,\\
\mathcal{T}_8 &= \bar{v}(p_2, m_e) \not{p}_3 \not{p}_5 u(p_1, m_e)(p_3 \cdot \varepsilon_4)\,.
\end{aligned}$$

[Dave, Paltrinieri, Petit Rosas, WJT (2026)]

$e^+e^- \to F^+F^-\gamma$ @ NLO

- **`Phokhara10`** provides fixed NLO theoretical predictions — we need NLO+
  - **Soft photon resummation** — YFS exponentiation

[Yennie, Frauschi, Suura (1961)]

YFS FF

$$\sigma_{\rm NLO}^{\rm resum} = \sum_{n_\gamma} \int_\Omega d{\rm LIPS}_{n_\gamma+2} \frac{e^{\alpha Y_{\rm FF}}}{n_\gamma!} \sum_{\rm hel} \left| \sum_{\phi\in[I,F]^{n_\gamma}} \left(\Pi_{i=1}^{n_\gamma} S_i(k_i)\,\Theta(k_i,\Omega)\right) \left(\beta_0 + \sum_{i=1}^{n_\gamma} \frac{\beta_1(k_i)}{S(k_i)} + \sum_{\substack{i,j=1\\ j<k}}^{n_\gamma} \frac{\beta_1(k_i,k_j)}{S(k_i)\,S(k_j)}\right)\right|^2$$

Regularised amplitudes

Resolved domain

Partitions (ISR/FSR)

Eikonal factors

- Virtual & real amplitudes inside ’s
- Control on ISR and FSR contributions!
- Need momenta mappings to map N-photon events to i-photon events for β_i.

- Warm up exercise :: Scan Mode NLO resummed, CMD scenario

Jeremy’s talk

- Work in progress :: Radiative return NLO resummed, KLOE scenarios

$e^+e^- \to \pi\pi\gamma$ @ 1L

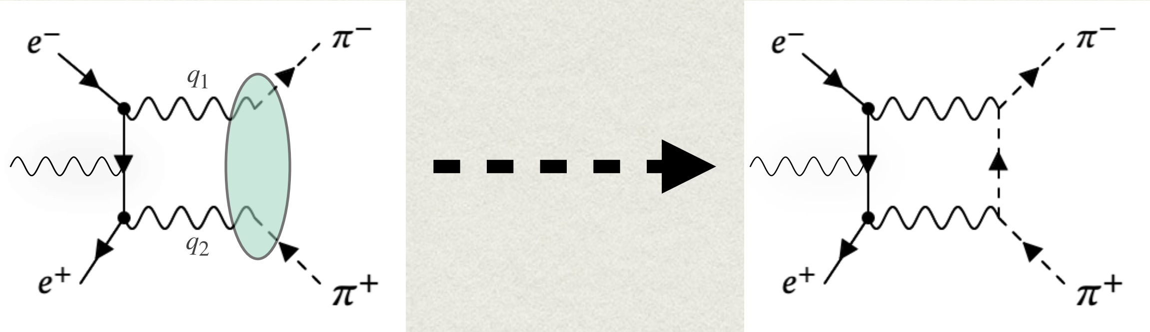
Pau's talk

- **`Phokhara10`** employs F × sQED to describe the interaction between pions and photons
  - Improve description between photon-pion interaction
    —> generalised vector-meson dominance (GVMD)

[Lee, Ignatov (2022)]

[Colangelo, Hoferichter, Monnard, Ruiz de Elvira (2022)]

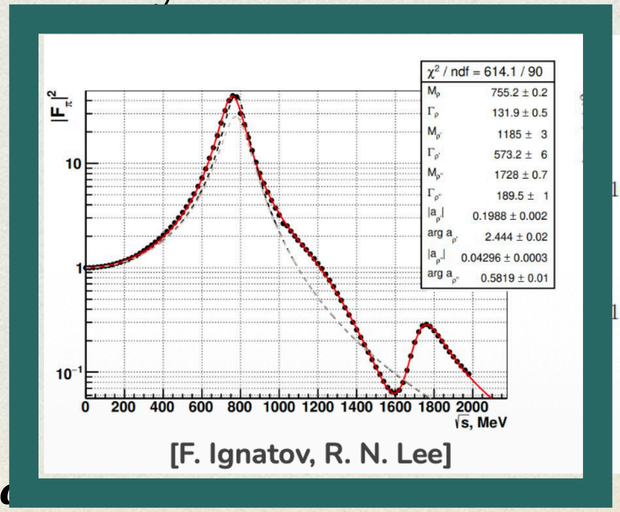
  - Insertion of GVMD in radiative return processes



