LECTURE 3.1: Classes of local and non-local higher-derivative theories of gravity

- GR as a QFT is unitary, but perturbatively non-renormalizable.
- Quadratic Gravity is perturbatively renormalizable, but (apparently) not unitary.
- How to reconcile perturbative renormalizability and unitarity in a QFT of gravity?

- We have learned that higher-order derivatives are unavoidable for a fundamental QFT description of the gravitational interaction.

- Is there any HD theory of gravity that is both unitary and renormalizable?
- Is there any HD theory of gravity that is ghost-free?
- Do we need to give up one of the "guiding principles" 1), 2), 3), 4) for a fundamental QFT description of gravity?

Warm-up: Classes of HD scalar models

- Make a "quite general" classification of HD models allowed by the Lagrangian formalism
- Then, investigate if any of these models can serve our purpose

- **General bare Lagrangian:** $\mathcal{L} = \frac{1}{2}\,\phi\, F(\Box)\,\phi - V(\phi)$
- **Kinetic operator:** $F(\Box)$ is an entire function of $\Box$ (no poles)
- **Propagator:** $G(-p^2) = \dfrac{i}{F(-p^2)}$
- **Weierstrass theorem:** if $F(z)$ is an entire function, we can factorize it as

$$F(z) = e^{\gamma(z)} \prod_{j=1}^{N} (z - m_j^2)^{r_j} \quad ,$$

where
- m_j^2 are real or complex zeroes of $F(z)$;
- r_j is the multiplicity of the j-th zero;
- N counts the # of zeroes ($< \infty$ or $= \infty$);
- $\gamma(z)$ is another entire function of z.

- In our case, $z = \Box$: $F(\Box) = e^{\gamma(\Box)} \prod_{j=1}^{N} (\Box - m_j^2)^{r_j}$
- In momentum space $z = -p^2$: $F(-p^2) = e^{\gamma(-p^2)} \prod_{j=1}^{N} (-1)^{r_j} (p^2 + m_j^2)^{r_j}$
- Momentum-space propagator : $G(-p^2) = i\, e^{-\gamma(-p^2)} \prod_{j=1}^{N} \left(\frac{-1}{p^2 + m_j^2} \right)^{r_j}$

Some remarks:

- If $N < \infty$ and $\gamma(\Box) = 0$ $\Rightarrow$ local HD kinetic operator
- if $N = \infty$ and/or $\gamma(\Box) \neq 0$ $\Rightarrow$ non-local (i.e. non-polynomial) HD kinetic operator
- In particular, if $\gamma(\Box) \neq 0$ $\Rightarrow$ $e^{\gamma(\Box)} = \sum_{m=0}^{\infty} f_m \Box^m$, $f_m \equiv \left. \frac{\partial^m e^{\gamma(z)}}{\partial z^m} \right|_{z=0}$

  $\Rightarrow$ infinite-order derivatives
- Since $\gamma(-p^2)$ is an entire function of p^2, $e^{\gamma(-p^2)}$ does not introduce additional poles.

Classification: local HD models

$$F(\Box) = e^{\gamma(\Box)} \prod_{j=1}^{N} (\Box - m_j^2)^{r_j} \quad , \quad G(-p^2) = i\, e^{-\gamma(-p^2)} \prod_{j=1}^{N} \left(\frac{-1}{p^2+m_j^2}\right)^{r_j}$$

(i) Ordinary two-derivative kinetic operator: $N=1$, $r_1=1$, $\gamma(\Box)=0$

$$\Longrightarrow \quad F(\Box) = \Box - m_1^2 \quad , \quad G(-p^2) = \frac{-i}{p^2+m_1^2}$$

(ii) Four-derivative kinetic operator: $N=2$, $r_1=r_2=1$, $\gamma(\Box)=0$

$$\Longrightarrow F(\Box) = (\Box - m_1^2)(\Box - m_2^2) \quad , \quad G(-p^2) = \frac{i}{(p^2+m_1^2)(p^2+m_2^2)} = \frac{i}{m_2^2-m_1^2}\left[\frac{1}{p^2+m_1^2} - \frac{1}{p^2+m_2^2}\right]$$

(iii) 2N-derivative kinetic operator (only real zeroes): N, $r_1=\dots=r_N=1$, $\gamma(\Box)=0$

$$F(\Box) = \prod_{j=1}^{N} (\Box - m_j^2) \quad , \quad G(-p^2) = i \prod_{j=1}^{N} \frac{-1}{p^2+m_j^2} = -i \sum_{j=1}^{N} \frac{c_j}{p^2+m_j^2} \quad ,$$

where $m_j^2 \in \mathbb{R}$ $\forall j$, $-i c_j$ is the residue at $p^2 = -m_j^2$ $\forall j$.

Can we avoid ghosts for some value of N? Namely, can we have $e_j \geq 0 \ \forall j$? NO!

Multiply by p^2 both sides of the propagator and take the limit $|p^2| \to \infty$:

$$\lim_{|p^2|\to\infty} p^2\, i \prod_{j=1}^{N} \frac{-1}{p^2+m_j^2} = \lim_{|p^2|\to\infty} p^2 (-i) \sum_{j=1}^{N} \frac{e_j}{p^2+m_j^2}$$

$$\Longleftrightarrow \quad 0 = \sum_{j=1}^{N} e_j = e_1 + e_2 + \cdots + e_N \iff \exists\ i, j\neq i : \ e_i > 0 \ \& \ e_j < 0.$$

$\Longrightarrow$ <u>At least one</u> of the components must be <u>ghost-like</u>!

