LECTURE 2.1: GR as a QFT - Interacting theory (Section 2.3 - arXiv: 2412.08690)

Canonical quantization (not always very convenient)

The quantum graviton field can be written as an infinite superposition of plane waves weighted by polarization tensors in the helicity basis and by the annihilation/creation operators:

$$h_{\mu\nu}(x) = \sum_{\lambda=+2,-2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}} \left(a_{\vec{p},\lambda}\, \epsilon^{(\lambda)}_{\mu\nu}\, e^{ip\cdot x} + a^{\dagger}_{\vec{p},\lambda}\, \epsilon^{(\lambda)*}_{\mu\nu}\, e^{-ip\cdot x} \right), \quad \omega_{\vec{p}} \equiv \sqrt{|\vec{p}|^2} = |\vec{p}|$$

- Commutation relations: $[a_{\vec{p},\lambda}, a_{\vec{p}',\lambda'}] = 0 = [a^{\dagger}_{\vec{p},\lambda}, a^{\dagger}_{\vec{p}',\lambda'}]$, $[a_{\vec{p},\lambda}, a^{\dagger}_{\vec{p}',\lambda'}] = 2\omega_{\vec{p}}\,(2\pi)^3\,\delta_{\lambda\lambda'}\,\delta^{(3)}(\vec{p}-\vec{p}')$

- (Free) vacuum: $a_{\vec{p},\lambda}\,|0\rangle = 0$.

- One-graviton state: $|\vec{p},\lambda\rangle \equiv a^{\dagger}_{\vec{p},\lambda}\,|0\rangle$.

Issues with self-interactions: gauge invariance, unitarity, Faddeev-Popov fields, ...

Faddeev-Popov fields (BRST quantization more convenient)

Total (bare) action : $S[g,\eta,h] = S_{EH}[g] + S_{gf}[\eta,h] + S_{FP}[g,\eta,h]$,

where $S_{EH}[g] = \frac{1}{2\kappa^2}\int d^4x\,\sqrt{-g}\,R$,

$S_{gf}[\eta,h] = -\frac{1}{\alpha}\int d^4x\, \mathcal{F}_\mu\,\eta^{\mu\nu}\,\mathcal{F}_\nu$, $\quad \mathcal{F}_\mu = \partial_\nu h^\nu{}_\mu - \partial_\nu h$

and S_{FP} is determined by looking how the the gauge-fixing transforms under diff.:

$$S_{FP}[g,\eta,h] = \int d^4x\ \bar{c}^\mu\ \frac{\delta \mathcal{F}_\mu}{\delta \xi_\nu}\ c_\nu$$

where c_μ is called FP field, while $\bar{c}_\mu$ anti-FP field; they are anti-commuting

(the construction can be generalized to generic backgrounds $\bar{g}_{\mu\nu}$)

Note that : $$\frac{\delta \mathcal{F}_\mu}{\delta \xi_\nu} = \frac{\delta \mathcal{F}_\mu}{\delta h_{\rho\sigma}} \frac{\delta h_{\rho\sigma}}{\delta \xi_\nu} = - \frac{\delta \mathcal{F}_\mu}{\delta h_{\rho\sigma}} \left(\delta_\sigma{}^\nu \nabla_\rho + \delta_\rho{}^\nu \nabla_\sigma \right)$$

$$\Rightarrow \quad S_{FP}[g, \eta, h] = - \int d^4x \, \bar{c}^\mu \, \frac{\delta \mathcal{F}_\mu}{\delta h_{\rho\sigma}} \left(\nabla_\rho c_\sigma + \nabla_\sigma c_\rho \right)$$

- de Donder gauge fixing : $\mathcal{F}_\mu = \partial_\nu h^\nu{}_\mu - \partial_\nu h = \left[\frac{1}{2} \left(\delta_\mu{}^\alpha \partial^\beta + \delta_\mu{}^\beta \partial^\alpha \right) - \frac{1}{2} \eta^{\alpha\beta} \partial_\mu \right] h_{\alpha\beta}$

$$\Rightarrow \quad S_{FP}[g, \eta, h] = \int d^4x \, \bar{c}^\nu \left[\partial^\rho \nabla_\rho c_\mu + \partial^\rho \nabla_\mu c_\rho - \partial_\mu \nabla^\rho c_\rho \right]$$

- Expansion around Minkowski :

$$S_{FP}[\eta, h] = \int d^4x \, \bar{c}^\mu \, \Box \, c_\mu + \mathcal{O}(n h \bar{c} c)$$

- Propagator of the FP fields : $\sim \frac{1}{p^2}$; Interactions : only cubic vertex $\frac{1}{M_P} h \bar{c} c$

Failure of perturbative renormalizability :

$$S = \int d^4x \, \frac{1}{2} \, h_{\mu\nu} \, \mathbb{K}^{\mu\nu\rho\sigma} \, h_{\rho\sigma} + \mathcal{O}(\kappa h^3)$$

- Generic self-interaction : $\kappa^{m-2} \, \partial^2 h^m$, $m \geq 3$.
- Mass dimension of the couplings: $\Delta_m \equiv [\kappa^{m-2}] = \left[\frac{1}{M_p^{m-2}}\right] = 2-m < 0$.

