LITP Workshop on Quantum Black Holes
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Fortuity and Quantum Corrections to Q-cohomology

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Outline

- A brief introduction to Q-cohomology and fortuity
- How does the Q-cohomology depend on the couplings?
- Quantum corrections to Q-cohomology from OPE and normal ordering
- Conclusion

Introduction

- The study of the supercharge Q**-cohomology** in supersymmetric field theories dates back to Witten's seminal paper: Constraints on supersymmetry breaking

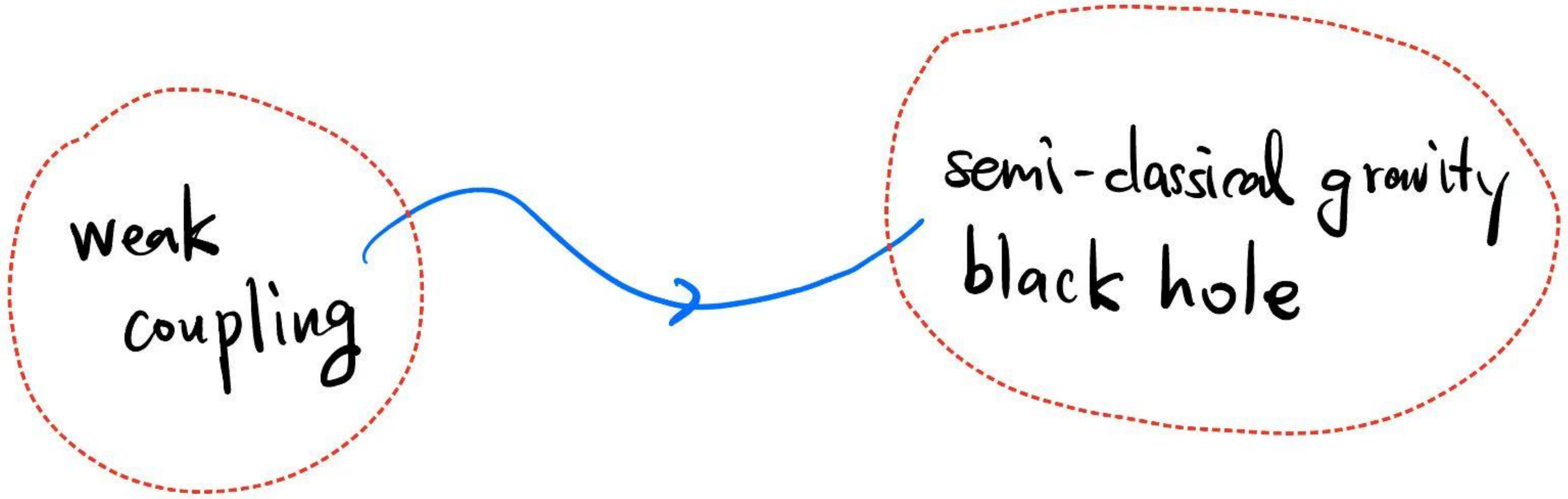
- Witten argued: BPS states $\overset{\text{1 to 1}}{\longleftrightarrow}$ Q-cohomology classes

- The Witten index is nothing but the Euler characteristic of the Q-cohomology, and Witten argued that it is independent of couplings.

- However, how the Q-cohomology itself depends on the couplings remains an open question.

Introduction

- This question is particularly important in holography.



- So far, computations of Q-cohomology can only be carried out at weak coupling, while under holography, the dual semi-classical gravity resides in the strong coupling region.

- If the coupling dependence of the Q-cohomology can be controlled, we can learn a lot about the physics in the gravity dual.

Introduction

- The Witten index counts the difference in the number between the bosonic and fermionic BPS states.

- It matches the Bekenstein-Hawking entropy of supersymmetric black holes: $S = \frac{A}{4G_N} = \log(\#\,\text{BPS states})$.

- The Q-cohomology carries more information, much more beyond just the number of BPS states.

- One can “follow” a Q-cohomology class as the rank N varies. (Roughly speaking, the rank N characterizes the number of d.o.f in the dual QFT.)

Introduction

- Under holography, the bulk Newton's constant is related to N as

$$G_N \sim \frac{1}{N^\alpha}, \quad \alpha > 0$$

- In the weak coupling limit $G_N \to 0$ ($N \to \infty$), the bulk Hilbert space is a Fock space consisting of multi-particle states (multiple particle creation operators acting on the AdS vacuum state).

- Holography: single-trace $\mathrm{Tr}\,(\cdots)$ $\leftrightarrow$ single-particle state

multi-trace $\mathrm{Tr}\,(\cdots)\mathrm{Tr}\,(\cdots)\mathrm{Tr}\,(\cdots)$ $\leftrightarrow$ multi-particle state

Introduction

- Holography predicts that, when G_N increases (N decreases), there are quantum relations among different multi-particle states.

- This is because, in the dual gauge theory, such relations are trace relations.

- For example, in an $\mathrm{SU}(2)$ gauge theory, two adjoint scalars X and Y must satisfy $\mathrm{Tr}\,(X^2Y^2) = \frac{1}{2}\mathrm{Tr}\,(X^2)\mathrm{Tr}\,(Y^2)$.

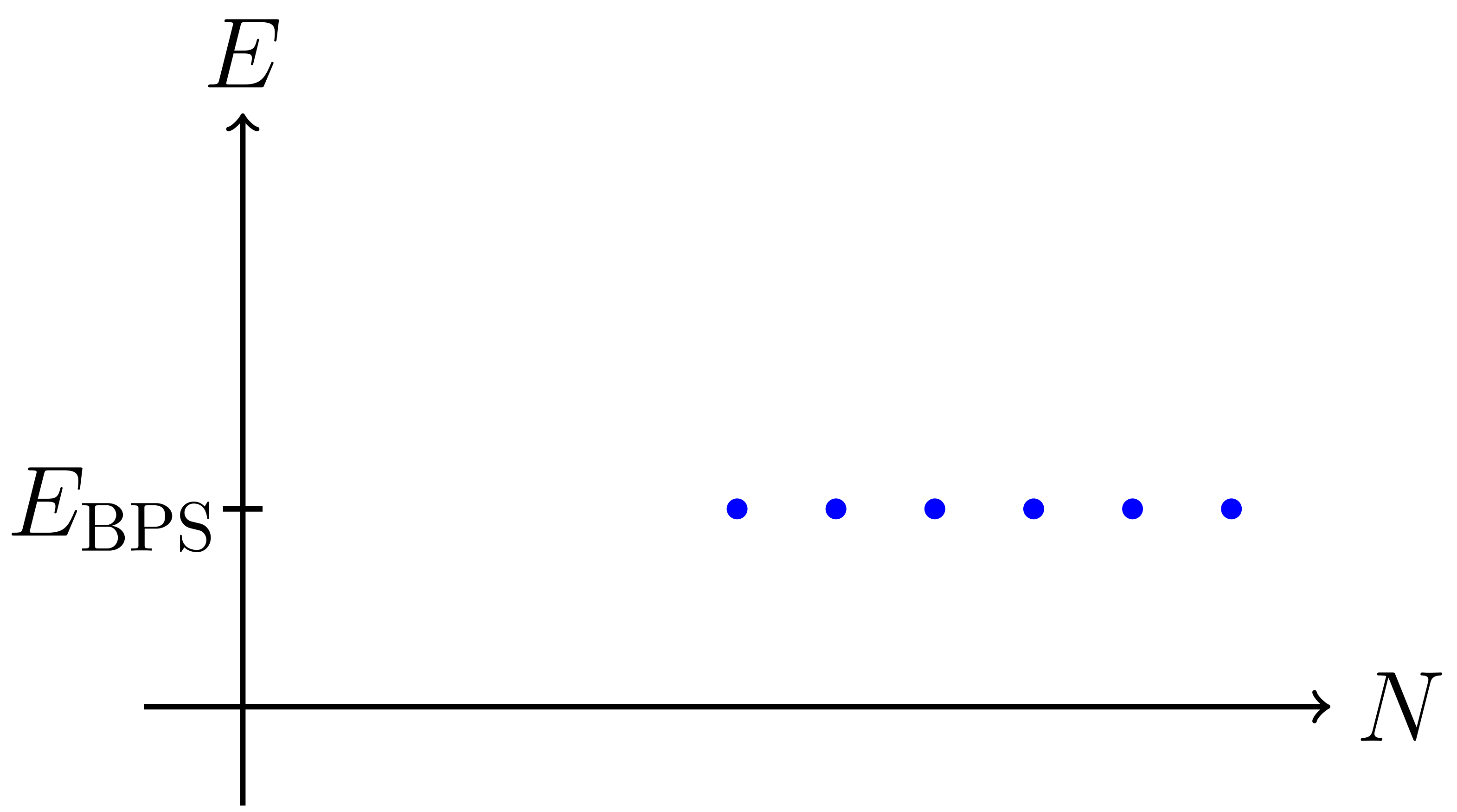
- Therefore, there are "fewer" states in the interacting gravity than in the free gravity.