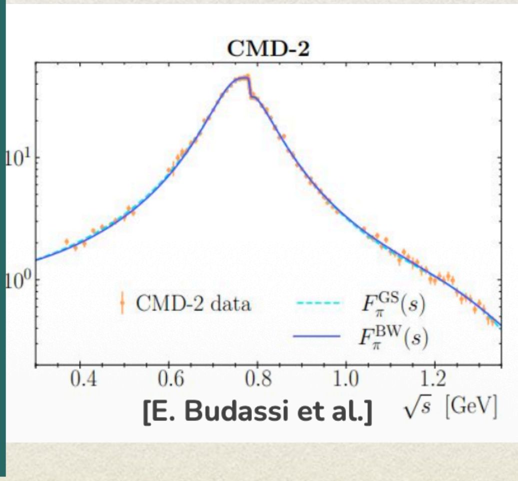
$$\times \left[F(M_{\pi\pi}) + F(q_1^2)F(q_2^2) - F(M_{\pi\pi})\right]$$

[Petit Rosas, Shekhovtsova, WJT (2026)]

  - Suitable to any vector form factor parametrisation



[F. Ignatov, R. N. Lee]



[E. Budassi et al.]

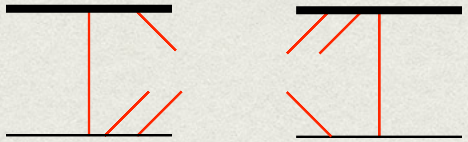
Validations

- ☑ Kloe scenarios: SA & LA
- ☑ Forward-Backward asymmetry in Kloe

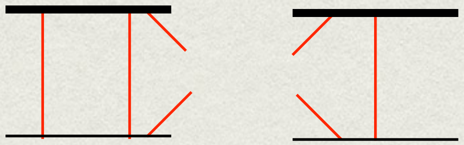
$e^+e^- \to F^+F^-\gamma$ @ NNLO

Anatomy @ NNLO

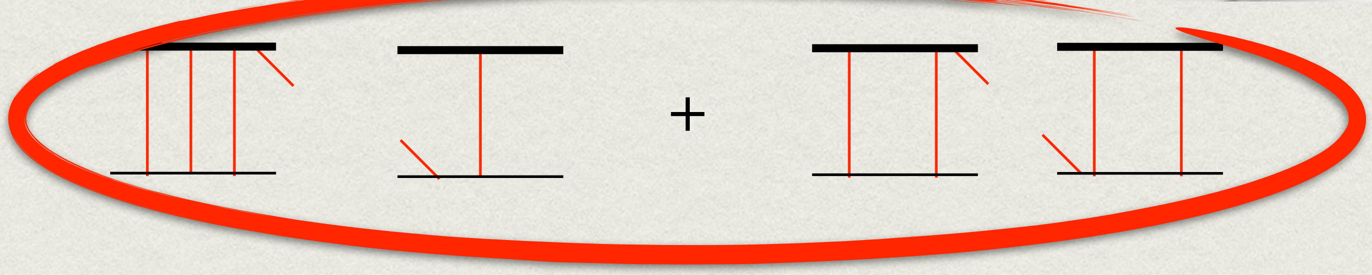
- Real-Real contribution
  Tree-level (*n*+2)-particles



- Real-Virtual Contribution
  one-loop (*n*+1)-particles



- Virtual-Virtual Contribution
  two-loop *n*-particles



☐ Harder (but doable) phase-space integration
☐ Extend numerical evaluation of one-loop Feynman integrals
☐ Basis of two-loop Feynman integral not known

☑ Kinematic invariant $(s_{12}, s_{23}, s_{34}, s_{45}, s_{51}, m_e^2, m_F^2)$
☑ one-loop polarised amplitudes to higher orders in ϵ
☑ Pronounced hierarchy of scales,

$$\frac{m_e^2}{s} \sim 10^{-7} - 10^{-8}$$

[Dave, Paltrinieri, Petit Rosas, WJT (2026)]