$\Rightarrow$ GR is perturbatively non-renormalizable !

- Superficial degree of divergence: $\delta(\mathcal{G}) = 4 - E - \sum_m V_m \Delta_m = 4 - E + \sum_m V_m |\Delta_m|$

$\Rightarrow$ $\delta(\mathcal{G})$ increases with the # of vertices (and of loops L).

- L-loop integral : (most divergent case)

$$\int \underbrace{d^4k_1 \dots d^4k_L}_{\text{L-loop}} \; \underbrace{\frac{1}{k^2} \dots \frac{1}{k^2}}_{\text{I-propagators}} \; \underbrace{k^2 \dots k^2}_{\text{V-vertices}} \sim k^{4L-2I+2V} = k^{2L+2}$$

$$\uparrow \quad (L - I + V = 1)$$

One-loop divergences (pure GR, no matter)

[Loop computations in GR are very hard. However, thanks to symmetry and power-counting arguments, we can determine the tensor structure of loop UV divergences. The full tedious calculation is only needed to find the exact numerical coefficients.]

- Let us consider one-loop diagrams with 2 external graviton legs and either gravitons or FP fields as internal propagators.

  - tadpole (): $PP\int d^4k\,\frac{1}{k^2}\sim k^2$; $P\int d^4k\,\frac{1}{k^2}k\sim k^3$; $\int d^4k\,\frac{1}{k^2}k^2\sim k^4$.

    $\Rightarrow$ power-law divergences; renormalize $M_P^2 R$ and $M_P^2\Lambda$.

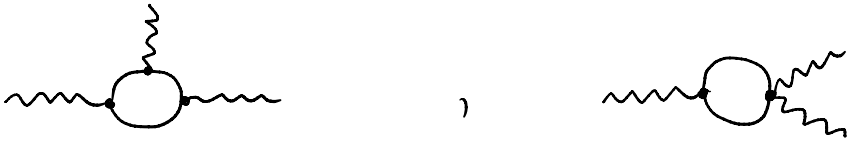
  - bubble (): both power-law

    and <u>logarithmic divergence</u> $PPPP\int d^4k\,\frac{1}{k^2}\frac{1}{k^2}\sim PPPP\int\frac{dk}{k}$

    ↳ <u>higher-derivative counterterms</u>: R^2, $R_{\mu\nu}R^{\mu\nu}$, $R_{\mu\rho\sigma\nu}R^{\mu\rho\sigma\nu}$

But, these terms are <u>not</u> contained in the original bare $\mathcal{L}$.

Similar UV divergences affect one-loop n-graviton vertices.
For example:



- **Tensor structure of 1-loop UV divergent part of the effective action**

$$\Gamma^{(1)}_{div}[g] = \frac{1}{\epsilon}\int d^4x\,\sqrt{-g}\,\Big[e_1 R^2 + e_2 R_{\mu\nu}R^{\mu\nu} + e_3 R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + e_4 R_{\mu\nu\rho\sigma}R^{\rho\sigma\mu\nu} + e_5 \nabla_\mu\nabla_\nu R^{\mu\nu} + e_6 \Box R\Big]\,;$$

$e_1,\dots,e_6$ constants; $\frac{1}{\epsilon} = \frac{1}{4-D} \equiv$ simple pole divergence in dimensional reg.

We now show that only <u>two</u> independent terms, out of the six, contribute.

- $\nabla_\mu \nabla_\nu R^{\mu\nu}$, $\Box R$ are total derivative $\longrightarrow$ we neglect boundary terms.

  (actually they are proportional : $\nabla_\mu \nabla_\nu R^{\mu\nu} = \frac{1}{2} \nabla_\mu \nabla^\mu R = \frac{1}{2} \Box R \quad (\nabla_\mu G^{\mu\nu} = 0)$)

- Third and fourth terms are also proportional :

$$R_{\mu[\rho\nu\sigma]} = 0 \quad \Rightarrow \quad R_{\mu\nu\rho\sigma} R^{\mu\rho\nu\sigma} = \frac{1}{2} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$$

- Using the Gauss-Bonnet identity valid in $D = 4$:

$\sqrt{-g} \left[R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 \right]$ = tot. derivative (See Pereeei's book for a proof)

$$\Rightarrow \quad \sqrt{-g}\, R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} = \sqrt{-g} \left[4 R_{\mu\nu} R^{\mu\nu} - R^2 \right] + \text{tot. derivative}$$