Introduction

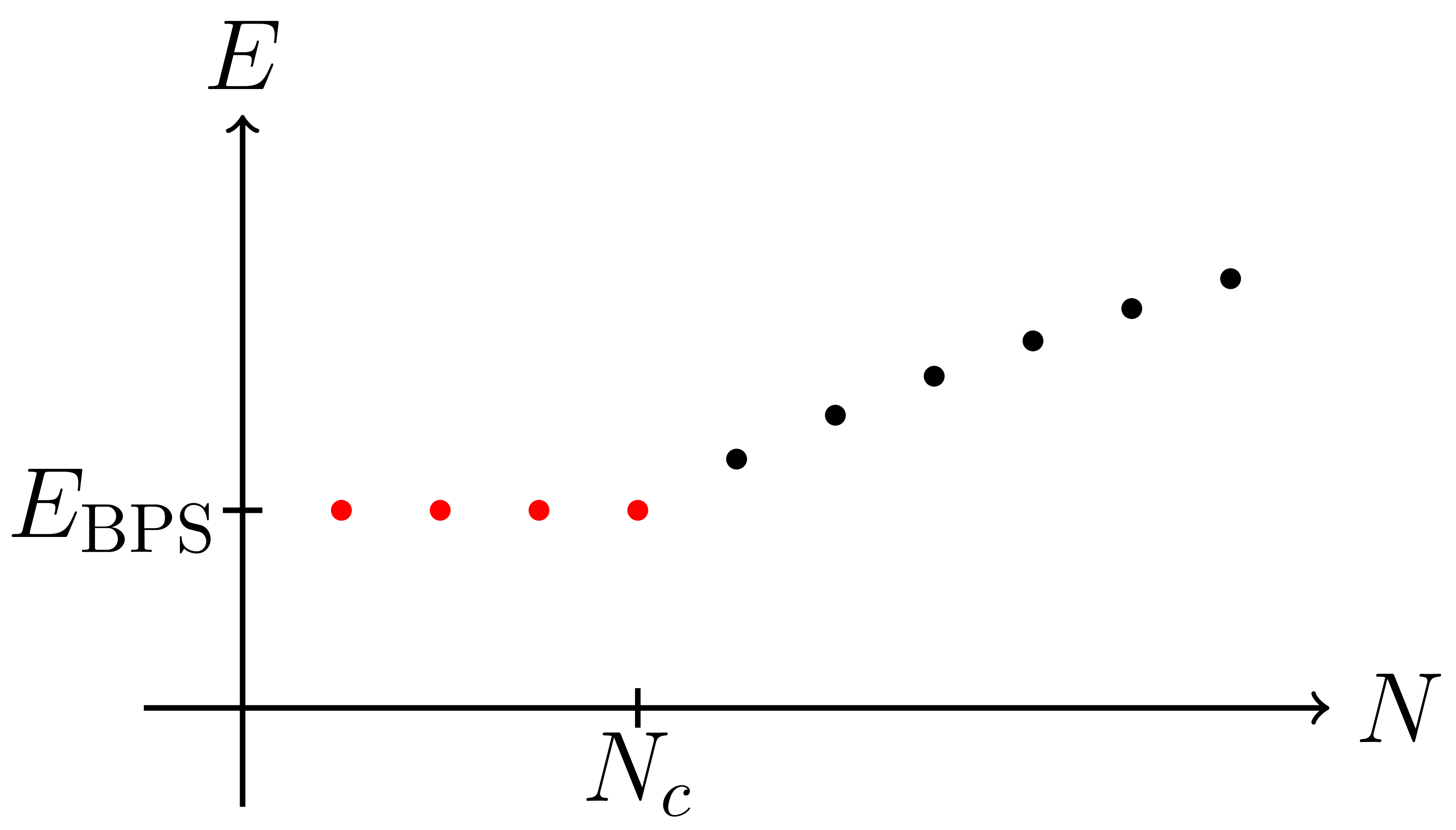
- However, this contradicts our intuition from semi-classical gravity.
- In any interacting gravity theory, there exist black holes, which are believed to be the objects with maximal entropy.
- The number of microstates of black holes should be much larger than the number of multi-particle states.
- To resolve this puzzle, let us focus on supersymmetric black holes, whose microstates are BPS states. Many of their properties are captured by the Q-cohomology.

Introduction

- For BPS states, we have the old mechanism that multi-particle BPS states can disappear by the quantum relations dual to trace relations.



- There is a new mechanism (called the fortuity mechanism) that non-BPS states become BPS when the gravitational interaction becomes strong (N decreases).



Introduction

- The fortuity mechanism can be easily understood by Q-cohomology.

- Q-cohomology classes fall into two categories:

  Let $\mathcal{O}$ be a formal multi-trace operator (an operator in the $N = \infty$ theory).

  - **Monotone classes:** $Q\mathcal{O} = 0$ without using any trace relation.

    Hence, $\mathcal{O}$ represents a cohomology class at any N.

  - **Fortuitous classes:** $Q\mathcal{O} =$ (a trace relation at $N \leq N_c$)

    $\mathcal{O}$ only represents a cohomology class at $N \leq N_c$.

Introduction

- The fortuitous classes are responsible for the black hole entropy. In fact, we can show that, at fixed energy $E = E_{\text{BPS}}(\text{charges, spins})$,

$$\#\ \text{fortuitous} \overset{\text{exponentially}}{\gg} \#\ \text{monotone}$$

- In other words, a typical black hole microstate must be fortuitous.

- Therefore, we hope that the study of fortuitous states can reveal interesting and important microscopic physics of black holes.

Introduction

- Maybe, the question we most like to know the answer is:

What is a (typical) black hole microstate?

- In semi-classical gravity, a black hole solution describes an emsemble of microstates. But, in a UV complete theory of gravity (for example, string theory), one should be able to describe a **single** typical black hole microstate.

- This is a question people also ask in the fuzzball program.

Introduction

- In the fuzzball program, many **smooth and horizonless** supergravity solutions are constructed. They are very close to black hole solutions outside the event horizon. [A nice review: Bena-Martinec-Mathur-Warner'24]

- This led people to wonder if the black hole microstates can already be captured in the low-energy supergravity without using the UV string theory.