(iii) Six-derivative kinetic operator (complex-conjugate zeroes): $N=3$, $r_1 = r_2 = r_3 = 1$, $\gamma(\Box) = 0$

$$F(\Box) = (\Box - m^2)(\Box + iM^2)(\Box - iM^2) = (\Box - m^2)(\Box^2 + M^4)$$

$$\mathcal{G}(-p^2) = \frac{-i}{(p^2+m^2)(p^2+iM^2)(p^2-iM^2)} = \frac{-i}{(p^2+m^2)((p^2)^2+M^4)}$$

- In this case we have 3 poles: $\begin{cases} 1 \text{ real pole } p^2 = -m^2, \\ \text{a complex-conjugate pair } p^2 = \pm i M^2 \end{cases}$.

$$G(-p^2) = (-i)\frac{1}{p^2+m^2}\frac{1}{2iM^2}\left[\frac{1}{p^2-iM^2} - \frac{1}{p^2+iM^2}\right]$$

$$= -\frac{1}{2M^2}\left[\frac{1}{p^2+m^2}\frac{1}{p^2-iM^2} - \frac{1}{p^2+m^2}\frac{1}{p^2+iM^2}\right]$$

$$= -\frac{1}{M^2}\left[\left(\frac{1}{p^2+m^2} - \frac{1}{p^2-iM^2}\right)\frac{-(m^2-iM^2)}{m^4+M^4} - \left(\frac{1}{p^2+m^2} - \frac{1}{p^2+iM^2}\right)\frac{-(m^2+iM^2)}{m^4+M^4}\right]$$

$$= \frac{-i}{m^4+M^4}\left[\frac{1}{p^2+m^2} + \frac{\alpha}{p^2+iM^2} - \frac{\alpha^*}{p^2-iM^2}\right], \quad \alpha \equiv 1 + i\frac{M^2}{m^2}$$

- <u>UV behavior</u>: $G(-p^2) \overset{UV}{\sim} \frac{1}{(p^2)^3}$

- Tree-level unitarity: let's check the sign of the imaginary part of $i\,\mathcal{G}(-p^2)$:

$$\mathrm{Im}\left[i\mathcal{G}(-p^2)\right] = \frac{1}{m^4+M^4}\,\mathrm{Im}\left[\frac{1}{p^2+m^2} + \frac{i\alpha}{p^2+iM^2} - \frac{i\alpha^*}{p^2-iM^2}\right]$$

$$= \frac{1}{m^2+M^4}\,\mathrm{Im}\left[\frac{1}{p^2+m^2}\right] = \frac{1}{m^4+M^4}\,\pi\,\delta(p^2+m^2) \geq 0$$

The additional complex-conjugate poles cancel $\Rightarrow$ the positivity constraint is satisfied.

- One can show that, using alternative contour deformations in the complex-energy plane, the optical theorem and so unitarity can be preserved at any loop order.

- For references on UV behavior and unitarity of 6-derivative QFTs with complex-conjugate poles, check papers by the following authors:

[Asorey, Shapiro, Modesto, Rachwal, Anselmi, Piva, Briscese, Calcagni, Giacchini, ...]

Classification: non-local HD models

- Motivation: avoid ghosts (ghost-free models), while keeping a good UV behavior
- This may happen in infinite-derivative models with $N=1$, $\kappa_1 = 1$, $\gamma(\Box) \neq 0$:

$$F(\Box) = e^{\gamma(\Box)}(\Box - m^2) \quad , \quad G(-p^2) = \frac{-i}{e^{\gamma(-p^2)}(p^2+m^2)} = \frac{-i\, e^{-\gamma(-p^2)}}{p^2+m^2}$$

Some remarks:

- Only 1 pole at $p^2 = -m^2 \Rightarrow \mathrm{Im}[iG(-p^2)] = \pi\, e^{-\gamma(m^2)}\, \delta(p^2+m^2) \geq 0$.
- The entire function $\gamma(-p^2)$ is dimensionless, therefore it must depend on $-p^2/\mu^2$, where μ is a mass/energy scale of new physics.
- In the regime $|p^2/\mu^2| << 1$: we must require $\gamma(-p^2/\mu^2) \simeq 0$ to recover the low-energy local physics.
- In the regime $|p^2/\mu^2| >> 1$: we want $e^{-\gamma(-p^2/\mu^2)}$ to give UV suppression in the propagator

Some interesting functional form for $\gamma(\Box/\mu^2)$

- **Monomial:** $\gamma(\Box/\mu^2) = \left(-\Box/\mu^2\right)^n$, n even

$$\Rightarrow \quad F(\Box) = e^{(-\Box/\mu^2)^n}(\Box - m^2) \quad , \quad G(-p^2) = \frac{-i \; e^{-\left(p^2/\mu^2\right)^n}}{p^2 + m^2}$$

$\Rightarrow$ the UV behavior is non-local all the way up to infinite energy

[References: see papers by Krasnikov (1987), Mazumdar, Biswas, Koshelv, Modesto, Calcagni, Briscese, Tokareva, L.B., ...]

- **Non-polymial:** $\gamma(\Box/\mu^2) = \frac{1}{2}\left[\Gamma(0, g(\Box)) + \gamma_E + \log(g(\Box))\right]$

where $\begin{cases} \Gamma(0, g(z)) \equiv \int_{g(z)}^{\infty} dt \, \frac{e^{-t}}{t} \quad , \quad g(z) \text{ is a polymial in } z \\ \gamma_E \equiv \text{Euler-Mascheroni constant} \end{cases}$

The most common choice for $g(\Box)$ is : $g(\Box) = \left(-\Box/\mu^2\right)^n$, n even

$$\Rightarrow \quad F(-p^2) = (p^2+m^2)\, e^{\frac{1}{2}\left[\Gamma\left(0, (p^2/\mu^2)^n\right) + \gamma_E + \log(p^2/\mu^2)^n\right]}$$

<u>UV behavior</u> : since for $z \gg 1$ we have $\Gamma(0,z) \simeq \frac{e^{-z}}{z} \to 0$ $\Rightarrow$

$$\Rightarrow \quad F(-p^2) \overset{UV}{\sim} (\text{constant}) \times p^2 \left(\frac{p^2}{\mu^2}\right)^{\frac{n}{2}}$$

$$\Rightarrow \quad G(-p^2) \overset{UV}{\sim} \frac{1}{(p^2)^{\frac{n}{2}+1}} \quad , \quad n \text{ even}$$

$\Rightarrow$ the UV behavior is <u>not</u> non-local all the way up to infinite energy, but after a non-local regime we have again a local (polynomial) behavior

[References: see papers by Kuzmin (1989), Tomboulis (1997), Modesto, Calcagni, Rachwal, Briscese, Giacchini, ...]