$\Rightarrow$ the <u>two</u> independent terms are : R^2 , $R_{\mu\nu} R^{\mu\nu}$

$$\Rightarrow \quad \boxed{\Gamma^{(1)}_{div}[g] = \frac{1}{\varepsilon}\int d^4x\,\sqrt{-g}\left[a\,R^2 + b\,R_{\mu\nu}R^{\mu\nu}\right]}$$

where $\quad a \equiv c_1 - c_3 - \frac{c_4}{2}\,, \quad b \equiv c_2 + 4\left(c_3 + \frac{c_4}{2}\right)$

- The exact numerical coefficients a, b were computed for the first time by G. 't Hooft and M. Veltman in 1974 in Feynman - de Donder gauge:

$$a = -\frac{1}{(4\pi)^2}\frac{1}{120}\,, \qquad b = -\frac{1}{(4\pi)^2}\frac{7}{20}$$

- Although we neglected the boundary terms, it is worth mentioning that they are important for the renormalization of the theory.

  Indeed, the Gauss-Bonnet term appears with coefficient $-\frac{1}{(4\pi)^2}\frac{1}{\varepsilon}\frac{53}{90}$.

  (Actually both Gauss-Bonnet and $\Box R$ have important physical implications as they contribute to the conformal anomaly when gravity is coupled to matter)

1-loop finiteness of pure GR: field redefinition

We now show that the 1-loop divergence above can be absorbed by a field redefinition of $g_{\mu\nu}$ (in pure GR with no matter couplings).

At one-loop we have: $\quad S^1[g] = S_{EH}[g] + \Gamma^{(1)}_{div}[g]$

We can find a field redefinition: $\quad g_{\mu\nu} \longrightarrow g'_{\mu\nu}(g,a,b) = g_{\mu\nu} + \Delta g_{\mu\nu}(a,b)$

such that, perturbatively in a and b, we have:

$$S_{EH}[g] + \Gamma^{(1)}_{div}[g] = S_{EH}[g+\Delta g] + \mathcal{O}(a^2, b^2, ab) = S_{EH}[g'] + \mathcal{O}(a^2, b^2, ab)$$

where $\mathcal{O}(a^2, b^2, ab)$ are expected to contribute at higher curvature/loop orders.

The field variation $\Delta g_{\mu\nu}(a,b)$ can be found by imposing:

$$S_{EH}[g+\Delta g] = S_{EH}[g] + \Delta S_{EH}[g] + \dots = S_{EH}[g] + \Gamma^{(1)}_{div}[g] + \dots$$

$$\Leftrightarrow \quad \Delta S_{EH}[g] = \Gamma^{(1)}_{div}[g]$$

Note that: $\Delta S_{EH}[g] \equiv \int d^4x \sqrt{-g}\, \frac{\delta S_{EH}[g]}{\delta g_{\mu\nu}} \Delta g_{\mu\nu}$

$$= \int d^4x \sqrt{-g}\, \frac{1}{2\kappa^2} \left(R^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} R\right) \Delta g_{\mu\nu}$$

Making the ansatz: $\Delta g_{\mu\nu} = A\, g_{\mu\nu} R + B\, R_{\mu\nu}$ we get:

$$\Delta S_{EH}[g] = \int d^4x \sqrt{-g}\, \frac{1}{2\kappa^2} \left(R^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} R\right)\left(A g_{\mu\nu} R + B R_{\mu\nu}\right)$$

$$= -\int d^4x \sqrt{-g}\, \frac{1}{2\kappa^2} \left[-\left(A + \tfrac{1}{2} B\right) R^2 + B\, R_{\mu\nu} R^{\mu\nu}\right]$$

$$\Delta S_{EH}[g] = \Gamma^{(1)}_{div}[g] \quad \Leftrightarrow \quad A = \frac{\kappa^2}{\varepsilon}(2a+b), \quad B = -\frac{\kappa^2}{\varepsilon} 2b$$

$$\Rightarrow \quad g'_{\mu\nu} = g_{\mu\nu} + \frac{\kappa^2}{10(4\pi)^2 \varepsilon}\left(\frac{11}{3} g_{\mu\nu} R - 7 R_{\mu\nu}\right)$$

- In the presence of a non-vanishing cosmological constant $\Lambda \neq 0$, pure GR is still 1-loop renormalizable because the UV divergence can be absorbed by a logarithmic renormalization of the metric field and of the Newton's constant off-shell:

$$S_{EH}^{(\Lambda\neq 0)}[g] + \Gamma^{(1)}_{div}[g] = S_{EH}^{(\Lambda\neq 0)}[g'] - \frac{23}{(4\pi)^2\, 30\, \varepsilon}\int d^4x\, \sqrt{-g'}\, \Lambda\, R(g')$$

1-loop divergences and coupling to matter

If matter couplings are switched on $\Rightarrow$ GR + matter is 1-loop non-renormalizable !