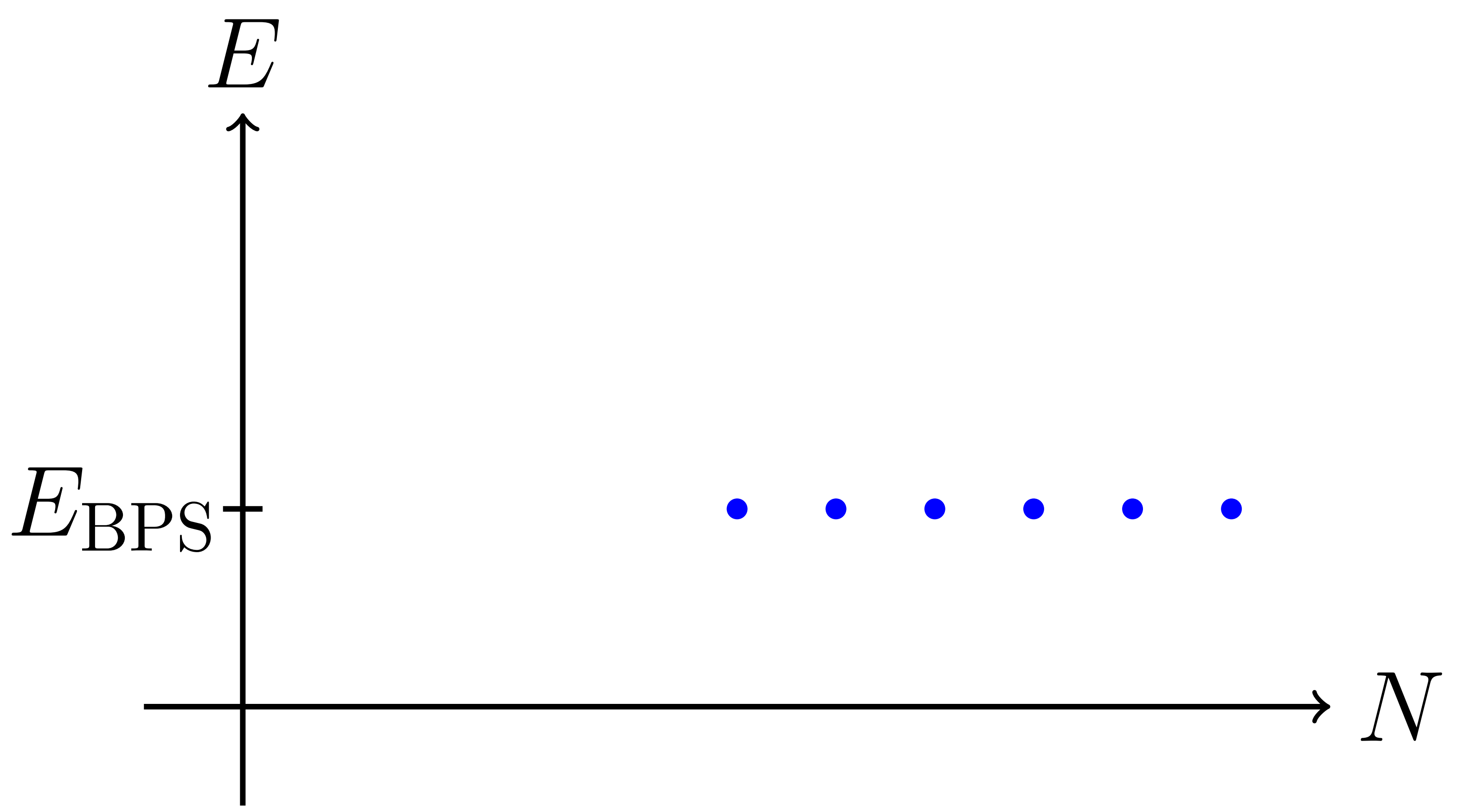
- However, the entropy given by quantizing and counting the smooth and horizonless solutions is much smaller than the black hole entropy. Therefore, they are not typical.

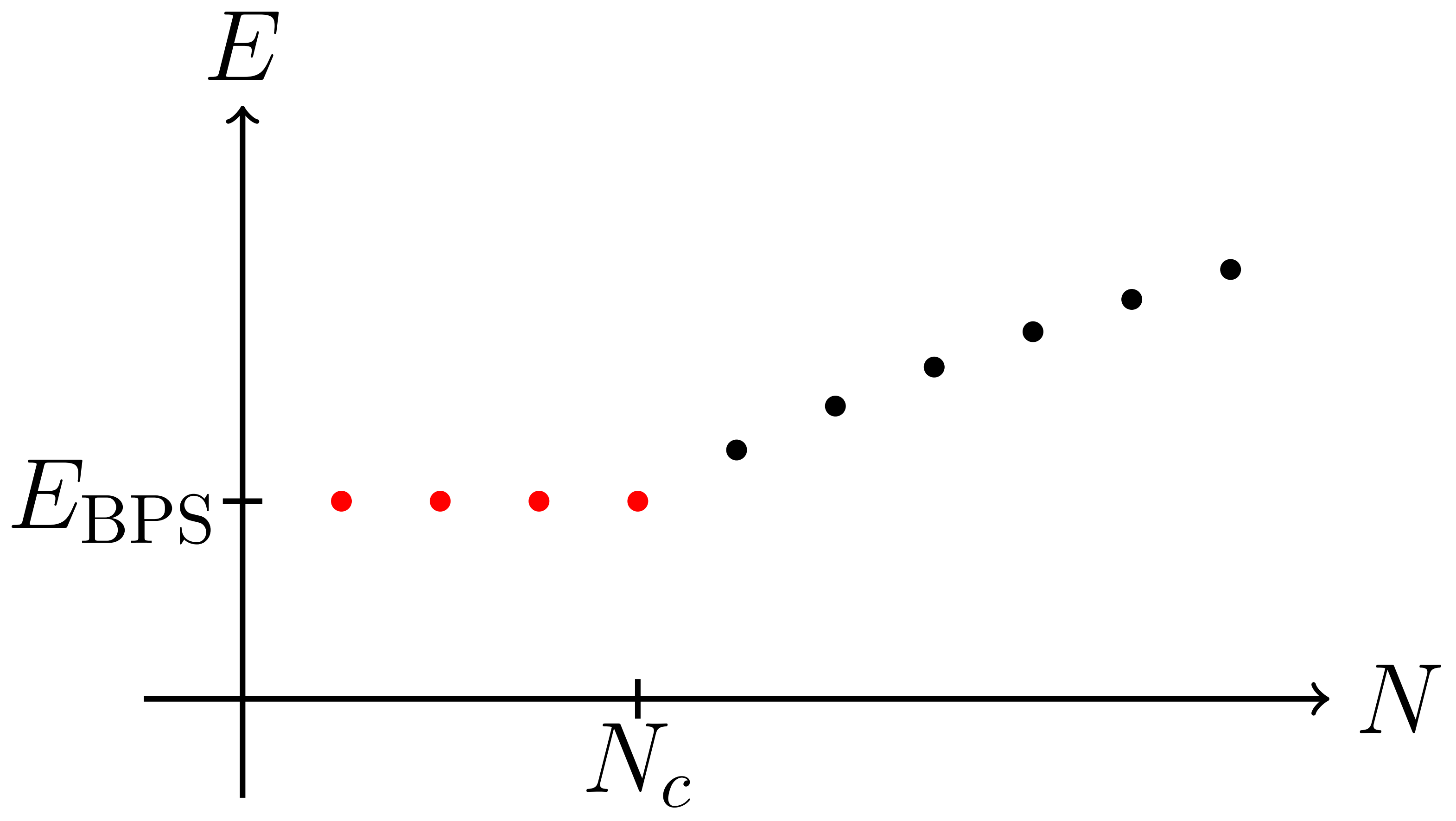
- The Q-cohomology and fortuity give a different perspective, explaining why smooth and horizonless solutions cannot be typical black hole microstates.