Classes of HD theories of gravity

- We now want to write down analogous Lagrangians for the gravitational interaction.
- To do so, we have to consider generalized higher-curvature Lagrangians.

General action: The most general action for a metric, torsion-free and parity-invariant theory of gravity is:

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2\kappa^2} R + R_{\mu\nu\rho\sigma} D^{\mu\nu\rho\sigma}_{\alpha\beta\gamma\omega} R^{\alpha\beta\gamma\omega} + \mathcal{O}(\mathcal{R}^3) \right] ,$$

where $D^{\mu\nu\rho\sigma}_{\alpha\beta\gamma\omega}$ is a generic (either local or non-local) differential operator, $\mathcal{O}(\mathcal{R}^3)$ accounts for higher-curvature terms.

One can manipulate the "quadratic-curvature" part and obtain:

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2\kappa^2} R + R\, F_1(\Box)\, R + R_{\mu\nu}\, F_2(\Box)\, R^{\mu\nu} + R_{\mu\nu\rho\sigma}\, F_3(\Box)\, R^{\mu\nu\rho\sigma} + \mathcal{O}(\mathcal{R}^3) \right]$$

where $F_i(\Box)$ $i = 1,2,3$ are generic (local or nonlocal differential operators).

Remarks:

- We assume that $F_i(z)$ are entire functions.
- If F_i are constants, we know that, using the Gauss-Bonnet identity, we can rewrite $R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$ in terms of $R_{\mu\nu} R^{\mu\nu}$ and R^2. Can we do something similar if $F_i \neq$ const.?
- The Gauss-Bonnet identity does not hold for $F_i \neq$ const. But, the following identity does hold:

$$\sqrt{g}\, R_{\mu\nu\rho\sigma} F_3(\Box) R^{\mu\nu\rho\sigma} = \sqrt{g} \left[4 R_{\mu\nu} F_3(\Box) R^{\mu\nu} - R\, F_3(\Box) R + \mathcal{O}(\mathcal{R}^3) \right] + \text{tot. derivatives}$$

In other words, the information about $F_3(\Box)$ can be partly moved to $\mathcal{O}(R^3)$ and partly to tot. derivatives.

- Proof:

Using the following properties:

(i) Since $F_3(\Box)$ is an entire function, we can write: $F_3(\Box) = \sum_{m=0}^{N} f_{3,m}\,\Box^m$, $N<\infty$ or $N=\infty$

(ii) $[\nabla,\nabla]\sim \mathcal{O}(R)$: commutation of covariant derivatives is proportional to a curvature tensor

(iii) $\nabla_{[\alpha} R_{\mu\nu]\rho\sigma} = 0 \iff \nabla_\alpha R_{\mu\nu\rho\sigma} + \nabla_\nu R_{\alpha\mu\rho\sigma} + \nabla_\mu R_{\nu\alpha\rho\sigma} = 0$

(iv) $\nabla_\mu R^\mu{}_{\nu\rho\sigma} = -\nabla_\sigma R_{\nu\rho} + \nabla_\rho R_{\nu\sigma}$

we can show:

$m=0$: $\quad R_{\mu\nu\rho\sigma}\,\Box^0\,R^{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = 4R_{\mu\nu}R^{\mu\nu} - R^2 + \text{tot. derivative}$

(Gauss-Bonnet identity)

$m \geq 1$:

$$R_{\mu\nu\rho\sigma}\,\Box^m R^{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma}\,\Box\,\Box^{m-1} R^{\mu\nu\rho\sigma}$$

$$= -\nabla_\alpha R_{\mu\nu\rho\sigma}\,\nabla^\alpha\,\Box^{m-1} R^{\mu\nu\rho\sigma} + \text{tot. derivative}$$

$$\overset{(ii)}{=} -\nabla_\alpha R_{\mu\nu\rho\sigma}\,\Box^{m-1}\nabla^\alpha R^{\mu\nu\rho\sigma} + \mathcal{O}(\mathcal{R}^3) + \text{tot. derivative}$$

$$\overset{(iii)}{=} +\left(\nabla_\nu R_{\alpha\mu\rho\sigma} + \nabla_\mu R_{\nu\alpha\rho\sigma}\right)\Box^{m-1}\nabla^\alpha R^{\mu\nu\rho\sigma} + \dots$$

$$\overset{(ii)}{=} -\left(\nabla_\nu\nabla^\alpha R_{\alpha\mu\rho\sigma} + \nabla_\mu\nabla^\alpha R_{\nu\alpha\rho\sigma}\right)\Box^{m-1} R^{\mu\nu\rho\sigma} + \dots$$

$$\overset{(iv)}{=} -\Big[\nabla_\nu\left(-\nabla_\sigma R_{\mu\rho} + \nabla_\rho R_{\mu\sigma}\right) - \nabla_\mu\left(-\nabla_\sigma R_{\nu\rho} + \nabla_\rho R_{\nu\sigma}\right)\Big]\Box^{m-1} R^{\mu\nu\rho\sigma} + \dots$$

$$\overset{(ii)}{=} -\left(\nabla_\sigma R_{\mu\rho} - \nabla_\rho R_{\mu\sigma}\right)\Box^{m-1}\nabla_\nu R^{\mu\nu\rho\sigma}$$

$$-\left(\nabla_\rho R_{\nu\sigma} - \nabla_\sigma R_{\nu\rho}\right)\Big]\Box^{m-1}\nabla_\mu R^{\mu\nu\rho\sigma} + \dots$$