Because the 1-loop divergence is no longer proportional to the field equations. Indeed, we have:

$$\Gamma^{(1)}_{div}[g] = \frac{1}{\varepsilon}\int d^4x\sqrt{-g}\left[a\,R^2 + b\,R_{\mu\nu}R^{\mu\nu}\right] \quad , \quad \frac{\delta(S_{EH}+S_m)}{\delta g_{\mu\nu}} = -\frac{1}{2\kappa^2}\sqrt{-g}\left[R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R - \kappa^2 T^{\mu\nu}\right]$$

if we make the metric redefinition above, we would generate terms like $\frac{\kappa^2}{\varepsilon}R_{\mu\nu}T^{\mu\nu}$, $\frac{\kappa^2}{\varepsilon}RT$.

2-loop divergences (pure GR)

Similar cancellations in pure GR do NOT occur at higher loops. In fact, pure GR is perturbatively non-renormalizable starting from 2 loops.

Some 2-loop divergences can still be absorbed by metric-field redefinitions, but not all of them. In the end, the gauge invariant 2-loop divergence is:

$$\Gamma^{(2)}_{div}[g] = \frac{209}{2880\,(4\pi)^4}\,\frac{\kappa^2}{\varepsilon}\int d^4x\sqrt{-g}\;R_{\mu\nu\rho\sigma}\,R^{\mu\nu}{}_{\alpha\beta}\,R^{\rho\sigma\alpha\beta} \qquad \text{(Goroff-Sagnotti 1985)}$$

Quantum GR as a non-renormalizable effective field theory (EFT) (Donoghue 1994)

$$S_{EFT}[g] = \int d^4x \sqrt{-g} \left[\frac{1}{2\kappa^2}(R - 2\Lambda) + a_1 R^2 + a_2 R_{\mu\nu}R^{\mu\nu} + a_3 \kappa^2 R^3 + a_4 \kappa^2 R_{\mu\nu\rho\sigma} R^{\rho\sigma}{}_{\alpha\beta} R^{\alpha\beta\mu\nu} + \dots \right] \quad (+ \text{matter})$$

- a_i are dimensionless and their renormalized value can be determined up to errors $\propto \left(\frac{E}{M_P}\right)^m$ where E is the characteristic energy scale of the physical process.
- When the couplings to matter are present, the EFT cutoff can be lowered ($\Lambda_{UV} < M_P$).
- The EFT of GR can make reliable quantum-gravitational predictions for $E << \Lambda_{UV} \leq M_P$.
- In the EFT of GR the only DOFs are the ± 2 helicity gravitons
- The higher curvature terms do not introduce new DOFs: they do not modify the propagator, but only the vertices.

Beyond the EFT of GR

Question:

- Is there new physics at some energy scale $E < M_p$ that cannot be described by the EFT of GR?
- Namely, does the EFT of GR need to be replaced by a new theory, or even a new framework, starting from some energy scale $E < M_p$?

Several Quantum Gravity approaches have been proposed since the late 70's, some of which are still very popular.

[Our aim is to remain within the framework of perturbative QFT and find a theory that is consistent with renormalizability and unitary (and that is also predictive)]

⇒ This necessarily leads us to Higher-Derivative Theories of (Quantum) Gravity.

LECTURE 2.2: Quadratic Gravity — Part I (Section 2.4 - arXiv: 2412.08690)

Question: Is there any strictly renormalizable QFT of gravity?

Answer: Yes, in $D=4$ there is a "unique" strictly renormalizable QFT of gravity:

Quadratic Gravity (QG)

Action: contains all possible operators $\mathcal{O}^{(i)}$ of mass dimensions up to four $[\mathcal{O}^{(i)}] \leq 4$:

$$S_{QG}[g] = \int d^4x \sqrt{-g} \sum_i b_i \mathcal{O}^{(i)}(g)$$

What are these operators? $\mathcal{O}^{(i)} = \{\Lambda, R, R^2, R_{\mu\nu}R^{\mu\nu}, R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}, \text{boundaries}\}$

$$[\Lambda] = 2, \quad [R] = 2, \quad [R^2] = 4, \quad \dots$$

$\Rightarrow$ Λ and R are multiplied by mass^2 coefficient (M_P^2), while R^2, $R_{\mu\nu}R^{\mu\nu}$, $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ by dimensionless coefficients

More explicitly we have:

$$S_{GB}[g] = \int d^4x \sqrt{-g} \left[\frac{M_P}{2}(R - 2\Lambda) + b_1 R^2 + b_2 R_{\mu\nu} R^{\mu\nu} + b_3 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \right] + (\text{boundary})$$

Recall that using the Gauss-Bonnet identity in $D=4$ we can eliminate $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ up to boundary terms:

$$\sqrt{-g}\, GB = \sqrt{-g} \left[R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 \right] = \text{tot. derivative}$$

Furthermore, we can choose to work in a basis of curvature-squared invariant different from $(R^2, R_{\mu\nu}R^{\mu\nu})$.