Introduction

- Smooth and horizonless geometries have a "smooth" $G_N \to 0$ limit — They can be viewed as coherent states of gravitons, and disassemble into non-interacting gravitons as $G_N \to 0$ $(\ell_P/\ell_{AdS} \to 0)$.

- OTOH, the duals of fortuitous states have non-perturbative quantum corrections (finite N effects).





Introduction

- **Conjecture** [CC-Lin'24]:
  - **Monotone classes** $\leftrightarrow$ **Microstates of horizonless geometries**
  - **Fortuitous classes** $\leftrightarrow$ **Typical microstates of black holes**
- **Evidence/checks**:
  - (Generalized) Lin-Lunin-Maldacen geometries $\subset$ monotone classes in $\mathcal{N} = 4$ SYM
  - Lunin-Mathur/superstrata geometries $\subset$ monotone classes in D1-D5 CFT

Introduction

- Let me give a partial summary of the studies of fortuity in various theories:

Theories	Explicit fortuitous classes	Ref
$\mathcal{N}=4$ SYM	$N=2,\,3$	1
$\mathcal{N}=1$ relevant deformation from $\mathcal{N}$ SYM	$N=2$	2
$\mathrm{U}(N)_k\times\mathrm{U}(N)_{-k}$ ABJM	$N=1,\,2$	3
$\mathrm{U}(N)_k\times\mathrm{U}(1)_{-k}$ ABJ	$N=2$	4
D1-D5 CFT/deformed $\mathrm{Sym}^N(T^4)$	$N=1,\,2,\,3$	5
SUSY SYK	$N\sim 10$ and higher	6
Matrix quantum mechanics	All N	Yiming's talk

(Partial) List of References

1. Chang, Yin, Lin, Choi^2, Kim^2, Lee^3, Park, de Mello Koch, Gadde, Raj, Tomar: 2209.06728, 2209.12696, 2304.10155, 2312.16443, 2402.10129, 2412.08695, 2506.13887, 2510.24008

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4. Kim, Lee^2, Oh: 2511.03105

5. Chang, Lin, Zhang: 2501.05448, 2511.23294. Hughes, Shigemori: 2505.14888

6. Chang, Chen, Sia, Yang: 2412.06902

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BPS states

- Consider a complex supercharge:

$$\{Q, Q^\dagger\} = H - E_{\mathrm{BPS}} \equiv \Delta\,, \quad Q^2 = 0 = Q^{\dagger 2} \Leftrightarrow \Delta = 0$$

- The BPS states $|\Psi\rangle$: $\quad H|\Psi\rangle = E_{\mathrm{BPS}}|\Psi\rangle \quad \Leftrightarrow \quad Q|\Psi\rangle = 0 = Q^\dagger|\Psi\rangle$

- Standard Hodge theory argument:

BPS states $\longleftrightarrow$ Q-cohomology classes $\quad \dfrac{\{\,|\Psi\rangle \,\big|\, Q|\Psi\rangle = 0\}}{\{\,|\Psi\rangle \,\big|\, |\Psi\rangle = Q|\Psi'\rangle\}}$

BPS operators in $\mathcal{N} = 4$ SYM

- State/operator correspondence: $O \leftrightarrow |O\rangle$
- $\mathcal{N} = 4$ SYM has 16 supercharges and 16 conformal supercharges
- Pick one supercharge $Q \equiv Q^4_-$ and one conformal supercharge $S = Q^\dagger$
- $E_{\text{BPS}} = J_1 + J_2 + q_1 + q_2 + q_3$ for $\mathcal{N} = 4$ SYM

BPS operators in $\mathcal{N} = 4$ SYM

- Consider $\mathcal{N} = 4$ SYM with $\mathrm{U}(N)$ gauge group.

- All operators are $\mathrm{U}(N)$ invariant composites of fundamental fields with covariant derivatives.

- Fundamental fields and derivatives (letters):

$$N \times N \text{ matrix:} \quad \Phi^{[IJ]}\,, \quad \Psi_{I\alpha}\,, \quad \overline{\Psi}^{I}_{\dot{\alpha}}\,, \quad A_\mu\,, \quad D_\mu = \partial_\mu - iA_\mu$$

$$\mathrm{SU}(4)_R : \; I = 1,\cdots,4\,, \quad \mathrm{SO}(1,3) : \; \mu = 0,\cdots,3, \quad \mathrm{SU}(2)\times\mathrm{SU}(2) : \; \alpha, \dot{\alpha} = \pm$$

BPS letters

- **BPS letters** ($\Delta = 0$):
  - Fields: $\phi^i \equiv \Phi^{4i}\,, \quad \psi_i \equiv -i\Psi_{i+}\,, \quad \lambda_{\dot\alpha} \equiv \overline{\Psi}^4_{\dot\alpha}\,, \quad f \equiv F_{\mu\nu}(\sigma^{\mu\nu})_{++}$
  - Derivatives: $D_{\dot\alpha} \equiv (\sigma^\mu)_{+\dot\alpha}D_\mu$ $\qquad (i = 1,2,3)$
- **BPS superfield** (a generating function) with auxiliary variables $(z^+, z^-, \theta_1, \theta_2, \theta_3)$: $z^\pm$ commuting, θ_i anti-commuting variables [Grant-Grassi-Kim-Minwalla'08, CC-Yin'13]

$$\Psi(z^+, z^-, \theta_1, \theta_2, \theta_3) = -i\sum_{n=0}^{\infty}\frac{(z^{\dot\alpha}D_{\dot\alpha})^n}{n!}\left[\frac{z^{\dot\beta}\lambda_{\dot\beta}}{n+1} + 2\theta_i\phi^i + \epsilon^{ijk}\theta_i\theta_j\psi_k + 4\theta_1\theta_2\theta_3 f\right]$$

- It satisfies $\Psi(z^\alpha, \theta_i)\,|_{z^\alpha=0,\theta_i=0} = 0$.

Q-action in free SYM

- At the free point $g_{\rm YM} = 0$ $(\tau = i\infty)$, any gauge invariant made out of Ψ is a BPS operator, because $Q\Psi = 0$.
- For example:
$$\mathrm{Tr}[\psi_1\phi_1]\,\mathrm{Tr}[\lambda_+ D_1 D_2 f] = \mathrm{Tr}[\partial_{\theta_2}\partial_{\theta_3}\Psi\partial_{\theta_1}\Psi]\,\mathrm{Tr}[\partial_{z^+}\Psi\partial_{z^+}\partial_{z^-}\partial_{\theta_1}\partial_{\theta_2}\partial_{\theta_3}\Psi]\Big|_{z^\alpha=\theta_i=0}$$

Q-action at tree-level

- At the tree-level (classical), the Q-action becomes

$$Q(\Psi) = \Psi^2$$

  and satisfies the Leibniz rule: $Q(AB) = Q(A)B + (-1)^{|A|}AQ(B)$.

- The tree-level cohomology differs from the free cohomology.

- For example, $\mathrm{Tr}\,(\phi_1\phi_2\{\phi_1, \phi_2\})$ is Q-closed but $\mathrm{Tr}\,(\phi_1\phi_2 f)$ is not.