$$= -2\left(\nabla_\rho R_{\nu\sigma} - \nabla_\sigma R_{\nu\rho}\right)\Box^{m-1}\nabla_\mu R^{\mu\nu\rho\sigma} + \dots$$

$$= -2\left(\nabla_\rho R_{\nu\sigma} - \nabla_\sigma R_{\nu\rho}\right)\Box^{m-1}\left(\nabla^\rho R^{\nu\sigma} - \nabla^\sigma R^{\nu\rho}\right) + \cdots$$

$$= -2\left[\nabla_\rho R_{\nu\sigma}\,\Box^{m-1}\nabla^\rho R^{\nu\sigma} + \nabla_\sigma R_{\nu\rho}\,\Box^{m-1}\nabla^\sigma R^{\nu\rho} - \nabla_\sigma R_{\nu\rho}\,\Box^{m-1}\nabla^\rho R^{\nu\sigma} - \nabla_\rho R_{\nu\sigma}\,\Box^{m-1}\nabla^\sigma R^{\nu\rho}\right] + \cdots$$

(ii) & $\nabla_\mu R^{\mu\nu} = \frac{1}{2}\nabla^\nu R$ $\quad = 4\,R_{\mu\nu}\,\Box^m R^{\mu\nu} - R\,\Box^m R + \mathcal{O}(R_\nu^3) + \text{tot. derivatives}$

$$\Rightarrow \sqrt{-g}\,R_{\mu\nu\rho\sigma}\,F_3(\Box)\,R^{\mu\nu\rho\sigma} = \sqrt{-g}\left[4R_{\mu\nu}\,F_3(\Box)\,R^{\mu\nu} - R\,F_3(\Box)\,R + \mathcal{O}(R_\nu^3)\right] + \text{tot. der.}$$

□

Furthermore, using the simple identity (no proof is really needed for this):

$$C_{\mu\nu\rho\sigma}\,F_3(\Box)\,C^{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma}\,F_3(\Box)\,R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}\,F_3(\Box)\,R^{\mu\nu} + \frac{1}{3}R\,F_3(\Box)\,R$$

we can write the action as follows:

$$S = \frac{1}{2}\int d^4x\,\sqrt{-g}\left[\frac{1}{\kappa^2}R + \frac{1}{6}R\,F_0(\Box)\,R - \frac{1}{2}C_{\mu\nu\rho\sigma}\,F_2(\Box)\,C^{\mu\nu\rho\sigma} + \mathcal{O}(\mathcal{R}^3)\right]$$

(where boundary terms are neglected).

Propagator

- We can compute the propagator for any $F_0(\Box)$, $F_2(\Box)$
- Since $g_{\mu\nu} = \eta_{\mu\nu} + 2h_{\mu\nu} \Rightarrow$ at order $\mathcal{O}(h^2)$ the term $\mathcal{O}(\mathcal{R}^3)$ does <u>not</u> contribute because $\mathcal{R}^3 \sim \mathcal{O}(h^3)$.
- The kinetic term is:

$$S^{(2)}[\eta, h] = \int d^4x\,\frac{1}{2}\,h_{\mu\nu}\,\mathbb{K}^{\mu\nu\rho\sigma}\,h_{\rho\sigma}$$

where the kinetic operator reads

$$
\begin{aligned}
\mathbb{K}^{\mu\nu\rho\sigma} \equiv\ & \frac{1}{2}\left(\eta^{\mu\rho}\eta^{\nu\sigma}+\eta^{\mu\sigma}\eta^{\nu\rho}\right)\left(\frac{1}{\kappa^2}-F_2(\Box)\,\Box\right)-\eta^{\mu\nu}\eta^{\rho\sigma}\left(\frac{1}{\kappa^2}-\frac{1}{3}\left(2F_0(\Box)+F_2(\Box)\right)\Box\right)\Box \\
& +\left(\eta^{\mu\nu}\partial^\rho\partial^\sigma+\eta^{\rho\sigma}\partial^\mu\partial^\nu\right)\left(\frac{1}{\kappa^2}-\frac{1}{3}\left(2F_0(\Box)+F_2(\Box)\right)\Box\right) \\
& -\frac{1}{2}\left(\eta^{\mu\rho}\partial^\nu\partial^\sigma+\eta^{\mu\sigma}\partial^\nu\partial^\rho+\eta^{\nu\rho}\partial^\mu\partial^\sigma+\eta^{\nu\sigma}\partial^\mu\partial^\rho\right)\left(\frac{1}{\kappa^2}-F_2(\Box)\,\Box\right) \\
& +\frac{2}{3}\left(F_0(\Box)-F_2(\Box)\right)\partial^\mu\partial^\nu\partial^\rho\partial^\sigma
\end{aligned}
$$

Going to momentum space and using the spin-projector formalism, we get:

$$
\mathbb{K}^{\mu\nu\rho\sigma} = -\frac{p^2}{\kappa^2}\left[\mathcal{P}^{(2)\mu\nu\rho\sigma}\left(1+\kappa^2 F_2(-p^2)p^2\right)-2\,\mathcal{P}^{(0,s)\mu\nu\rho\sigma}\left(1+\kappa^2 F_0(-p^2)\,p^2\right)\right]
$$

The kinetic operator is not invertible, so we need to introduce a gauge fixing. The latter can be either local or nonlocal in order to match the same derivative order of $\mathbb{K}^{\mu\nu\rho\sigma}$.

Since we are only interested in the part of the propagator that survives after contraction with a conserved source, we can forget about the gauge-fixing in what follows.