It is useful to work with the basis $(R^2, C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma})$ where $C_{\mu\nu\rho\sigma}$ is the Weyl tensor, which is equal to the trace-free part of $R_{\mu\nu\rho\sigma}$:

$$C_{\mu\nu\rho\sigma} \equiv R_{\mu\nu\rho\sigma} + \frac{1}{2}(g_{\mu\sigma} R_{\nu\rho} - g_{\mu\rho} R_{\nu\sigma} + g_{\nu\sigma} R_{\mu\rho} - g_{\nu\sigma} R_{\mu\sigma}) + \frac{1}{6}(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\sigma})\, R$$

Its square equals: $C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 2 R_{\mu\nu} R^{\mu\nu} + \frac{1}{3} R^2$

Therefore, we have two equations for 4 curvature invariants and we can always choose to work with two independent of them:

$$\sqrt{-g}\, R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} = \sqrt{-g}\,[\, 4 R_{\mu\nu} R^{\mu\nu} - R^2 + GB\,] \Rightarrow \sqrt{-g}\, R_{\mu\nu} R^{\mu\nu} = \sqrt{-g}\left[\frac{1}{2} C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + \frac{1}{3} R^2 - \frac{1}{2} GB\right]$$

$\Rightarrow$ Classical action of QG: (up to boundaries)

$$S_{QG}[g] = \frac{1}{2} \int d^4x \sqrt{-g} \left[\frac{1}{\kappa^2}(R - 2\Lambda) + \frac{c_0}{6} R^2 - \frac{c_2}{2} C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \right]$$

$$[\kappa] = \left[\sqrt{8\pi G_N}\right] = \left[1/M_P\right] = -1\,, \quad [c_0] = 0 = [c_2]\,.$$

(The convention for the numerical coefficients and their signs has been chosen for convenience as we will see in a moment)

Field equations

$$E_{\mu\nu} \equiv \frac{1}{\chi^2}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu}\right) + \frac{2}{3}\left(\frac{e_0}{2} + e_2\right)\left(g_{\mu\nu}\Box R - \nabla_\mu\nabla_\nu R + RR_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R^2\right)$$
$$- e_2\left(\Box R_{\mu\nu} + \frac{1}{2}g_{\mu\nu}\Box R - \nabla_\rho\nabla_\mu R^\rho{}_\nu - \nabla_\rho\nabla_\nu R^\rho{}_\mu + 2R_\mu{}^\rho R_{\rho\nu} - \frac{1}{2}g_{\mu\nu}R_{\rho\sigma}R^{\rho\sigma}\right) = T_{\mu\nu} \ .$$

- Bianchi identity and conservation law : $\nabla_\mu E^{\mu\nu} = 0$, $\nabla_\mu T^{\mu\nu} = 0$.
- Fourth-order derivatives : $R \sim \partial\partial g \Rightarrow \nabla\nabla R \sim \partial\partial\partial\partial g$
- Additional DOFs : ± 2 graviton $+ \cdots$?
- New physics below the Planck-energy scale : $E \ll M_p$?
- Why is S_{GB} strictly renormalizable ?
- What is the physical meaning of e_0 and e_2 ?

} QUESTIONS ?

Action expansion

Again, we assume $\Lambda \simeq 0$ and expand around Minkowski:

$$g_{\mu\nu} = \eta_{\mu\nu} + 2h_{\mu\nu} \quad \Rightarrow \quad S_{GR}[\eta + 2h] = S^{(2)}_{GR}[\eta, h] + S^{(m\geq 3)}_{GR}[\eta, h]$$

kinetic term:

$$S^{(2)}_{GR}[\eta, h] = \int d^4x \; \frac{1}{2} h_{\mu\nu} \mathbb{K}^{\mu\nu\rho\sigma}_{GR} h_{\rho\sigma} \,,$$

where

$$\begin{aligned}
\mathbb{K}^{\mu\nu\rho\sigma}_{GR} \equiv & \frac{1}{2}\left(\eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}\right)\left(\frac{1}{\kappa^2} - c_2\,\Box\right) - \eta^{\mu\nu}\eta^{\rho\sigma}\left(\frac{1}{\kappa^2} - \frac{1}{3}(2c_0 + c_2)\Box\right)\Box \\
& + \left(\eta^{\mu\nu}\partial^\rho\partial^\sigma + \eta^{\rho\sigma}\partial^\mu\partial^\nu\right)\left(\frac{1}{\kappa^2} - \frac{1}{3}(2c_0 + c_2)\Box\right) \\
& - \frac{1}{2}\left(\eta^{\mu\rho}\partial^\nu\partial^\sigma + \eta^{\mu\sigma}\partial^\nu\partial^\rho + \eta^{\nu\rho}\partial^\mu\partial^\sigma + \eta^{\nu\sigma}\partial^\mu\partial^\rho\right)\left(\frac{1}{\kappa^2} - c_2\,\Box\right) \\
& + \frac{2}{3}(c_0 - c_2)\,\partial^\mu\partial^\nu\partial^\rho\partial^\sigma
\end{aligned}$$

interaction terms:

$$S^{(m\geq 3)}_{GR}[\eta, h] \sim \frac{1}{\kappa^2}\,\partial^2 h^m \,, \; c_0\,\partial^4 h^m \,, \; c_2\,\partial^4 h^m$$