Monotone and fortuitous classes

- Monotone classes: $O_{m_1} \cdots O_{m_n}$ multitraces

$$O_m = \partial^{p_1}_{z^+} \partial^{p_2}_{z^-} \partial^{s_1}_{\theta_1} \partial^{s_2}_{\theta_2} \partial^{s_3}_{\theta_3} \, \mathrm{Tr} \underbrace{\left[(\partial_{z^+}\Psi)^{k_1} (\partial_{z^-}\Psi)^{k_2} (\partial_{\theta_1}\Psi)^{n_1} (\partial_{\theta_2}\Psi)^{n_2} (\partial_{\theta_3}\Psi)^{n_3} \right]}_{\text{symmetrize}} \Big|_{z^\alpha = 0 = \theta_i}$$

  - At $N = \infty$, they are dual to BPS multi-graviton states. [CC-Yin'13]

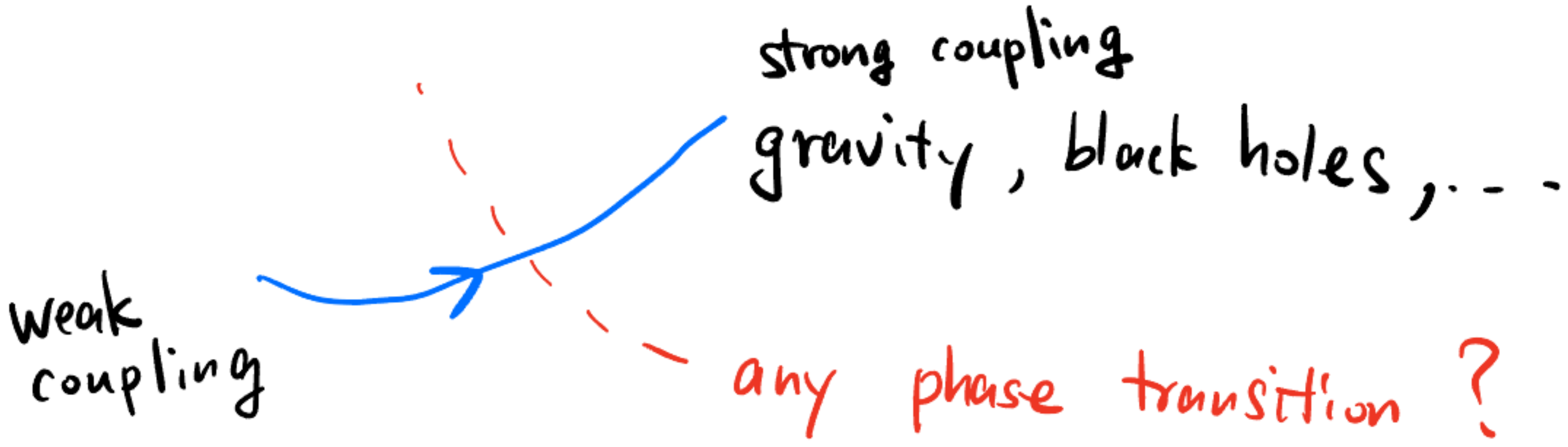
- First fortuitous classes: in SU(2) theory $(\partial^{i_1 \cdots i_n} \equiv \partial_{\theta_{i_1}} \cdots \partial_{\theta_{i_n}})$

$$O = \epsilon_{i_1 i_2 i_3} \epsilon_{j_1 j_2 j_3} \epsilon_{l_1 l_2 l_3} \epsilon_{m_1 m_2 m_3} \epsilon^{k_1 l_1 m_1} \mathrm{Tr}(\partial^{i_1}\Psi \partial^{k_2 k_3}\Psi) \mathrm{Tr}(\partial^{j_1}\Psi \partial^{l_2 l_3}\Psi) \mathrm{Tr}(\partial^{i_2 i_3}\Psi \partial^{j_2 j_3}\Psi \partial^{m_2 m_3}\Psi)$$

  - SU(2) and SU(3): multiple infinite towers of fortuitous classes

[CC-Lin '22, Choi-Kim-Lee-Park '22, Choi^2-Kim-Lee^3-Park '22, '23]

Non-renormalization conjectures



- Weak conjecture: There is no (codimension one) phase boundary. Phase transition can only occur at discrete points, usually free points.

- Strong conjecture: The Q-cohomology is tree-level (classically) exact. ([Kinney-Maldacena-Minwalla-Raju'05, Grant-Grassi-Kim-Minwalla'08, ...] for $\mathcal{N} = 4$ SYM)

$\{Q_{\text{tree}}, Q^{\dagger}_{\text{tree}}\} = \Delta_{1-\text{loop}}$, so the BPS spectrum is 1-loop exact.

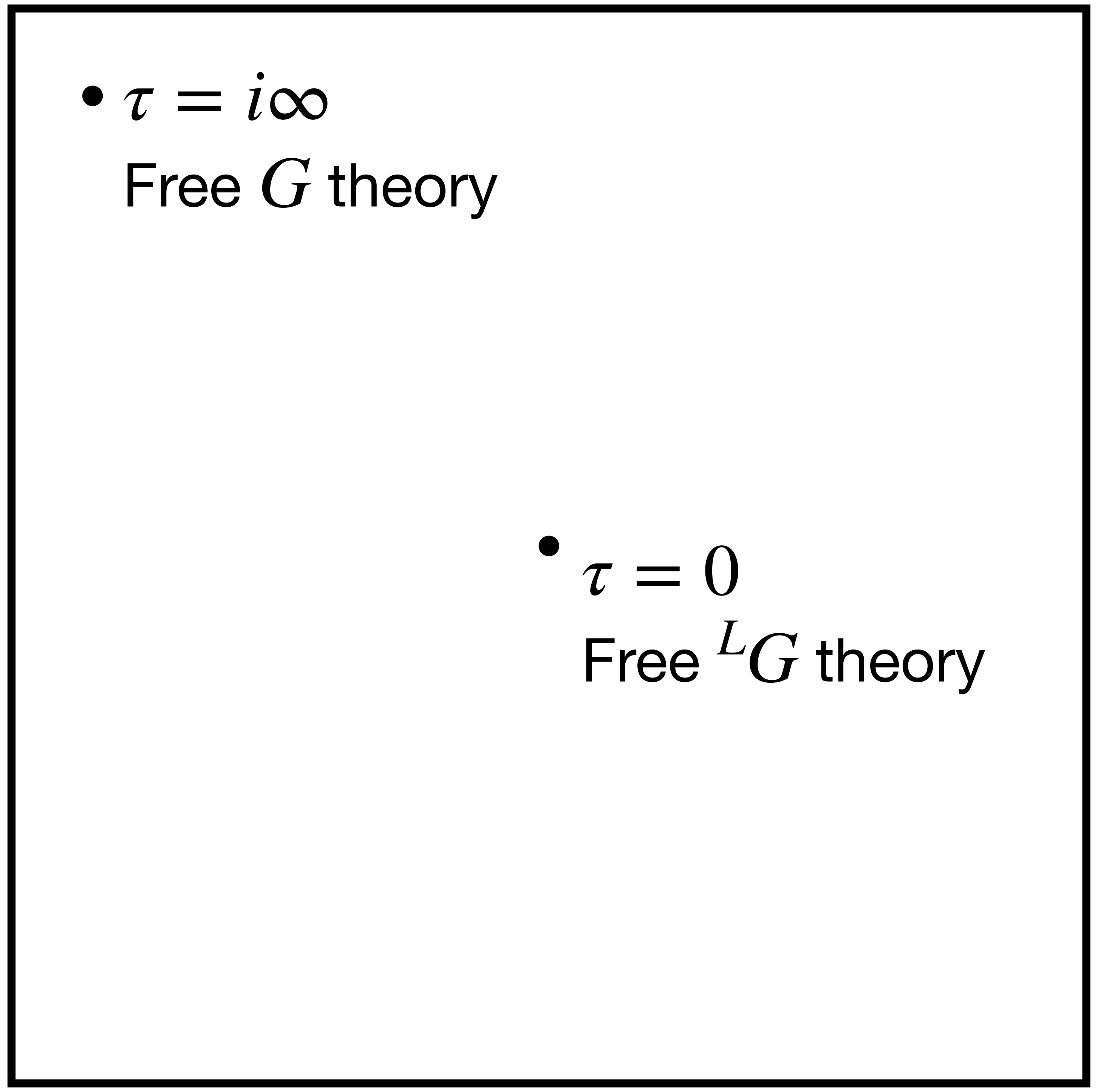
S-duality test

- Let us test the conjectures using the S-duality.