The relevant propagator components are:

$$G_{\mu\nu\rho\sigma}(-p^2) = -i\kappa^2\left[\frac{\mathcal{P}^{(2)}_{\mu\nu\rho\sigma}}{p^2\left(1+\kappa^2 F_2(-p^2)p^2\right)} - \frac{1}{2}\frac{\mathcal{P}^{(0,s)}_{\mu\nu\rho\sigma}}{p^2\left(1+\kappa^2 F_0(-p^2)p^2\right)}\right]$$

- Note that, it is again the same spin structure as in GR and Quadratic Gravity.
- If F_0 and F_2 are constants, we recover the Quadratic Gravity case.
- Depending on the functional form of $F_0(-p^2)$ and $F_2(-p^2)$ we can have different UV behaviors and different # of gravitational DOFs.

- UV behavior: if $F_0 \neq 0$, $F_2 \neq 0$ we have:

$$\mathcal{G}_{\mu\nu\rho\sigma}(p) \overset{UV}{\sim} \frac{1}{F_2(-p^2)\,p^4}\,,\ \frac{1}{F_0(-p^2)\,p^4}$$

- # of DOFs: it depends on the # of gauge-invariant poles:

$$\begin{cases} p^2 = 0 \rightarrow \text{we expect to always have the massless spin-2 graviton} \\ 1 + \kappa^2 F_2(-p^2)\,p^2 = 0 \rightarrow \text{additional massive spin-2 poles} \\ 1 + \kappa^2 F_0(-p^2)\,p^2 = 0 \rightarrow \text{additional massive spin-0 poles} \end{cases}$$

Let us now make a classification of local and non-local HD theories of gravity

Classification: local HD theories of gravity

- Einstein's GR: $F_0 = F_2 = 0$ & $\mathcal{O}(\mathcal{R}^3) = 0$.
- Quadric Gravity: $F_0 = e_0$, $F_2 = e_2$ & $\mathcal{O}(\mathcal{R}^3) = 0$.
- Six-derivative gravity: The action contains all operators of mass dimension up to 6:

$$\begin{cases} F_0(\Box) = f_0^{(0)} + f_0^{(1)}\,\Box\,, \quad F_2(\Box) = f_2^{(0)} + f_2^{(1)}\,\Box\,, \quad [f_0^{(1)}] = [f_2^{(1)}] = -2 \\ \mathcal{O}(\mathcal{R}^3) = \{\text{all cubic curvature contractions with no covariant derivatives}\} \end{cases}$$

$$\Rightarrow \quad S_6[g] = \frac{1}{2}\int d^6x\,\sqrt{-g}\Big[\frac{1}{\kappa^2}R + \frac{1}{6}f_0^{(0)}R^2 - \frac{1}{2}f_2^{(0)}\,C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \frac{1}{6}f_0^{(1)}R\Box R - \frac{1}{2}f_2^{(1)}\,C_{\mu\nu\rho\sigma}\Box C^{\mu\nu\rho\sigma} + g_1 R^3 + g_2 R\,R_{\mu\nu}R^{\mu\nu} + \dots\Big]$$

- The pole equations are: $1 + \kappa^2 \left(f_i^{(0)} - f_i^{(1)} p^2 \right) p^2 = 0$, $i = 0, 2$.

  $\Rightarrow$ we have two massive spin-0 poles and two massive spin-2 poles.

- UV behavior of the propagator: $G_{\mu\nu\rho\sigma}(p) \overset{UV}{\sim} \dfrac{1}{f_i^{(1)} \, p^6}$, $i = 0, 2$

- UV behavior of the vertices: $S_6^{(n \geq 3)} \overset{UV}{\sim} f_i^{(1)} \partial^6 h^n$, $g_j \partial^6 h^n$

  $\longrightarrow$ in momentum space: $f_i^{(1)} p^6$, $g_j p^6$

- UV behavior of most divergent L-loop integral:

$$\int \underbrace{d^4k \ldots d^4k}_{\text{L-loop}} \cdot \underbrace{\frac{1}{k^6} \ldots \frac{1}{k^6}}_{\text{I-propagator}} \underbrace{k^6 \ldots k^6}_{\text{V-vertices}} \sim k^{4L - 6I + 6V} = k^{-2L + 6(L - I + V)} = k^{6 - 2L}$$

$\Rightarrow$ The superficial degree of divergence decreases by increasing L.

- To check the formula $\delta(G) = 4 - EF - \sum_m V_m \Delta_m$ we first have to canonically normalize the kinetic term in the UV.

- The kinitic term in the UV behaves as $\sim f_i^{(1)} \, h \, \partial^6 h$

  To canonically normalize it, we have to rescale: $h \to \frac{1}{\sqrt{f_i^{(1)}}} h$

  $\Rightarrow h \partial^6 h \,, \quad [h] = -1$

- After canonical normalization the vertices in the UV schematically behave as

$$f_i^{(1)} \left(\frac{1}{\sqrt{f_j^{(1)}}} \right)^m \partial^6 h^m \,, \quad m \geq 3$$

The mass dimension of the interaction couplings is:

$$\left[f_i^{(1)} \left(\frac{1}{\sqrt{f_j^{(1)}}} \right)^m \right] = \left[f_i^{(1)} \right] - \frac{m}{2} \left[f_j^{(1)} \right] = -2 + m \geq 1$$

Therefore, since $F = [h] = -1$, $\Delta_n \geq 1$

$$\Rightarrow \quad \delta(G) = 4 + E - \sum_n V_n \Delta_n$$

For sufficiently large V_n $\delta(G)$ becomes negative.

$\Rightarrow$ The six-derivative theory of gravity is <u>super-renormalizable</u>!