Propagator

Using the spin-projector formalism and going to momentum space we have:

$$\mathbb{K}^{\mu\nu\rho\sigma}_{\alpha\beta} = -\frac{p^2}{\kappa^2}\left[\mathcal{P}^{(2)\,\mu\nu\rho\sigma}\left(1+\kappa^2 c_2 p^2\right) - 2\,\mathcal{P}^{(0,s)\,\mu\nu\rho\sigma}\left(1+\kappa^2 c_0 p^2\right)\right]$$

Like in GR, it is <u>not</u> invertible. We need to add a gauge-fixing term:

$$S_{gf}[\eta,h] = -\frac{1}{\alpha\kappa^2}\int d^4x\, \mathcal{F}_\mu\, \Upsilon^{\mu\nu}\, \mathcal{F}_\nu \,, \qquad \mathcal{F}_\mu \equiv \partial_\nu h^\nu{}_\mu - \frac{1+\beta}{4}\partial_\mu h$$

$$\Upsilon^{\mu\nu} \equiv \eta^{\mu\nu}\left(1+\gamma\,\Box\right) + w\,\partial^\mu\partial^\nu$$

In principle, we can work with more gauge parameters (α,β,γ,w) and then choose their values based on what is most convenient for us.

Since we are only interested in the propagator components that survive after contractions with a conserved source $(p_\mu T^{\mu\nu}=0)$, we can forget about the gauge-fixing terms in what follows.

In the end, the propagator components we are interested in are given by

$$\mathcal{G}_{\alpha\beta\mu\nu\rho\sigma}(p) = -i\kappa^2\left[\frac{m_2^2\,\mathcal{P}^{(2)}_{\mu\nu\rho\sigma}}{p^2(p^2+m_2^2)} - \frac{1}{2}\frac{m_0^2\,\mathcal{P}^{(0,s)}_{\mu\nu\rho\sigma}}{p^2(p^2+m_0^2)}\right]$$

where $m_0 \equiv \frac{M_P}{\sqrt{c_0}}$, $m_2 \equiv \frac{M_P}{\sqrt{c_2}}$

Remarks:

- 4th-order derivatives in the kinetic term $\Rightarrow \mathcal{G}_{\alpha\beta\mu\nu\rho\sigma}(p) \overset{UV}{\sim} \frac{1}{p^4}$
- Using partial fraction decomposition, $\frac{m^2}{p^2(p^2+m^2)} = \frac{1}{p^2} - \frac{1}{p^2+m^2}$, we can write:

$$\frac{1}{\kappa^2}\mathcal{G}_{\alpha\beta\mu\nu\rho\sigma}(p) = -i\left[\frac{\mathcal{P}^{(2)}_{\mu\nu\rho\sigma}}{p^2} - \frac{1}{2}\frac{\mathcal{P}^{(0,s)}_{\mu\nu\rho\sigma}}{p^2}\right] + \frac{(-i)}{2}\frac{\mathcal{P}^{(0,s)}_{\mu\nu\rho\sigma}}{p^2+m_0^2} - \frac{(-i)\,\mathcal{P}^{(2)}_{\mu\nu\rho\sigma}}{p^2+m_2^2}$$

$$= \mathcal{G}_{EH\,\mu\nu\rho\sigma}(p) + \frac{(-i)}{2}\frac{\mathcal{P}^{(0,s)}_{\mu\nu\rho\sigma}}{p^2+m_0^2} - \frac{(-i)\,\mathcal{P}^{(2)}_{\mu\nu\rho\sigma}}{p^2+m_2^2}$$

- In addition to the ± 2 massless graviton we have $\begin{cases} \text{a massive spin-0} \\ \text{a massive spin-2} \end{cases}$

- While the massive spin-0 has the "ordinary sign", the massive spin-2 has the "wrong (minus) sign" and for this reason it is called ghost

- The total # of DOFs is: $2 + 1 + 5 = 8$

- We must require $c_0 > 0$, $c_2 > 0$ to avoid tachyons

- if $c_0 > 1$, $c_2 > 1$ $\Rightarrow$ $m_0 = \frac{M_P}{\sqrt{c_0}} < M_P$, $m_2 = \frac{M_P}{\sqrt{c_2}} < M_P$.