- The $\mathcal{N} = 4$ SYM theory enjoys the S-duality, which maps the theory with gauge group G and complexified gauge coupling

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g_{\mathrm{YM}}^2}$$

to the theory with gauge group ${}^L G$ (the Langlands dual of G) and complexified gauge coupling $-1/\tau$.



S-duality test

- The dual pair with gauge groups $\mathrm{SU}(N)$ and $\mathrm{PSU}(N)$ does not give any non-trivial checks because the classical Q-cohomology depends only on the Lie algebra of the gauge group.

- To perform nontrivial checks, we consider the gauge groups $\mathrm{SO}(2N+1)$ and $\mathrm{USp}(2N)$.

- Smallest nontrivial N: $\mathrm{SO}(7)$ and $\mathrm{USp}(6)$.

Search for differences

- Focusing on the BMN sector (letters with $J_1 = J_2$), Gadde-Lee-Raj-Tomar found the first fortuitous (non-Coulomb-branch) class in the $\mathrm{SO}(7)$ theory with charges $(J_1, J_2, q_1, q_2, q_3) = (\frac{1}{2}, \frac{1}{2}, \frac{5}{2}, \frac{5}{2}, \frac{5}{2})$, which has no S-dual in the $\mathrm{USp}(6)$ theory in the BMN sector.

$$
\begin{aligned}
\mathcal{O} = &\ \mathrm{Tr}[Y^2]\mathrm{Tr}[X\psi_1]\mathrm{Tr}[XZ]^2 - 4\mathrm{Tr}[Y^2]\mathrm{Tr}[XZ]\mathrm{Tr}[ZXZ\psi_3] - \mathrm{Tr}[XZ]^2\mathrm{Tr}[ZY\psi_3Y] \\
&- 4\mathrm{Tr}[XZ]^2\mathrm{Tr}[ZY^2\psi_3] + 8\mathrm{Tr}[XZ]\mathrm{Tr}[ZXY^2Z\psi_3] + 4\mathrm{Tr}[XZ]\mathrm{Tr}[ZXZY\psi_3Y] \\
&+ 16\mathrm{Tr}[XZ]\mathrm{Tr}[ZXZY^2\psi_3] - 4\mathrm{Tr}[Z\psi_3]\mathrm{Tr}[ZX]\mathrm{Tr}[ZXY^2] + 8\mathrm{Tr}[ZXZ\psi_3]\mathrm{Tr}[XYZY] \\
&- 2\mathrm{Tr}[Y^2]\mathrm{Tr}[Z\psi_3]\mathrm{Tr}[ZXZX] + 8\mathrm{Tr}[ZXZX]\mathrm{Tr}[Y^2Z\psi_3] + 2\mathrm{Tr}[ZXZX]\mathrm{Tr}[YZY\psi_3] \\
&+ 16\mathrm{Tr}[YZ\psi_3]\mathrm{Tr}[ZXZXY] + 8\mathrm{Tr}[Z\psi_3]\mathrm{Tr}[ZXZXY^2] + 8\mathrm{Tr}[Y^2]\mathrm{Tr}[ZXZXZ\psi_3] \\
&+ 16\mathrm{Tr}[ZXZXYZY\psi_3] - 8\mathrm{Tr}[ZXZXZY\psi_3Y] - 32\mathrm{Tr}[ZXZXZY^2\psi_3] - 16\mathrm{Tr}[ZXZYXYZ\psi_3] \\
&- 16\mathrm{Tr}[ZXZYXZY\psi_3] - 16\mathrm{Tr}[ZXZY^2XZ\psi_3] - (grav)
\end{aligned}
$$

$$
\begin{aligned}
&\partial_1\Psi = X, \partial_2\Psi = Y, \partial_3\Psi = Z \\
&\partial_{12}\Psi = \psi_3, \partial_{23}\Psi = \psi_1, \partial_{31}\Psi = \psi_2
\end{aligned}
$$

- However, the BMN sector is not a complete charge sector. They did not search outside the BMN sector.

Violation of S-duality

- Up to $L = 3J_1 + 3J_2 + 2q_1 + 2q_2 + 2q_3 = 18$, the $\mathrm{USp}(6)$ and $\mathrm{SO}(7)$ classical cohomology aggee up to two additional classes in $\mathrm{SO}(7)$: [Chang-Lin'25]

BMN

chiral ring

$(J_1, J_2, q_1, q_2, q_3)$	$(\frac{1}{2}, \frac{1}{2}, \frac{5}{2}, \frac{5}{2}, \frac{5}{2})$		(0,0,3,3,3)	
$(n_+, n_-, n_1, n_2, n_3, n)$	(0,0,3,3,3,8)		(1,1,2,2,2,8)	
gauge group	SO(7)	USp(6)	SO(7)	USp(6)
all states	903	903	826	825
non-Q-closed states	220	221	0	0
Q-exact states	559	559	741	741
monotone classes	123	123	85	84
fortuitous classes	1	0	0	0

O_{BMN}

$O^i_{\mathrm{CR}},\ i = 1,\cdots,85$

Chiral ring sector

- Due to the 85 vs. 84, the classical chiral ring is not S-duality invariant.
- What's special about the 1 out of the 85 chiral ring elements?
- The chiral ring is generated by the $\mathcal{N}=1$ chiral superfields Φ_i and W_α, whose bottom components are $\phi_i = \partial_{\theta_i}\Psi$ and $\lambda_\alpha = \partial_{z^\alpha}\Psi$.
- The superpotential gives the relations $\partial W/\partial(\Phi_i, W_\alpha) = 0$:

$$[\phi_i, \phi_j] = [\phi_i, \lambda_\alpha] = \{\lambda_\alpha, \lambda_\beta\} = 0.$$

Chiral ring sector

- The chiral ring is $G_{\mathbb{C}}$ invarinat polynomail ring:

$$\mathbb{C}[\mathfrak{g}_{\mathbb{C}}^{3|2}/(\text{commutators})]^{G_{\mathbb{C}}}$$

G is the gauge group, and $\mathfrak{g}$ is the Lie algebra of G.

- A common lore: $\mathbb{C}[\mathfrak{g}_{\mathbb{C}}^{3|2}/(\text{commutators})]^{G_{\mathbb{C}}} \overset{?}{\cong} \mathbb{C}[\mathfrak{h}_{\mathbb{C}}^{3|2}]^{W_{\mathfrak{g}}}$.

- A naive argument: Since the ϕ_i, λ_α are mutually commuting, we can put them into the Cartan subalgebra $\mathfrak{h}$ using the gauge symmetry. The residual gauge symmetry is the Weyl group $W_{\mathfrak{g}}$. [Kinney-Maldacena-Minwalla-Raju'05]

$$\mathbb{C}[\mathfrak{g}_{\mathbb{C}}^{3|2}/(\text{commutators})]^{G_{\mathbb{C}}} \overset{?}{\cong} \mathbb{C}[\mathfrak{h}_{\mathbb{C}}^{3|2}]^{W_{\mathfrak{g}}}.$$