$e^+e^- \to \pi\pi\gamma$ @ 2L

Amplitude generation

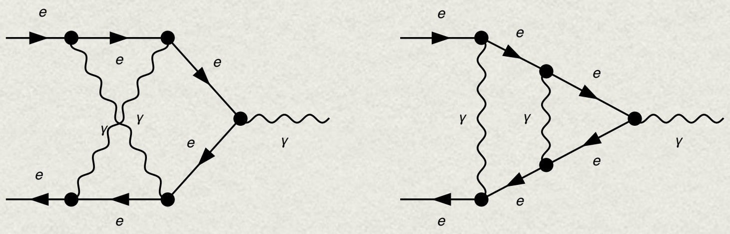
Algebraic decomposition

Loop integral evaluation

MC input

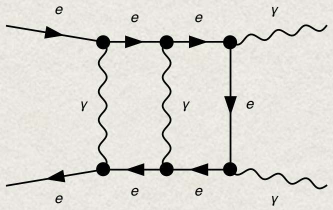
- Two-loop gauge invariant pieces

$f^+f^- \to \gamma^* \to F^+F^- + \gamma$



☑ Easy (m_f^2, s)

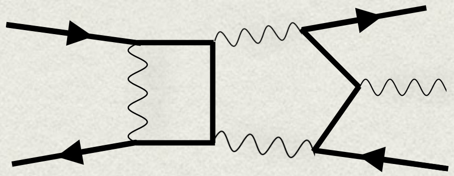
$f^+f^- \to \gamma\,\gamma^* \to F^+F^-$



$+ \ldots$

☐ "Normal" (s, t, m_e^2, q^2)