- $(2N+4)$ - derivative theories of gravity:

$$\begin{cases} F_0(\Box) = f_0^{(0)} + f_0^{(1)}\Box + \cdots + f_0^{(N)}\Box^N = \sum_{n=0}^{N} f_0^{(n)}\Box^n, \quad F_2(\Box) = f_2^{(0)} + f_2^{(1)}\Box + \cdots + f_2^{(N)}\Box^N = \sum_{n=0}^{N} f_2^{(n)}\Box^n \\ [f_0^{(N)}] = [f_2^{(N)}] = -2N \\ \mathcal{O}(\mathcal{R}^3) = \left\{ \begin{array}{l} \text{all higher-order curvature and covariant derivative contractions} \\ \text{up to } N\text{-th order, i.e. up to operators of mass dimension } 2N+4 \end{array} \right\} \end{cases}$$

$$\Rightarrow \quad S_6[g] = \frac{1}{2}\int d^4x\,\sqrt{-g}\,\Big[\,\frac{1}{\kappa^2}R + \frac{1}{6}f_0^{(0)}R^2 - \frac{1}{2}f_2^{(0)}\,C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \frac{1}{6}R\Big(\sum_{n=1}^{N} f_0^{(n)}\Box^n\Big)R - \frac{1}{2}C_{\mu\nu\rho\sigma}\Big(\sum_{n=1}^{N} f_2^{(n)}\Box^n\Big)C^{\mu\nu\rho\sigma} + \mathcal{O}(R^3)\,\Big]$$

- The pole equations are: $1 + \kappa^2\Big(\sum_{n=0}^{N} f_i^{(n)}(-p^2)^n\Big)p^2 = 0$, $i = 0, 2$.

  $\Rightarrow$ we have N+1 massive spin-0 poles and N+1 massive spin-2 poles.

- UV behavior of the propagator: $G_{\mu\nu\rho\sigma}(p) \overset{UV}{\sim} \dfrac{1}{f_i^{(N)}\,p^{2N+4}}$, $i = 0, 2$

- UV behavior of the vertices: $S_6^{(n\geq 3)} \overset{UV}{\sim} f_i^{(N)}\,\partial^{2N+4}\,h^n$

  (before canonical normalization) $\longrightarrow$ in momentum space: $f_i^{(N)}\,p^{2N+4}$

- UV behavior of most divergent L-loop integral :

$$\int \underbrace{d^4k \dots d^4k}_{\text{L-loop}} \underbrace{\frac{1}{k^{2N+4}} \dots \frac{1}{k^{2N+4}}}_{\text{I-propagator}} \underbrace{k^{2N+4} \dots k^{2N+4}}_{\text{V-vertices}} \sim k^{4L-(2N+4)I+(2N+4)V} = k^{4-2N(I-V)} = k^{4-2N(L-1)}$$

$\Rightarrow$ The superficial degree of divergence decreases by increasing L.

$\Rightarrow$ $(2N+2)$-derivative theories of gravity can be super-renormalizable

- Remark: What about unitarity in these super-renormalizable HD theories ?

In some cases it is possible to replace the spin-two ghost-like real massive pole with complex-conjugate pairs of poles.

[Then, the validity of unitarity is claimed by implementing alternative diagrammatic rules and contour deformation in the complex-energy plane]

[See papers by: Anselmi, Piva, Modesto, Calcagni, Briscese, ...]

Classification: non-local HD theories of gravity

Recall the form of the propagator for generic $F_0(-p^2)$ and $F_2(-p^2)$:

$$G_{\mu\nu\rho\sigma}(-p^2) = -i\kappa^2\left[\frac{\mathcal{P}^{(2)}_{\mu\nu\rho\sigma}}{p^2\left(1+\kappa^2 F_2(-p^2)p^2\right)} - \frac{1}{2}\frac{\mathcal{P}^{(0,s)}_{\mu\nu\rho\sigma}}{p^2\left(1+\kappa^2 F_0(-p^2)p^2\right)}\right]$$

Can we choose $F_0(-p^2)$ and $F_2(-p^2)$ in such a way that the HD theory is ghost-free?

Yes, there exists an infinite class of non-polynomial F_i such that the tree-level propagator is ghost-free!

- Poles and DOFs: we only want the ± 2 massless graviton and at most an additional (non-ghost) massive spin-0.

- We can achieve this by requiring:

$$1+\kappa^2 F_2(-p^2)\,p^2 = e^{\gamma_2(-p^2)}\,, \qquad 1+\kappa^2 F_0(-p^2)\,p^2 = \left(1+\frac{p^2}{m_0^2}\right) e^{\gamma_0(-p^2)}$$

$$\Leftrightarrow \quad F_2(-p^2) = \frac{1}{\kappa^2}\,\frac{e^{\gamma_2(-p^2)}-1}{p^2}\,, \qquad F_0(-p^2) = \frac{(1+p^2/m_0^2)\,e^{\gamma_0(-p^2)}-1}{\kappa^2\,p^2}$$

where $\gamma_2(-p^2)$ and $\gamma_0(-p^2)$ are <u>entire functions</u> (no additional poles from $e^{\gamma_i(-p^2)}$)

- Action:

$$F_2(\Box) = \frac{1}{\kappa^2}\,\frac{1-e^{\gamma_2(\Box)}}{\Box}\,, \qquad F_0(\Box) = \frac{1-\left(1-\frac{\Box}{m_0^2}\right)e^{\gamma_0(\Box)}}{\kappa^2\,\Box}$$

$$\Rightarrow \quad S = \frac{1}{2}\int d^4x\sqrt{-g}\left\{\frac{1}{\kappa^2}R + R\left[\frac{1-(1-\Box/m_0^2)\,e^{\gamma_0(\Box)}}{6\kappa^2\Box}\right]R - C_{\mu\nu\rho\sigma}\left[\frac{1-e^{\gamma_2(\Box)}}{2\kappa^2\Box}\right]C^{\mu\nu\rho\sigma} + O(R^3)\right\}$$

Remarks:

- Also in this case we can have super-renormalizable or even finite theories by suitably choosing the entire functions $\gamma_0(\Box)$ and $\gamma_2(\Box)$

- In particular, this happens for entire functions that behave polynomially in the deep UV like the ones we saw in the scalar-field analysis.

For example:

$$\gamma_i(-p^2) = \frac{1}{2}\left[\Gamma\left(0, g_i(\Box)\right) + \gamma_E + \log\left(g_i(\Box)\right)\right] , \qquad i = 0, 2 ,$$

where $g_i(\Box) \sim \left(-\Box/\mu\right)^n$.