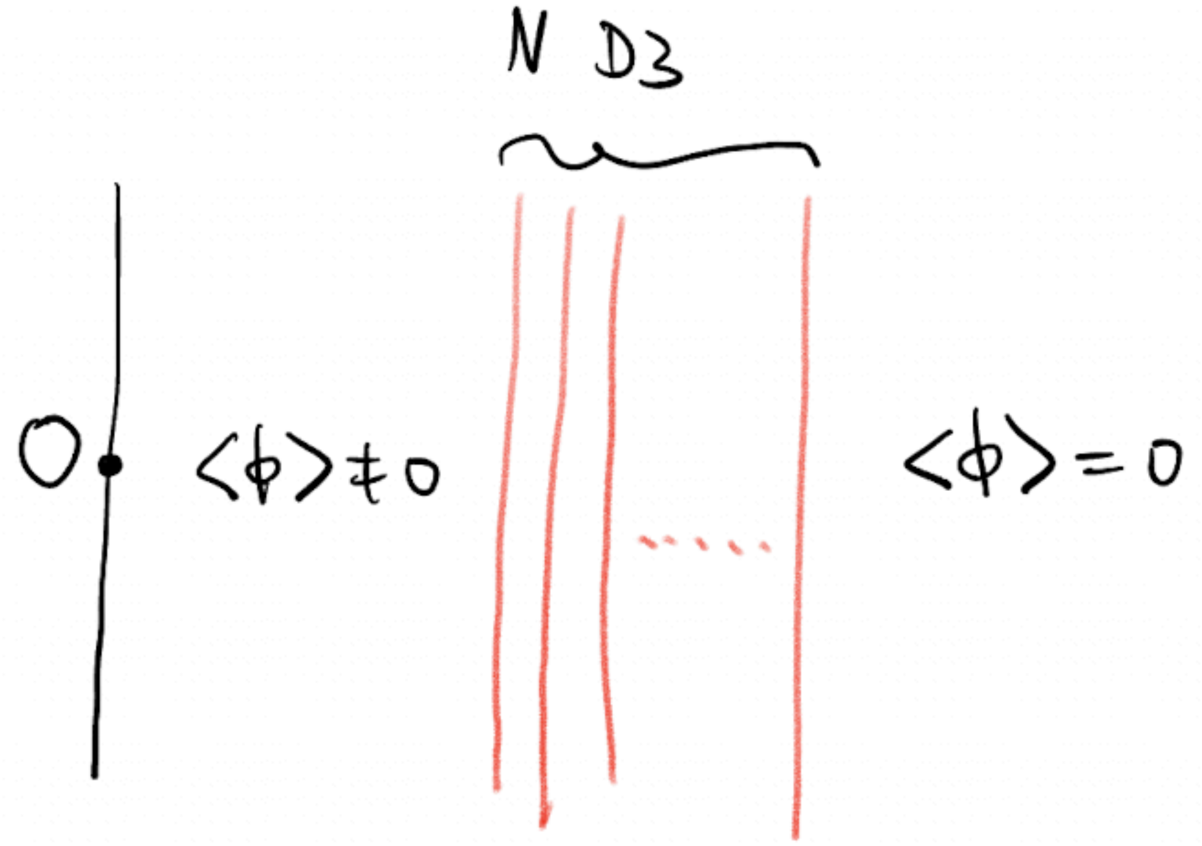
- $\mathbb{C}[\mathfrak{h}_{\mathbb{C}}^{3|2}]^{W_{\mathfrak{g}}}$ is the chiral ring of the IR theory on the Coulomb branch.
- With only bosonic elements (replacing $\mathfrak{g}_{\mathbb{C}}^{3|2}$ by $\mathfrak{g}_{\mathbb{C}}^{n}$), this isomorphism was rigorously proven. [Vaccarino'07 for $\mathfrak{su}$, Chen-Chau'21 for $\mathfrak{sp}$, Song-Xia-Xu'22 for $\mathfrak{so}$]
- However, this isomorphism cannot be true, because if it is true, the chiral ring would be S-duality invariant due to $\mathfrak{h} \cong \mathfrak{h}^L$ and $W_{\mathfrak{g}} \cong W_{\mathfrak{g}^L}$.
- In general, the abelianization map $\mathbb{C}[\mathfrak{g}_{\mathbb{C}}^{3|2}/(\text{commutators})]^{G_{\mathbb{C}}} \to \mathbb{C}[\mathfrak{h}_{\mathbb{C}}^{3|2}]^{W_{\mathfrak{g}}}$ has a nontrivial kernel.
- The 1 (out of the 85) resides in this kernel, i.e., $O_{\text{CR}}^{85}|_{\text{cartan}} = 0$ in a good basis.

Some comments

- In the $\mathrm{SO}(7)$ example, the non-Coulomb element is responsible for the quantum correction to the cohomology. Is this a general relation?

- Coulomb vs. non-Coulomb can be defined for the full cohomology (not just in the chiral ring). All known fortuitous operators are non-Coulomb.

- Bulk picture of a non-Coulomb operator: the dual bulk field is screened by the D3-branes,



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Quantum correction

- Quickly after our work, two groups of people (Choi-Lee and Buzik-Kulp) found the 1-loop corrected Q, which exactly lifts the pair of tree-level Q-cohomology classes in $SO(7)$.

$$Q_{1-\text{loop}} O_{\text{BMN}} = O^{85}_{\text{CR}}$$

- They use very different methods:

  - Choi-Lee: the generalized Konishi anomaly

  - Buzik-Klup: holomorphic twist theory of the $\mathcal{N} = 4$ SYM

Quantum correction

- However, both approaches have some weak points:
  - Generalized Konishi anomaly: It is not a systematic approach, not very clear whether it captures all the quantum corrections.
  - Holomorphic twist theory: The relation between the perturbative expansions in the twisted and untwisted (physical) theories is not clear.

Quantum correction

- Find a new way of computing the quantum correction. That is systematic, non-perturbative, and in the physical theory.

- Idea: Operators are composites of fundamental fields.

- They are well-defined classically, but quantum mechanically, there are coincident point singularities (OPE singularities) between fundamental fields.

- Point splitting regularization: subtracting off the OPE singularities

Chiral OPE of BPS operators

- Let us introduce the complex coordinates:

$$z_{\dot{\alpha}} = (x^\mu \sigma_\mu)_{-\dot{\alpha}} = (x^0 + x^3, x^1 - ix^2)\,, \quad \bar{z}_{\dot{\alpha}} = (x^\mu \sigma_\mu)_{+\dot{\alpha}} = (x^1 + ix^2, x^0 - x^3)$$

$$-(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 = z^{\dot{\alpha}} \bar{z}_{\dot{\alpha}} \equiv |z|^2$$

- We have $\Delta(z_{\dot{\alpha}}) = 0$, $\Delta(\bar{z}_{\dot{\alpha}}) = -2$, $\Delta(|z|^2) = -2$, $\quad (\Delta = H - E_{\mathrm{BPS}})$
- We can constrain the OPE between BPS operators $\mathcal{O}_1, \mathcal{O}_2$ to take the form

$$\mathcal{O}_1(z, \bar{z}) \mathcal{O}_2(0) \sim \sum_{\Delta \geq 0} \sum_{\tilde{\ell}=1}^{\infty} \sum_{\ell=0}^{\tilde{\ell}-\Delta} \frac{z_{\dot{\alpha}_1} \cdots z_{\dot{\alpha}_\ell} \bar{z}_{\dot{\beta}_1} \cdots \bar{z}_{\dot{\beta}_{\tilde{\ell}}}}{|z|^{2\tilde{\ell}-\Delta}} \mathcal{O}_{\Delta, \dot{\alpha}_1 \cdots \dot{\alpha}_\ell \dot{\beta}_1 \cdots \dot{\beta}_{\tilde{\ell}}}(0)$$

("$\sim$": ignoring regular terms)

- Taking $\bar{z} \to 0$ while fixing z, all non-BPS terms ($\Delta > 0$) vanish.