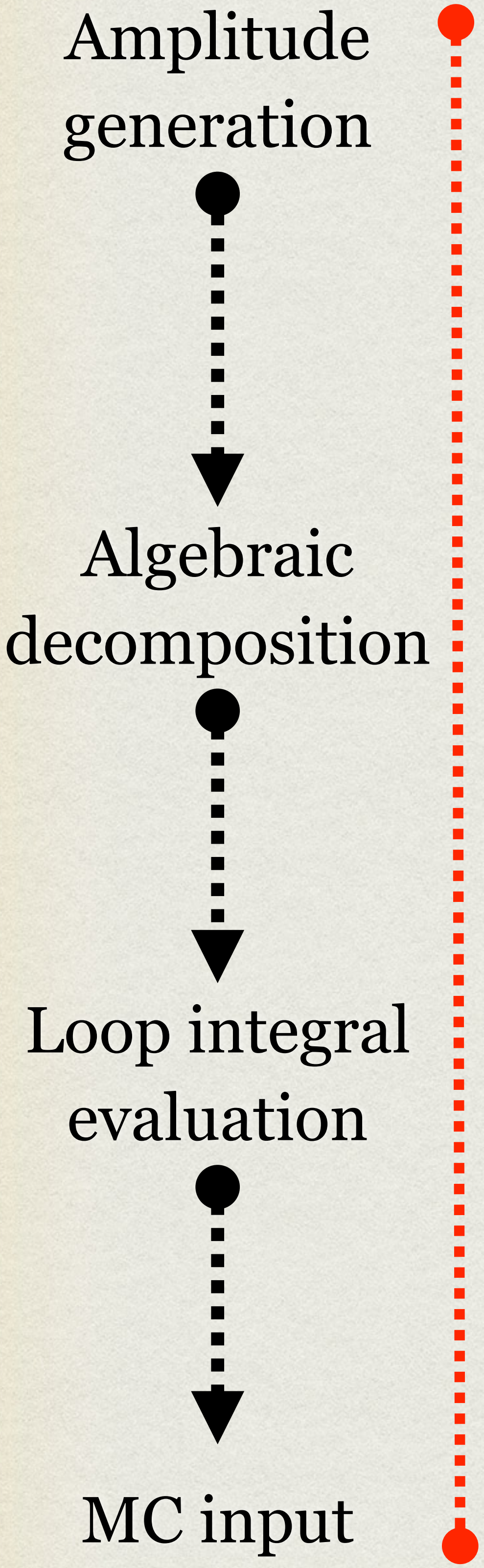
$f^+f^- \to F^+F^-\gamma$



$+ \ldots$

☐ Hard $(s_{12}, s_{23}, s_{34}, s_{45}, s_{51}, m_f^2, m_F^2)$

$e^+e^- \to \pi\pi\gamma$ @ 2L



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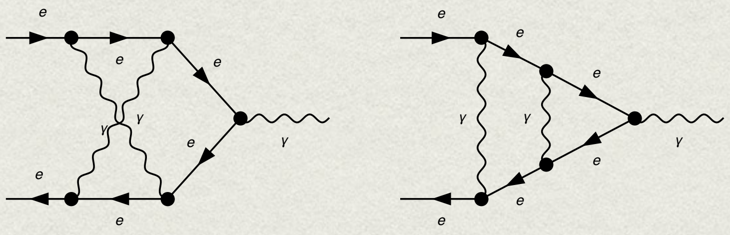
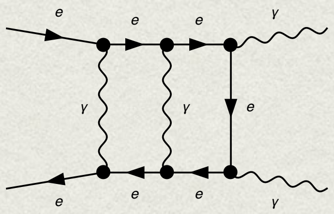


☑ Easy (m_f^2, s)

$f^+f^- \to \gamma\,\gamma^* \to F^+F^-$

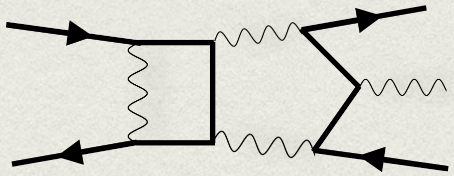


$+\ldots$

☐ "Normal" (s, t, m_e^2, q^2)

