Higher-Derivative (Quantum) Gravity

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- **Lecture 1:**
  - 1.1: Introduction, Elements of perturbative QFT & classical GR.
  - 1.2: GR as a QFT - Free theory
- **Lecture 2:**
  - 2.1: GR as a QFT - Interacting theory
  - 2.2: Quadratic Gravity - Part I
- **Lecture 3:**
  - 3.1: Classes of local and non-local HD theories of gravity
  - 3.2: Quadratic Gravity - Part II

References:

- Section 2. of "Lectures in QG", arXiv: 2412.08690
- Percacci, "An Introduction to Covariant QG and Asymptotic Safety", (2017)
- Buchbinder and Shapiro, "Introduction to QFT and applications to QG", (2023).
- Research papers

Conventions & Notations

Spacetime dimensions : $D = 4 = 1 + 3$

Spacetime metric signature : $(\eta_{\mu\nu}) = \text{diag}\,(-1, +1, +1, +1)$

Units : $\begin{cases} \hbar = 1 = c \\ M_P \equiv \dfrac{1}{\sqrt{8\pi G_N}} \end{cases} \Rightarrow$

$$\begin{cases} \left(-\frac{1}{c^2}\frac{\partial}{\partial t^2} + \nabla^2 + \frac{m^2c^2}{\hbar^2}\right)\phi(x) = \left(-\frac{\partial^2}{\partial t^2} + \nabla^2 + m^2\right)\phi(x) = (\Box - m^2)\phi(x) \\ R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G_N}{c^4}T_{\mu\nu} = \frac{1}{M_P^2}T_{\mu\nu} \\ [\phi] = 1\,, \quad [g_{\mu\nu}] = 0\,, \quad [R_{\mu\nu}] = 2\,, \ldots \end{cases}$$

Fourier transform : $f(x) = \int \frac{d^4p}{(2\pi)^4}\,\tilde{f}(p)\,e^{ip\cdot x} \Rightarrow \partial_\mu \to ip \Rightarrow \Box = \partial_\mu\partial^\mu \to -p^2$

Curvature tensors:

- Christoffel symbol: $\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} (\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}) = \Gamma^{\rho}_{\nu\mu}$
  (symmetric $\Rightarrow$ zero torsion)
- Covariant derivatives: $\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^{\nu}_{\mu\rho} V^\rho$, $\nabla_\mu V_\nu = \partial_\mu V_\nu - \Gamma^{\rho}_{\mu\nu} V_\rho$
- Metric Compatibility: $\nabla_\rho g_{\mu\nu} = 0$
- Commutation relations: $[\nabla_\nu, \nabla_\rho] = V^\sigma R^{\sigma}_{\ \mu\nu\rho}$, $[\nabla_\nu, \nabla_\rho] V_\mu = - V_\sigma R^{\sigma}_{\ \mu\nu\rho}$
- Riemann tensor: $R^{\sigma}_{\ \mu\nu\rho} = \partial_\nu \Gamma^{\sigma}_{\mu\rho} - \partial_\rho \Gamma^{\sigma}_{\mu\nu} + \Gamma^{\sigma}_{\alpha\nu} \Gamma^{\alpha}_{\mu\rho} - \Gamma^{\sigma}_{\alpha\rho} \Gamma^{\alpha}_{\mu\nu}$
- Ricci tensor and Ricci scalar: $R_{\nu\sigma} = R^{\rho}_{\ \nu\rho\sigma} = \delta_\mu^{\ \rho} R^{\mu}_{\ \nu\rho\sigma} = g^{\mu\rho} R_{\mu\nu\rho\sigma}$, $R = R^{\mu}_{\ \mu} = g^{\mu\nu} R_{\mu\nu}$

LECTURE 1.1

Elements of perturbative QFT (Section 2.A - arXiv: 2412.08690)

- Perturbative QFT framework : $\mathcal{L} = K - V$, " $|V| << |K|$ "

- Useful guiding "principles":

1) Locality of bare Lagrangians : $\mathcal{L} \equiv \mathcal{L}\left(\phi, \partial\phi, \partial^2\phi, \dots, \partial^{(m)}\phi\right)$, $m < \infty$.

2) Symmetries: global and local internal symmetries & external spacetime symmetries

3) Unitarity : conservation of the inner products under time-evolution $\left(U^\dagger(t)U(t) = \mathbb{1}\right)$

4) (Strict) renormalizability : UV power counting governed by couplings g_i with $[g_i] \geq 0$

Question :

Is there any physically consistent QFT of gravity compatible with 1)+2)+3)+4) ?