[See papers by: Kuzmin (1989), Tomboulis (1997), Modesto, Rachwal, Calcagni, Kosheler, Kumar, ...]

Problems (at least, in my opinion)

- Both local and non-local HD theories analyzed in this lecture are characterized by some violation of causality: either due to non-locality or to the new type of diagrammatics in local theories with complex-conjugate poles.

(In principle a violation of micro-causality can be fine, if really confined to very short time scales and somehow unobservable. But, this is still not fully clear.)

- In principle, resummation of radiative corrections can violate the ghost-freeness condition: the dressed propagator may not respect the tree-level ghost-free structure [Shapiro (2015)]

- The most serious problem (in my opinion) is the lack of uniqueness!

  In both cases, we have an infinite class of super-renormalizable theories. No additional guiding principle to uniquely select a Lagrangian!

  ⇒ This also implies lack of predictivity!

LECTURE 3.2: Quadratic Gravity - Part II

(Section 2.3 - arXiv: 2412.08690
L.B., arXiv: 2504.05900)

- We noticed that requiring super-renormalizability as a "guiding principle" opens up an infinite class of local HD theories of gravity: (2N+2)-derivative theories.
- Giving up the locality of the bare Lagrangian is even worse in this respect because the class of possibilities is even wider: an infinite type of entire functions $\gamma_0(\Box)$ and $\gamma_2(\Box)$ may work.

$\Rightarrow$ [This means that if we start with one of these theories, and one day we manage to falsify it, then we can simply choose another theory in this infinite class without changing the starting guiding principles.]

$\Rightarrow$ [This is <u>not</u> good because we would have no way to make predictions, nor to understand which QFT can physically describe quantum-gravitational features beyond the EFT of GR.]

- Are there reasonable guiding principles that lead to a more unique theory ?

The answer is : Yes !

- If we consider again the starting "guiding principles" introduced in the 1st lecture, in particular strict renormalizability, in $D=4$ there exists a unique theory of gravity that

  - is local
  - respects the symmetries of GR (diff. and parity invariance)
  - respects the geometric structure of GR (metric compatibility and zero torsion)
  - is strictly renormalizable

$\Rightarrow$ this unique theory is Quadratic Gravity !

- **What about unitarity ?**

In the 2nd lecture we said that the ghost-like component of a four-derivative propagator can violate unitarity. But, actually, this is <u>not</u> always true !

Unitarity is the statement of conservation of inner products under time-evolution. This can be true even when ghost-like "minus" signs are present.

- Recall that the optical theorem tells us that:

$$i\left[\langle b|T^{\dagger}|a\rangle - \langle b|T|a\rangle\right] = \sum_{\{m\}} \sigma_m \langle b|T^{\dagger}|m\rangle\langle m|T|a\rangle$$

Unitarity can be still respected if one of the following two assumptions are made:

[(i) alternative diagrammatic rules are introduced (<u>Fakeon approach</u>)

or (ii) the positive-norm Hilbert space is replaced by an <u>indefinite-norm vector space</u>]

(i) Fakeon approach [See papers by Anselmi, Piva, Calcagni, ...]

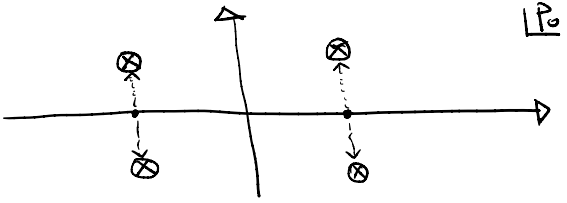
The idea is to introduce new diagrammatic rules and new prescriptions to shift ghost-like poles and deform contours in the complex-energy plane in order to render the ghost a purely virtual particle (never appearing on-shell)

At tree-level this means that the ghost-like component of the propagator has to be prescribed in a different way:

$$\frac{-(-i)}{p^2+m^2} \longrightarrow -(-i)\,\frac{p^2+m^2}{(p^2+m^2)^2+\varepsilon^2} = -(-i)\,\frac{1}{2}\left[\frac{1}{p^2+m^2+i\varepsilon}+\frac{1}{p^2+m^2-i\varepsilon}\right]$$

with $\varepsilon \to 0^+$

This means that poles are not simply shifted but are a splitted in two pairs of complex-conjugate poles:



$\Rightarrow$ The full four-derivative propagator is:

$$G(-p^2) = (-i)\left[\frac{1}{p^2 - i\epsilon} - \frac{1}{2}\left(\frac{1}{p^2 + m^2 + i\xi} + \frac{1}{p^2 + m^2 - i\xi}\right)\right], \quad \epsilon \to 0^+, \ \xi \to 0^+$$

If we compute the imaginary part of the corresponding amplitude we get:

$$\mathrm{Im}\left[i\,G(-p^2)\right] = \pi\,\delta(p^2) + 0 = \pi\,\delta(p^2) \geq 0$$

$\Rightarrow$ the fakeon ghost does <u>not</u> contribute to the imaginary part!

$\Rightarrow$ consequently, if we make a compatible projection in the vector space:

$$\mathcal{H} \longrightarrow \overline{\mathcal{H}} \subset \mathcal{H} \Rightarrow \mathbb{1} = \sum_{|m\rangle \in \mathcal{H}} \sigma_m |m\rangle\langle m| \longrightarrow \overline{\mathbb{1}} = \sum_{|m\rangle \in \overline{\mathcal{H}}} \overline{\sigma}_m |m\rangle\langle m|, \quad \overline{\sigma}_m > 0.$$

where $\overline{\mathcal{H}}$ is the "physical" Hilbert space according to the Fakeon approach, in which no on-shell ghost is present, then we can ensure that the optical theorem is satisfied, namely LHS = RHS.

We have just shown a tree-level example for a scalar field.

However, the whole construction has been generalized to all loop orders in perturbation and to the case of Quadratic Gravity.