Progress on these Feynman integrals

$f^+f^- \to F^+F^-\gamma$



$+\ldots$

☐ Hard $(s_{12}, s_{23}, s_{34}, s_{45}, s_{51}, m_f^2, m_F^2)$

William J. Torres Bobadilla

$e^+e^- \to \pi\pi\gamma$ @ 2L

○ **Initial state contributions**

✓ Amplitude generation

↓

✓ Algebraic decomposition

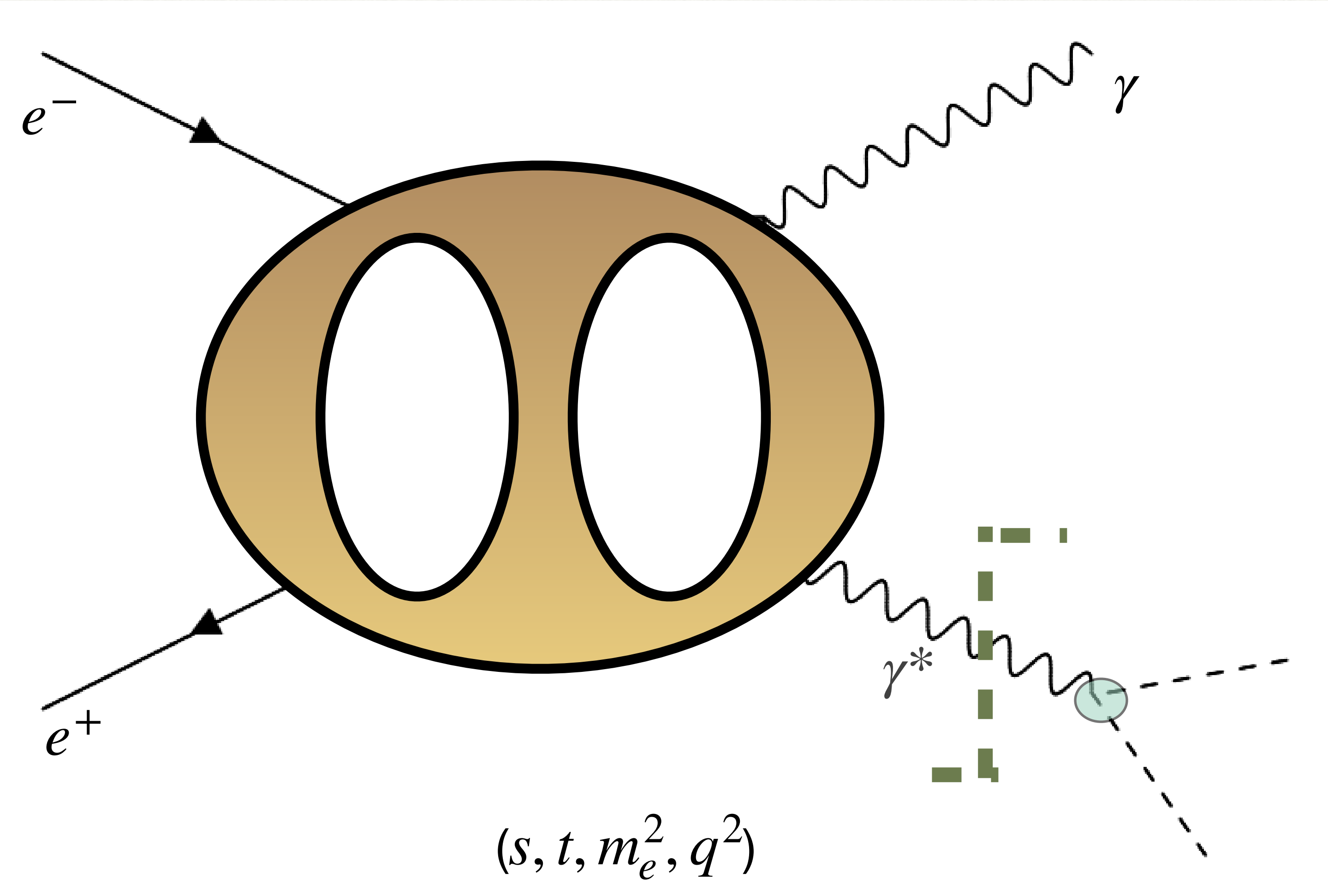
↓

↻ Loop integral evaluation

↓

↻ MC input

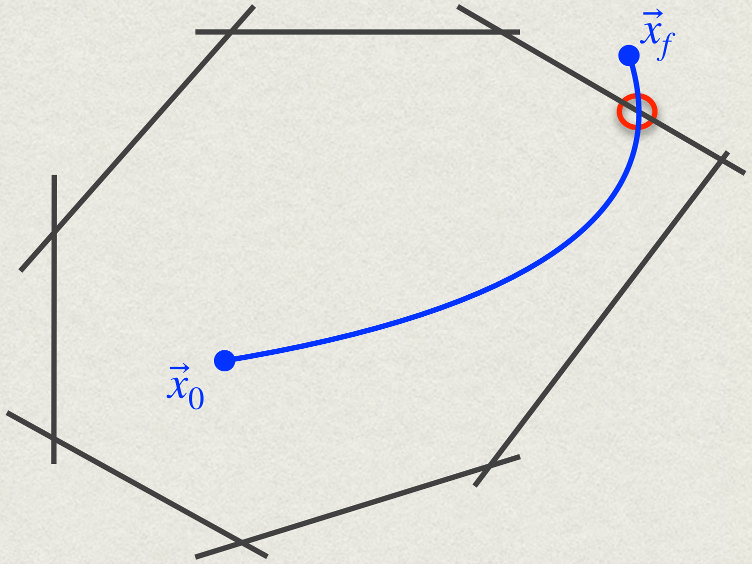
$$e^+e^- \to \gamma\gamma^* \to \pi\pi$$



Evaluation of Feynman integrals

- Evaluation of Feynman integrals by the method of differential equations

$$\partial_x \vec{I}(\vec{x};\epsilon) = A_x(\vec{x};\epsilon)\,\vec{I}(\vec{x};\epsilon)$$



- Find DEQs of the form

$$\mathrm{d}\,A^{(\mathsf{X})}(\vec{x};\varepsilon) = \sum_{k=0}^{2} \varepsilon^k \left[\sum_\alpha c_{k\alpha}^{(\mathsf{X})} \,\mathrm{dlog}\left(W_\alpha(\vec{x})\right) + \sum_\beta d_{k\beta}^{(\mathsf{X})} \omega_\beta(\vec{x}) \right]$$

$$\omega_\beta(\vec{x}) = \omega_\beta^{(s)}(\vec{x})\mathrm{d}s + \omega_\beta^{(t)}(\vec{x})\mathrm{d}t + \omega_\beta^{(m^2)}(\vec{x})\mathrm{d}m^2 + \omega_\beta^{(q^2)}(\vec{x})\mathrm{d}q^2$$

Move **difficult** integrals to very late stages

Available `Mathematica` implementations (DiffExp, …) but slow for MC evaluations and costly to generate grids