- The GR propagator is recovered in the limits:

$$c_0 \to 0,\ c_2 \to 0 \iff \left(\frac{m_0}{M_P}\right)^2 \to \infty,\ \left(\frac{m_2}{M_P}\right)^2 \to \infty \implies G_{\text{ab μνρσ}}(p) \to G_{\text{EH μνρσ}}(p).$$

Perturbative (strict) renormalizability (original paper: K. Stelle, PRD 1977)

- **Power counting:** Recall the formula for the superficial degree of divergence:

$$\delta(\mathcal{G}) = 4 - EF - \sum_i V_i \Delta_i \quad ; \quad [F] = 0, \ \Delta_i = 0$$

$$\Rightarrow \boxed{\delta(\mathcal{G}) = 4}$$

UV divergences do NOT get worse
⇒ strict renormalizability!

- in Quadratic Gravity the UV behavior is controlled by a dimensionless field $h_{\mu\nu}$:

  kinetic term $\overset{UV}{\sim}$ $c_0\, h\, \partial^4 h$, $c_2\, h\, \partial^4 h$

- the interaction terms in the UV are governed by dimensionless couplings c_0, c_2

- **Explicit loop integral power counting:**

$$\int \underbrace{d^4k \cdots d^4k}_{L\text{- loops}} \ \underbrace{\frac{1}{k^4} \cdots \frac{1}{k^4}}_{I\text{- propagators}} \ \underbrace{k^4 \cdots k^4}_{V\text{- vertices}} \sim k^{4(L-I+V)} = k^4$$

- **Tensor structure of UV divergences / counterterms**

Question: Do the counterterms have the same tensor structure as those contained in the starting bare Lagrangian? YES!

In $D=4$, operators of mass dimension higher than four need to be multiplied by couplings of negative mass dimension.

For example: $\int d^4x \; a \; R^3$, $[a] = -2 < 0$.

Can loop contributions from Quadratic Gravity generate such a UV divergence? NO!
Indeed, schematically the UV behavior of the propagator including Λ is:

$$\frac{1}{\Lambda M_P^2 + M_P^2 p^2 + c_0 p^4 + c_2 p^4} \overset{UV}{\sim} \frac{1}{c_0 p^4 + c_2 p^4}$$

Moreover, the vertices are schematically proportional to: ΛM_P^2, $M_P^2 p^2$, $c_0 p^4$, $c_2 p^4$.
Therefore, any UV-divergence CANNOT be proportional to couplings with negative mass dimension.

The "only" possible counterterms are:

$$\{\ \lambda\mathbb{1}\ ,\ R\ ,\ R^2\ ,\ C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}\ \}$$

Note that the renormalization procedure also generates boundary terms:

$$\sqrt{-g}\,GB\ ,\quad \Box R$$

However, these do <u>not</u> feed back into the renormalization of the other terms, thus they can be safely neglected for our purposes.

- <u>Matter couplings</u>: Quadratic Gravity coupled to the Standard Model of particle physics gives rise to a strictly renormalizable QFT of gravity + matter iff the following non-minimal coupling is added to the Lagrangian:

$$\int d^4x\sqrt{-g}\ \xi\,|H|^2R\ ,\quad [\xi]=0\ .$$

- <u>Finite # of parameters</u>: $\{$SM parameters $+\ \Lambda, G_N, c_0, c_2, \xi\}$

Quadratic Gravity versus EFT of GR (see also L.B., arXiv: 2504.05900)

EFT of GR up to order $\mathcal{O}(R^2)$:

$$S_{EFT} \simeq \int d^4x \sqrt{-g}\left[\frac{1}{2\kappa^2}R + a_1 R^2 + a_2 R_{\mu\nu}R^{\mu\nu}\right]$$
$$\simeq \int d^4x \sqrt{-g}\left[\frac{1}{2\kappa^2}R + \left(a_1 + \frac{a_2}{3}\right)R^2 + \frac{a_2}{2}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}\right]$$

Quadratic Gravity:

$$S_{QG} = \frac{1}{2}\int d^4x \sqrt{-g}\left[\frac{1}{2\kappa^2}R + \frac{c_0}{6}R^2 - \frac{c_2}{2}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}\right]$$

Why do they describe different QFTs ?

- Different perturbative UV expansions: S_{EFT} governed by $\left(\frac{1}{M_P^2}\right)^{L-1}$, a_i, $\frac{1}{p^2}$; S_{QG} by $\left(\frac{1}{c_i}\right)^{L-1}$, $\frac{1}{p^4}$
- Perturbative renormalizability: NO ; YES !
- Different meanings of the quadratic-curvature coefficients: $\begin{cases} a_i \lesssim 1 \ ; \\ c_i \text{ are free renormalized parameters.} \end{cases}$
- $[h_{\mu\nu}] = 1$ in the EFT of GR ; $[h_{\mu\nu}] = 0$ in the UV in Quadratic Gravity
- Different # of DOFs: $\begin{cases} S_{EFT} \to \pm 2 \text{ massless graviton} \\ S_{QG} \to \pm 2 \text{ massless graviton, massive spin-0, massive spin-2} \end{cases}$