[See papers by Anselmi and Piva]

Remarks:

- The use of "ad hoc" new diagrammatics is very useful to save unitarity. However, this procedure makes the whole approach to deviate from the standard formalism of local QFT.
- In my opinion, because of these new diagrammatics and of the non-analiticities that it introduces already at tree-level, the perturbative renormalizability of the theory is <u>not</u> automatically guaranteed in both momentum and position space.
- In my opinion, the Fakeon approach is an <u>hybrid</u> between QFT and S-matrix approaches.

$\Rightarrow$ [For these reasons, in my opinion, if we want to be as conservative as possible and do not deviate from the framework of local QFT, the approach (ii) is the only way to quantize Quadratic Gravity, compatibly with all four "guiding principles".]

(ii) Indefinite-norm quantization [See papers by Salvio, Strumia, Holdom, Donoghue, Menezes, L.B., ...]

- At least perturbatively, we do not change anything about the LHS of the optical theorem; namely, we keep the same Feynman prescription:

$$G(-p^2) = -i\left[\frac{1}{p^2 - i\epsilon} - \frac{1}{p^2 + m^2 - i\epsilon}\right] \Rightarrow \mathrm{Im}\left[iG(-p^2)\right] = \pi\left[\delta(p^2) - \delta(p^2 + m^2)\right]$$

- Although the LHS of the optical theorem can be negative, unitarity would be still satisfied if also the RHS could be negative in such a way to compensate the negative sign of LHS and lead to LHS = RHS.

- This can be achieved by working in an <u>indefinite-norm vector space</u>. In particular we assume that the completeness relation reads:

$$\mathbb{1} = \sum_{|m\rangle} \sigma_m \, |m\rangle\langle m| \quad , \quad \begin{cases} \sigma_m < 0 & \text{for states containing an odd \# of ghosts} \\ \sigma_m > 0 & \text{otherwise} \end{cases}$$

$\Rightarrow$ 1-particle ghost states must have <u>negative norm</u>.

$\Rightarrow$ This ensures that the optical theorem, and so unitarity, are satisfied:

$$S^\dagger \mathbb{1} S = \mathbb{1}$$

Note that, if the vector space has indefinite-norm states, then the pseudo-unitarity equation shown at the end of the 2nd lecture becomes equivalent to unitarity: $S^\dagger \mathbb{1} S = \mathbb{1}$.

[Unitarity does <u>not</u> say anything about the sign of the norm of the state.
Indeed, we can have a unitary quantization even if negative-norm states are present.]

Question: Does the presence of negative-norm states imply the existence of observable negative probabilities ?

- If the answer were yes, then the theory would be pathological !
- Actually, works in the last decade have been supporting the physical consistency of four-derivative theories, and so Quadratic Gravity.

  [See papers by Salvio, Holdom, Donoghue, Menezes, Turok, L.B., ...]

- I will not comment on these works, nor on my recent ones (arXiv: 2605.29047, 2606.18349)

- I just want to say that the asymptotic dynamics of ghost fields is highly non-trivial and that most likely no particle interpretation can be assigned to ghosts asymptotically.

Final comments (my personal view on the topic; see also L.B., arXiv: 2504.05900)

$$S_{QG}[g] = \frac{1}{2}\int d^4x\sqrt{-g}\left[\frac{1}{\kappa^2}(R-2\Lambda) + \frac{c_0}{6}R^2 - \frac{c_2}{2}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}\right]$$

- Quadratic Gravity is the most conservative QFT of gravity
- It is a unique strictly renormalizable QFT, compatible with the guiding principles that proved to be very successful for the formulation of the SM of particle physics
- If $c_0 > 1$, $c_2 > 1$ $\Rightarrow$ $m_0 = \frac{M_P}{\sqrt{c_0}} < M_P$, $m_2 = \frac{M_P}{\sqrt{c_2}} < M_P$

  $\Rightarrow$ new physics in the sub-Planckian regime!
- In fact, the massive spin-0 (also known as the scalaron) provides one of the best candidates for the inflaton field and for explaining CMB anisotropies.

  If $c_0 \sim 10^9 \Leftrightarrow m_0 \sim 10^{13}$ GeV, QG can explain CMB anysotropies

- Therefore, without even asking for it, just requiring consistency with the four guiding principles 1), 2), 3), 4) within the perturbative QFT framework we ended up explaining new physics beyond Einstein's GR.

- Future satellite missions aim to observe primordial gravitational waves. The relevant physical quantity to be measured is the <u>tensor-to-scalar ratio</u>, which in Quadratic Gravity, at leading order in the slow-roll expansion, is

$$r = \frac{24}{N_e^2} \frac{m_2^2}{m_0^2 + 2 m_2^2} \,, \qquad N_e \simeq 60 \rightarrow \#\text{ of e-folds}$$

Therefore, a future measurement of r will also constrain m_2 (i.e. c_2).

- Actually, from high-energy physics arguments, we may expect:

$$m_2 \gtrsim m_0 \qquad \Rightarrow \qquad 3 \times 10^{-3} \lesssim r \lesssim 3.3 \times 10^{-3}$$

Current constraint on r is $\lesssim O(10^{-2})$.

Future satellite missions (like LiteBird planned to be launched in 2032) will probe values of order $O(10^{-3})$, exactly the reasonable order of magnitude we expect in Quadratic Gravity.

- This is very exciting because we can strongly constrain Quadratic Gravity even observationally, and not just theoretically, in the near future.
- We don't know whether Quadratic Gravity and the paradigm of strict renormalizability are the end of the story. But, certainly we can say that both history and Nature are suggesting that we cannot proceed without them.
- In my opinion, it looks very likely that Quadratic Gravity can describe a QFT layer of reality that includes also new physics beyond GR in the sub-Planckian regime.
- All these interesting theoretical and phenomenological features of Quadratic Gravity are the main reasons that motivate me to investigate this unique gravitational QFT further!