- Our numerical approach $\mathcal{O}(10 - 100\text{ms})$ per point

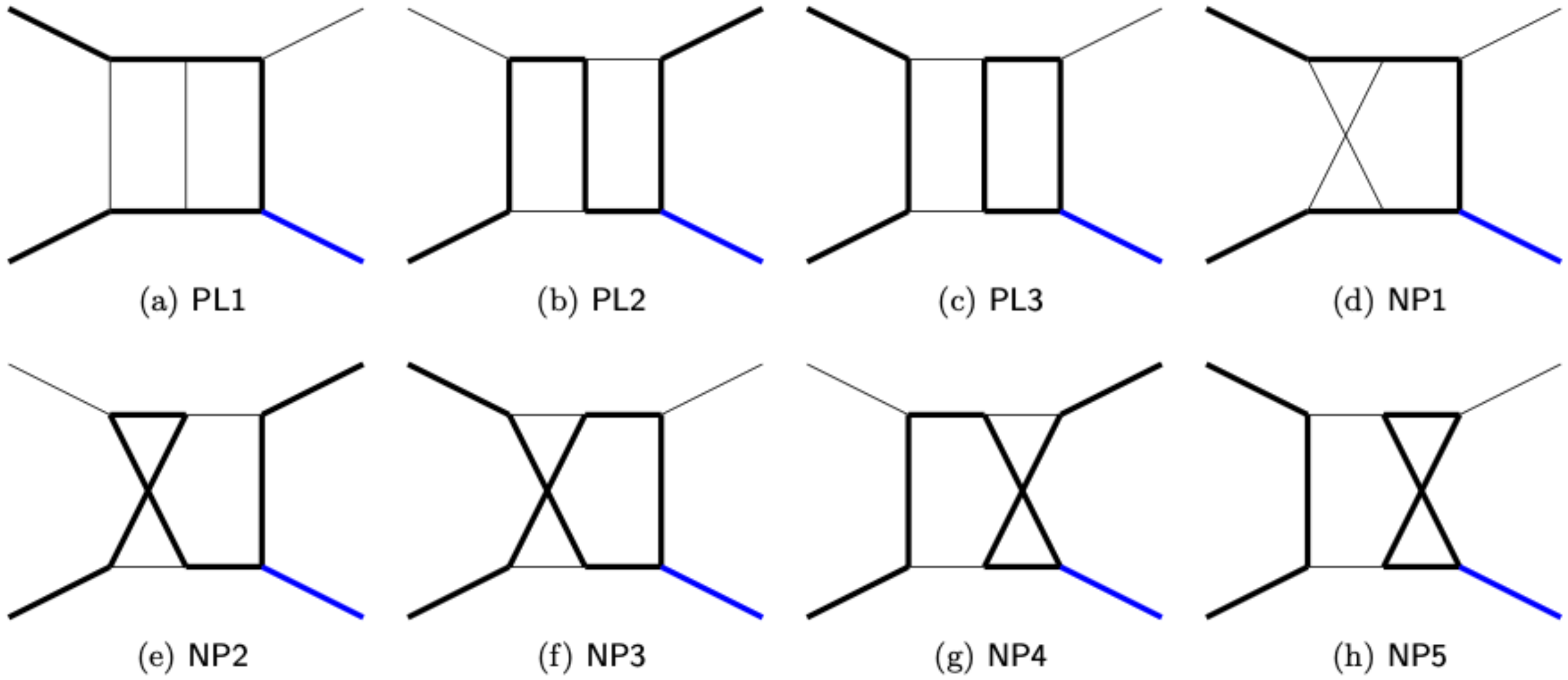
[Petit Rosas, WJT (2025)]

☑ Fast integrator in a low-level language
☑ Precise, but no need for 50 significant figures!
☑ Ideally, fast enough to not use grids

Validations

☑ $e^+e^- \to \pi^+\pi^-\gamma$ up $\mathcal{O}(\epsilon^2)$
☑ $pp \to ttj$ @ 2L
☑ **$e^+e^- \to \gamma\gamma^*$ @ 2L**

Evaluation of Feynman integrals for $e^+e^- \to \gamma\gamma^*$ *@ 2L*


(a) PL1 (b) PL2 (c) PL3 (d) NP1

(e) NP2 (f) NP3 (g) NP4 (h) NP5

Figure 1: Two-loop integral families. Thin lines correspond to massless external momenta and propagators, thick black lines account for propagators and momenta with mass m^2, and blue ones for the external momentum of mass q^2.

