Towards a Bulk Dual of Toda Theory through $O(N)$ Vector Models

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- A_{n-1} **Toda enhances the chiral algebra with conserved holomorphic currents of spins** $2, 3, \ldots, N$
  - Generalizes Liouville to a theory of an $(n-1)$-component scalar field with exponential interactions determined by the A_{n-1} roots
- **General structure:** a tower of higher-spin currents strongly suggesting a bulk description with higher-spin gauge fields (beyond pure Einstein gravity).

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  - → These constructions realize sectors of theories with W_n symmetry (aka Minimal Model Holography), but **do not** recover the full spectrum nor provide a complete bulk dual of Toda.

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- ⇒ What kinds of objects appear in the bulk dual if these were the only criteria?
- ⇒ Need a concrete construction if these questions are to be well-posed.

Toda Action and Correlators

$$S_T = \int d^2x\sqrt{g}\left(\frac{1}{4\pi}\langle\partial\phi,\partial\phi\rangle + \frac{R}{4\pi}\langle Q,\phi\rangle + \mu\sum_{i=1}^{N} e^{b\langle e_i,\phi\rangle}\right)$$

- e_i are the *simple roots* of A_{N-1}; $\langle\cdot,\cdot\rangle$ is the inner product on root space.

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- e_i are the *simple roots* of A_{N-1}; $\langle \cdot, \cdot \rangle$ is the inner product on root space.
- ϕ is an N-component field: $\phi = \sum_i \phi_i f_i$, with $\{f_i\}$ orthonormal $\Rightarrow$ kinetic term diagonal/normalized.

Toda Action and Correlators

The A_N Toda correlators admit a Coulomb gas representation

$$\langle \mathcal{V} \rangle \propto \int \mathcal{D}X_n \, e^{-S_{red}(X)} \underbrace{\Big(\text{Vertex Operators}\Big)}_{e^{2\langle \alpha, \phi \rangle}} \underbrace{\Big(\text{Screening Insertions}\Big)}_{\left(\int d^2x \sqrt{g} \, e^{\cdots}\right)^{w_a}}$$

with a reduced free action

$$S_{\text{red}} = \int d^2x \sqrt{g} \left[\frac{1}{4\pi} \sum_{a=1}^{n} \partial_\mu X_a \partial^\mu X_a \right].$$

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Structural takeaways

- **Free part:** S_{red} is Gaussian, local, and (after the redefinitions) has a **vector-model-like** structure.

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Structural takeaways

- **Free part:** S_{red} is Gaussian, local, and (after the redefinitions) has a **vector-model-like** structure.
- **Insertions:** heavy operator insertions (screening-charge-type objects) are **nonlocal** and couple different X_a's.
- **Key obstacle:** the insertions *break* the $O(N)$ invariance one would exploit in standard collective-field holography.

Vector Models in the Bilocal Language

- Free $O(N)$ model:

$$S = \frac{1}{2} \int d^2x \sqrt{g} \sum_{I=1}^{N} \partial_\mu \Phi_I \, \partial^\mu \Phi_I .$$

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- The singlet sector is described by the bilocal $O(N)$ invariant

$$G(x_1,x_2)=\frac{1}{N^r}\sum_{I=1}^{N}\Phi_I(x_1)\Phi_I(x_2),$$

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- Fluctuations of G encode the tower of singlet primaries [Jevicki-Sakita, 1980; de Mello Koch-Jevicki-Suzuki-Yoon, 2011]
- Allows for a direct bulk reconstruction (through the embedding space formalism) of bulk higher spin fields [Aharony–Chester–Urbach, 2021].