Chiral OPE of Ψ's

- The BPS superfield can be viewed as covariantly translating operators from the origin to $(z, \bar{z})$:

$$\Psi(\theta, z) = e^{z^{\dot\alpha} D_{\dot\alpha}} \cdot \Psi(\theta, 0) = e^{-i\int_0^z A_{\dot\alpha} dz^{\dot\alpha}} \cdot e^{z^{\dot\alpha}\partial_{\dot\alpha}} \Psi(\theta, 0),$$ (" $\cdot$ ": adjoint action)

- $\Psi_{a_1}(\theta_1, z_1)\cdots\Psi_{a_n}(\theta_n, z_n)$ transforms under the gauge group at $z = 0$.

- The OPE between Ψ's can be constrained by the charges

a: adjoint index

$$\Psi_a(\theta, z)\Psi_a(\theta', 0) \sim \kappa f_{abc} \delta^3(\theta - \theta') \frac{\bar{z}^{\dot\alpha}}{|z|^2} \frac{\partial \Psi_c}{\partial z_{\dot\alpha}}(\theta, 0)$$

	E	J_L	J_R	q_i
Ψ	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$
$z_{\dot\alpha}$	-1	$-\frac{1}{2}$	$\pm\frac{1}{2}$	0
$\bar{z}_{\dot\alpha}$	-1	$\frac{1}{2}$	$\pm\frac{1}{2}$	0
θ_j	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2} - \delta_{ij}$

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Normal ordering

- We can define the normal ordered operators (regularized operators) by subtracting off the OPE singularities. For example:

$$: \Psi_a(\theta', z)\Psi_b(\theta,0) : \equiv \Psi_a(\theta', z)\Psi_b(\theta,0) - \kappa f_{abc}\delta^3(\theta - \theta')\frac{\bar{z}^{\dot{\alpha}}}{|z|^2}\frac{\partial \Psi_c}{\partial z_{\dot{\alpha}}}(\theta,0)$$

- The Q-action on a single Ψ is more properly defined as

$$Q\Psi_a = \frac{1}{2}f_{abc} : \Psi_b \Psi_c :$$

- Q does **not** obey the Leibniz rule when acting on $: \Psi_a \Psi_b :$

Q-action on composites

$$
\begin{aligned}
&Q :\Psi_a(\theta',z')\Psi_b(\theta,z): \\
&= Q\left[\Psi_a(\theta',z')\Psi_b(\theta,z) - \kappa f_{abc}\delta^3(\theta'-\theta)\frac{(\bar z'-\bar z)^{\dot\alpha}}{|z'-z|^2}\partial_{\dot\alpha}\Psi_c(\theta,z)\right] \\
&= \frac{1}{2}f_{acd}\Psi_b(\theta,z) :\Psi_c\Psi_d: (\theta',z') - \frac{1}{2}\Psi_a(\theta',z')f_{bcd} :\Psi_c\Psi_d: (\theta,z) \\
&\quad + \frac{1}{2}\kappa f_{abc}\delta^3(\theta'-\theta)\frac{(\bar z'-\bar z)^{\dot\alpha}}{|z'-z|^2}f_{cde}\partial_{\dot\alpha}(:\Psi_d\Psi_e:)(\theta,z) \\
&= \frac{1}{2}f_{acd} :\Psi_b(\theta,z) :\Psi_c\Psi_d: (\theta',z') : + \kappa f_{acd}f_{bce}\delta^3(\theta-\theta')\frac{(\bar z-\bar z')^{\dot\alpha}}{|z-z'|^2} :\partial_{\dot\alpha}\Psi_e\Psi_d: (\theta',z') \\
&\quad - \frac{1}{2}f_{bcd} :\Psi_a(\theta',z') :\Psi_c\Psi_d: (\theta,z) - \kappa f_{bcd}f_{ace}\delta^3(\theta'-\theta)\frac{(\bar z'-\bar z)^{\dot\alpha}}{|z'-z|^2} :\partial_{\dot\alpha}\Psi_e\Psi_d: (\theta,z) \\
&\quad + \kappa f_{abc}\delta^3(\theta'-\theta)\frac{(\bar z'-\bar z)^{\dot\alpha}}{|z'-z|^2}f_{cde} :\Psi_d\partial_{\dot\alpha}\Psi_e: (\theta,z) \\
&= \boxed{\frac{1}{2}f_{acd} :\Psi_b(\theta,z) :\Psi_c\Psi_d: (\theta',z') : - \frac{1}{2}f_{bcd} :\Psi_a(\theta',z') :\Psi_c\Psi_d: (\theta,z) :} \equiv Q_C :\Psi_a(\theta',z')\Psi_b(\theta,z): \\
&\quad - \kappa f_{acd}f_{bce}\delta^3(\theta'-\theta)\frac{(\bar z'-\bar z)^{\dot\alpha}}{|z'-z|^2}\left[:\partial_{\dot\alpha}\Psi_e\Psi_d: (\theta',z') - :\partial_{\dot\alpha}\Psi_e\Psi_d: (\theta',z)\right] ,
\end{aligned}
$$

where we have used

$$Q\Psi_a = \frac{1}{2}f_{abc} :\Psi_b\Psi_c:$$

$$
\begin{aligned}
&Q : \Psi_a(\theta', z')\Psi_b(\theta, z) : \\
&= Q_C : \Psi_a(\theta', z')\Psi_b(\theta, z) : -\kappa f_{acd} f_{bce} \delta^3(\theta' - \theta) \frac{(\bar{z}' - \bar{z})^{\dot{\alpha}}}{|z' - z|^2} \left[: \partial_{\dot{\alpha}} \Psi_e \Psi_d : (\theta', z') - : \partial_{\dot{\alpha}} \Psi_e \Psi_d : (\theta', z) \right] ,
\end{aligned}
$$

Let us set $z' = 0$ and take the limit $z \to 0$, and we find

$$
\begin{aligned}
&Q : \Psi_a(\theta', 0)\Psi_b(\theta, 0) : \\
&= Q_C : \Psi_a(\theta', 0)\Psi_b(\theta, 0) : -\kappa f_{acd} f_{bce} \delta^3(\theta' - \theta) \lim_{x \to 0} \frac{\bar{z}^{\dot{\alpha}} z^{\dot{\beta}}}{|z|^2} \partial_{\dot{\beta}} (: \partial_{\dot{\alpha}} \Psi_e \Psi_d :)(\theta', 0) \\
&= Q_C : \Psi_a(\theta', 0)\Psi_b(\theta, 0) : + \frac{1}{2} \kappa f_{acd} f_{bce} \delta^3(\theta' - \theta) : \partial^{\dot{\alpha}} \Psi_e \partial_{\dot{\alpha}} \Psi_d : (\theta', 0) .
\end{aligned}
$$

where we have used

$$
\lim_{x \to 0} \frac{\bar{z}^{\dot{\alpha}} z^{\dot{\beta}}}{|z|^2} = \frac{1}{2} \epsilon^{\dot{\alpha}\dot{\beta}} .
$$

← This is still a bit subtle …

Q-action on composites

- In summary: $Q : \Psi_a \Psi_b : = \; Q_c : \Psi_a \Psi_b : + Q_q : \Psi_a \Psi_b :$

- Q_c is the classical Q-action, satisfying the Leibniz rule. The quantum correction Q_q agrees with the results of Choi-Lee and Buzik-Kulp.

- Future directions:

  - We can further compute the Q-action on composits of more Ψ's, for example, $Q : \Psi_a : \Psi_b \Psi_c ::$.

  - Chiral OPE between monotone and fortuitous operators.

Concluding remarks

- There are quantum corrections to the Q-cohomology in the $\mathcal{N} = 4$ SYM with $\mathrm{SO}(2N + 1)$ gauge group.

- It is still unclear whether there are similar loop corrections in theories with other gauge groups.

- The weak conjecture can still be correct, i.e., there is no “wall”.

- Q-cohomology is a powerful tool, and the study of fortuity has taught us interesting lessons about black holes. More in the later talks today!

Thank you