Mattia's talk

- ✎ Presence of Elliptic and hyper elliptic integrals
- ✓ Planar integrals (PL1, PL2, PL3) completed
- ↻ Non planar integrals in the making

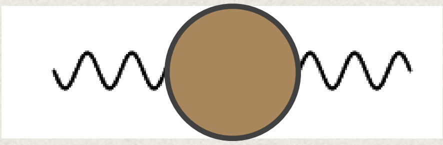
[Pozzoli, WJT (to appear)]

$e^+e^- \to \gamma\gamma^*$ @ 2L

[Dave, Petit Rosas, Pozzoli, WJT (in preparation)]

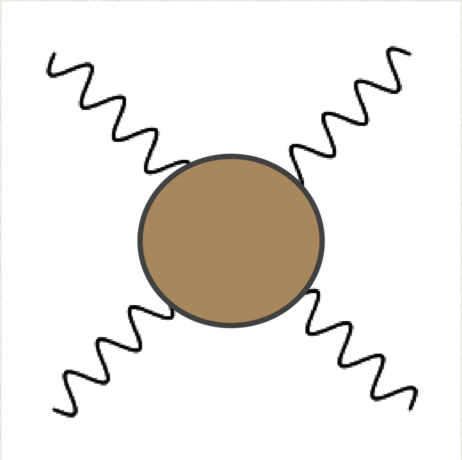
Decomposition of the two-loop amplitudes (from leptonic to hadronic corrections)

$$\mathscr{A}^{(2)} = A^{(2)} + N_F\, B^{(2)}$$

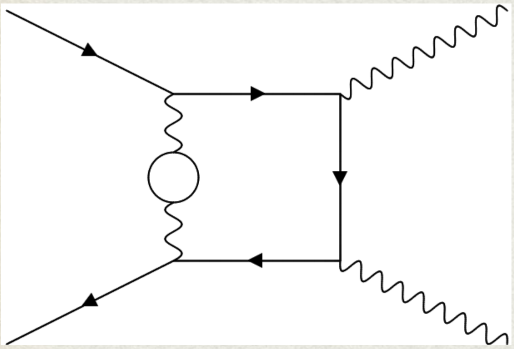


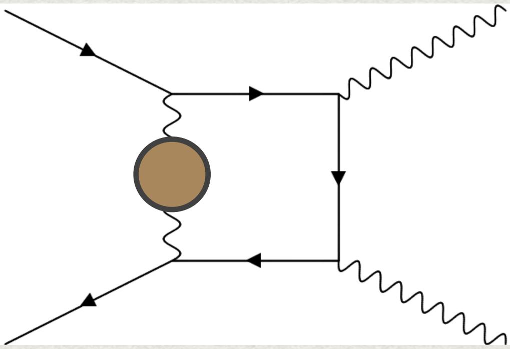
Photon correction

$$\Pi^{(\text{had})}\left(q^2\right) = -\frac{q^2}{\pi}\int_0^\infty \frac{d\Lambda}{\Lambda^2}\frac{\widetilde{\Pi}^{(\text{had})}\left(\Lambda^2\right)}{q^2-\Lambda^2+\imath 0}$$



Light-by-light





$$-\frac{1}{\pi}\int_0^\infty \frac{d\Lambda}{\Lambda^2}\widetilde{\Pi}^{(\text{had})}\left(\Lambda^2\right)\times[1\text{L}]$$

$e^+e^- \to \gamma\gamma^*$ @ 2L

Decomposition of the two-loop amplitudes (from leptonic to hadronic corrections)

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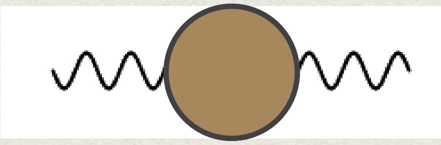

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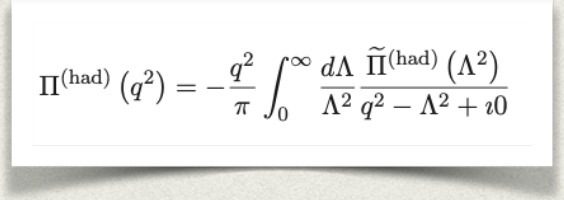

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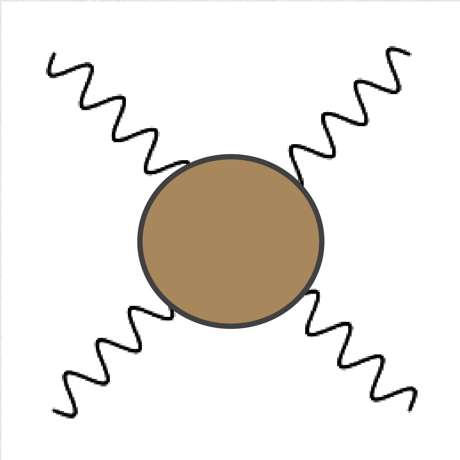