Vector Models in the Bilocal Language

Goal

Rewrite the free-field sector *and* the nonlocal insertions of Toda using **order-N singlets** that can survive the symmetry breaking of $O(N)$

Vector Models in the Bilocal Language

- Define $O(N-2)$ and $O(N-1)$ singlets

$$G_I^{N-2}(x_1, x_2) = \frac{1}{(N-2)^r} \sum_{a \neq I, I+1} \Phi_a(x_1)\Phi_a(x_2)$$

$$G_I^{N-1}(x_1, x_2) = \frac{1}{(N-1)^r}\Big(\Phi_{I+1}(x_1)\Phi_{I+1}(x_2) + (N-2)^r G_I^{N-2}(x_1, x_2)\Big)$$

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- Image of this map is an N-**dimensional submanifold** inside a $2N$-dimensional field space.
- The Jacobian will enforce the embedding via N **independent functional relations** (selecting the physical submanifold).

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$$S = \frac{1}{2} \int d^2x \sqrt{g} \sum_{I=1}^{N} \left[\nabla_{x_1} \cdot \nabla_{x_2} \left((N-1)^r G_I^{N-1} - (N-2)^r G_I^{N-2} \right) \right]_{x_1 = x_2}$$

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  - The partition function becomes

$$Z = \int \mathcal{D}G^{N-1} \mathcal{D}G^{N-2} \, J[G^{N-1}, G^{N-2}] \, e^{-S[G^{N-1}, G^{N-2}]}$$

where J is defined by inserting two delta-functionals per I (one for G^{N-2} and one constraining $\Phi_{I+1}\Phi_{I+1}$).

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    where J is defined by inserting two delta-functionals per I (one for G^{N-2} and one constraining $\Phi_{I+1}\Phi_{I+1}$).
  - The Jacobian organizes schematically as

$$J \;\propto\; \prod_{I=1}^{N} \left| \det D_I \right|^{\frac{1}{2} - V} \left| \det \mathcal{T} \right|^{-V},$$

    so that

$$Z = \int \mathcal{D}G^{N-1} \mathcal{D}G^{N-2} \, e^{-S_{\text{eff}}}.$$

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  - Understand the heavy operators ($\Delta \sim N$) in this bilocal language
  - Control over Toda correlators (e.g. through probabilistic techniques) suggests that the partition function is tractable
  - We compute only *microscopic* (flavor-resolved) correlators

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$$G_{I,0}^{N-1}(x,y) = (N-1)^{1-r}\, G_0(x,y)$$
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  - Expand around the background:

$$G_I^{\alpha} = G_{I,0}^{\alpha} + \frac{1}{N^k}\eta_I^{\alpha}$$

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  - Ensure vanishing of loop corrections - cancellations are non-trivial, but they occur sector by sector

$$\boxed{\langle \eta_I^\alpha \eta_I^\beta \rangle^{\text{3-pt}}_{\text{1-loop}} + \langle \eta_I^\alpha \eta_I^\beta \rangle^{\text{4-pt}}_{\text{1-loop}} + \langle \eta_I^\alpha \eta_I^\beta \rangle^{ct}_{\text{1-loop}} = 0}$$

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- Armed with large-N singlets, we map the theory to a conformal basis and employ the embedding space formalism

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- Study the spectrum and thermodynamics of the bulk.

Mechanism in $d = 2$: $\mathrm{AdS}_3 \times S^1$ from bilocal kinematics

- A bilocal depends on two boundary points:

$$(x_1^\mu, x_2^\mu) \in (\mathbb{R}^2)^2 \quad \Rightarrow \quad \text{4 real dof.}$$

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$$x = \tfrac{1}{2}(x_1 + x_2) \quad (2\text{ dof}), \qquad r = x_1 - x_2 \quad (2\text{ dof}).$$

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- So the unconstrained bilocal phase space naturally organizes like $\mathbf{AdS}_3 \times S^1$.

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and reinterpret the result as a bulk field $\tilde{H}(p, p_z, \theta)$; inverse Fourier transform gives $H(x, z, \theta)$ [de Mello Koch–Jevicki–Suzuki–Yoon, 2011].

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- Aharony et al. describe the same map in a conformal/embedding-space basis (same content, different organization) [Aharony–Chester–Urbach, 2021].