The spin-2 massive ghost and the unitarity puzzle

Quadratic-gravity propagator: $\frac{i}{k^2}\mathcal{G}_{\text{QG}\,\mu\nu\rho\sigma}(p) = \frac{m_2^2\,\mathcal{P}^{(2)}_{\mu\nu\rho\sigma}}{p^2(p^2+m_2^2)} - \frac{1}{2}\frac{m_0^2\,\mathcal{P}^{(0,s)}_{\mu\nu\rho\sigma}}{p^2(p^2+m_0^2)}$

$$= i\,\mathcal{G}_{\text{EH}\,\mu\nu\rho\sigma} + \frac{1}{2}\frac{\mathcal{P}^{(0,s)}_{\mu\nu\rho\sigma}}{p^2+m_0^2} - \frac{\mathcal{P}^{(2)}_{\mu\nu\rho\sigma}}{p^2+m_2^2}$$

Because of the "minus" sign in front of $\frac{\mathcal{P}^{(2)}_{\mu\nu\rho\sigma}}{p^2+m^2}$, the massive spin-2 is a ghost!

[More generally, in QFT a ghost is a field whose kinetic term has the opposite sign to that of ordinary fields. Consequently, the propagator will also have an opposite sign.]

(do NOT get confused with Faddeev-Popov ghosts, those are different things, although the same name "ghost" is used)

- Ghost-like "minus" signs should worry us !

  They "may" lead to classical instabilities and violation of unitarity.

- Recall the positivity constraint obtained from the optical theorem:

$$2\,\mathrm{Im}\,\langle a|\mathcal{M}|a\rangle = \sum_{\{n\}}\prod_{\ell=1}^{n}\int\frac{d^3k_\ell}{(2\pi)^3}\frac{(2\pi)^4}{2\omega_\ell}\,\delta^{(4)}\Big(P_a-\sum_{\ell=1}^{n}k_\ell\Big)\,\big|\langle\{k_\ell\}|\mathcal{M}|a\rangle\big|^2 \geq 0$$

- The simplest tree-level amplitude is the propagator itself:

  $(-i)$ ●——● $= (-i)\,(-i)^2\,\varepsilon^{*\mu\nu}\,\mathcal{G}_{ab\,\mu\nu\rho\sigma}\,\varepsilon^{\rho\sigma} = \varepsilon^{*\mu\nu}\,i\,\mathcal{G}_{ab\,\mu\nu\rho\sigma}\,\varepsilon^{\rho\sigma}$; $\varepsilon^{\mu\nu} \equiv$ polarization tensor

- Positivity constraint: $\mathrm{Im}\big[\varepsilon^{*\mu\nu}\,i\,\mathcal{G}_{ab\,\mu\nu\rho\sigma}(p)\,\varepsilon^{\rho\sigma}\big] \overset{?}{\geq} 0$? ? ?

Scalar 4-derivative toy model

To understand the implications of the ghost-like "minus", we now consider a scalar toy model that mimic the pole structure of the spin-2 sector of QG

- Lagrangian: $\mathcal{L} = \frac{1}{2}\phi\,\Box\left(1 - \frac{\Box}{m^2}\right)\phi - V(\phi)$
- Propagator: $G(p) = (-i)\,\frac{m^2}{p^2(p^2+m^2)} = (-i)\left[\frac{1}{p^2} - \frac{1}{p^2+m^2}\right]$
- Feynman prescription: $G(p) = (-i)\left[\frac{1}{p^2 - i\epsilon} - \frac{1}{p^2+m^2-i\epsilon}\right]$, $\epsilon \to 0^+$.
- Imaginary part of the amplitude: $\mathrm{Im}\left[i\,G(p)\right] = \pi\left[\delta(p^2) - \delta(p^2+m^2)\right]$

$\Rightarrow$ it can be negative $\Rightarrow$ the optical theorem, and so unitarity, can be violated!

- Indeed, given the assumptions made so far and the results obtained unitarity is violated:

$$2\,\mathrm{Im}\,T \neq T^+T \quad \Longleftrightarrow \quad S^+S \neq \mathbb{1}$$

- However, we can still introduce a <u>pseudo-unitarity</u> equation:

$$i\left[\langle b|T^+|a\rangle - \langle b|T|a\rangle\right] = \sum_m \sigma_m \langle b|T^+|m\rangle\langle m|T|a\rangle$$

where $\sigma_m = \begin{cases} +1 & \text{for ordinary states} \\ -1 & \text{whenever the LHS is negative for an elastic process.} \end{cases}$

<u>NB:</u> without additional assumptions, pseudo-unitarity ~~$\Longleftrightarrow$~~ unitarity !!!

- <u>Diatribe:</u> we achieved perturbative renormalizability <u>BUT</u>, apparently, we lost unitarity!

How to solve this puzzle?