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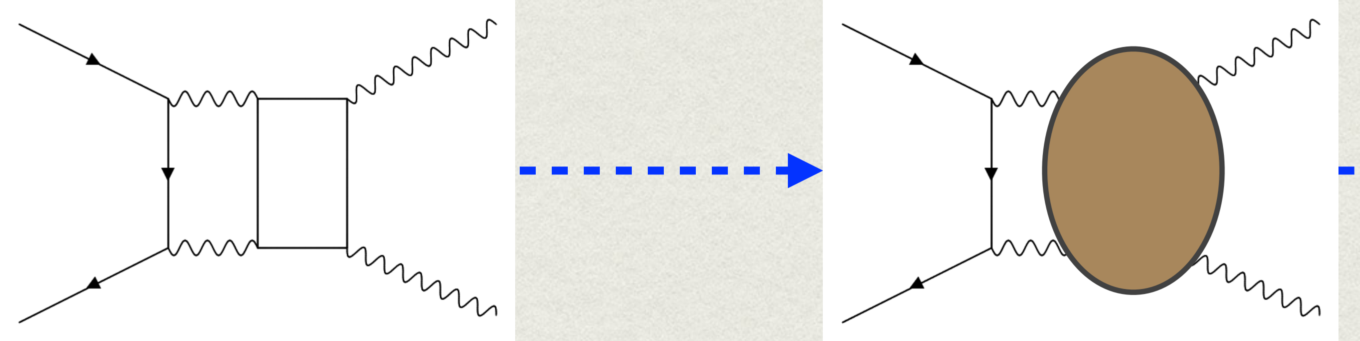
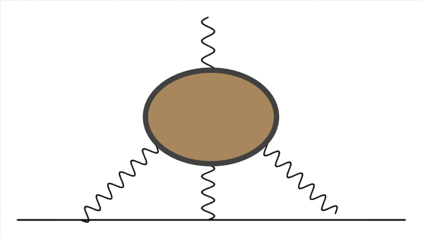


Can we borrow knowledge from$(g-2)_\mu$?



$e^+e^- \to \gamma\gamma^*$ @ 2L

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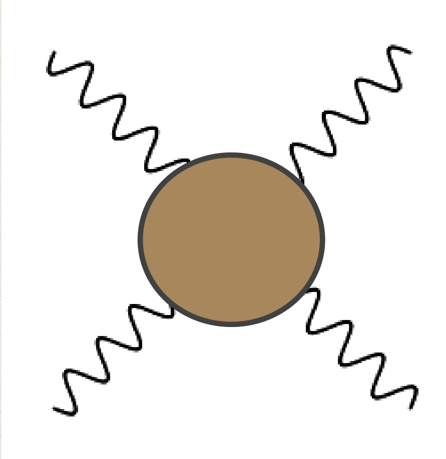

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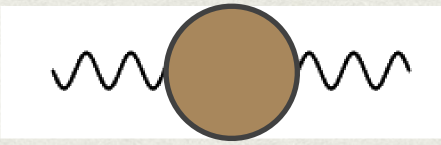

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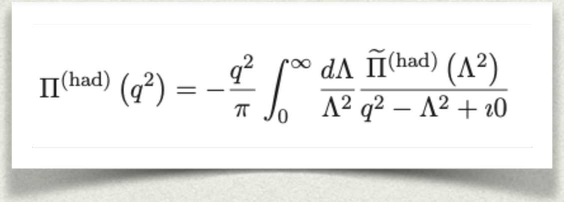

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Getting a dispersive representation in terms of one-loop amplitudes will immediately a check for $B^{(2)}$ calculation and show possible pitfalls in the calculation of $A^{(2)}$

Conclusions

- We have reached (and will be discussed in more details):

☑ Analytic study of radiative return processes
☑ First improvements and validation of the `Phokhara` generator
GVMD + Soft-photon resummation
☑ First look at the evaluation of two-loop Feynman integrals for $e^+e^- \to \gamma\,\gamma^*$

- Open questions & future directions

☐ Complete radiative return NLO resummed for KLOE scenarios
☐ Perform comparison against FsQED
☐ Complete calculation of two-loop Feynman integrals for $e^+e^- \to \gamma\,\gamma^*$

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Thanks