Unitarity :

(initial state, $t = -\infty$) $|a\rangle \longrightarrow S \longrightarrow S|a\rangle = |b\rangle$ (final state, $t = +\infty$)

$(-\infty < t < +\infty)$

Norm conservation : $\langle a|a\rangle = \langle b|b\rangle = \langle a|S^\dagger S|a\rangle \Rightarrow S^\dagger S = \mathbb{1}$

(unitary time-evolution)

$S = \mathbb{1} + iT \Rightarrow \mathbb{1} = (\mathbb{1} - iT^\dagger)(\mathbb{1} + iT) = \mathbb{1} + i(T - T^\dagger) + T^\dagger T$

$\Leftrightarrow -i(T - T^\dagger) = T^\dagger T$ optical theorem (operator form)

Insert the completeness relation : $\mathbb{1} = \sum_m \sigma_m |m\rangle\langle m|$, $\langle m|m\rangle = \delta_{mm}\sigma_m$

$\Rightarrow -i\left[\langle b|T|a\rangle - \langle b|T^\dagger|a\rangle\right] = \sum_m \sigma_m \langle b|T^\dagger|m\rangle\langle m|T|a\rangle$

- More explicit form for computations:

$\mathcal{M} \equiv$ Feynman amplitude operator , $\quad \langle b|T|a\rangle = (2\pi)^4 \delta^{(4)}(P_b - P_a)\langle b|\mathcal{M}|a\rangle$

Explicit completeness relation: $\mathbb{1} = \sum_{\{m\}} \sigma_m \prod_{l=1}^{m} \int \frac{d^3k_l}{(2\pi)^3} \frac{1}{2\omega_l} |\{k_l\}\rangle\langle\{k_l\}|$; $\omega_l \equiv \sqrt{\vec{k}_l^2 + m_l^2}$.

$\Rightarrow$ Optical theorem:

$$i\left[\langle b|\mathcal{M}^\dagger|a\rangle - \langle b|\mathcal{M}|a\rangle\right] = \sum_{\{m\}} \sigma_m \prod_{l=1}^{m} \int \frac{d^3k_l}{(2\pi)^3} \frac{(2\pi)^4}{2\omega_l} \delta^{(4)}\left(P_a - \sum_{l=1}^{m} k_l\right) \times \langle b|\mathcal{M}^\dagger|\{k_l\}\rangle\langle\{k_l\}|\mathcal{M}|a\rangle$$

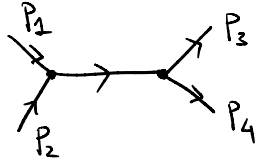
- Elastic process ($|b\rangle = |a\rangle$) & $\sigma_m > 0$ ($\sigma_m = 1$) $\Rightarrow$ Positivity constraint

$$2\,\mathrm{Im}\,\langle a|\mathcal{M}|a\rangle = \sum_{\{m\}} \prod_{l=1}^{m} \int \frac{d^3k_l}{(2\pi)^3} \frac{(2\pi)^4}{2\omega_l} \delta^{(4)}\left(P_a - \sum_{l=1}^{m} k_l\right) \left|\langle\{k_l\}|\mathcal{M}|a\rangle\right|^2 \geq 0$$

- Example: $\mathcal{L}(\phi) = \frac{1}{2}\phi(\Box - m^2)\phi - \frac{g_3}{3!}\phi^3$

Feynman rules: $G(-p^2) = \frac{-i}{p^2 + m^2 - i\epsilon}$, $\epsilon \to 0^+$; $V = -i\lambda$

Elastic tree-level $2 \to 2$ scattering:



$p_1 + p_2 = p_3 + p_4 \equiv p$; $|a\rangle = |p_1 p_2\rangle$, $|b\rangle = |p_3 p_4\rangle$

If $|b\rangle = |a\rangle \Rightarrow \langle a|\mathcal{M}|a\rangle = \langle p_1 p_2|\mathcal{M}|p_1 p_2\rangle$

$= (-i)$  $= (-i)(-ig_3)^2 G(-p^2) = \frac{g_3^2}{p^2 + m^2 - i\epsilon}$

Using $\frac{1}{x \pm i\epsilon} = \text{P.V.}\left(\frac{1}{x}\right) \mp i\pi\delta(x)$:

$$\text{Im}\left[\langle a|\mathcal{M}|a\rangle\right] = \text{Im}\left[\frac{g_3^2}{p^2 + m^2 - i\epsilon}\right] = g_3^2\,\text{Im}\left[\text{P.V.}\left(\frac{1}{p^2 + m^2}\right) + i\pi\delta(p^2 + m^2)\right] = g_3^2\pi\delta(p^2 + m^2) \geq 0$$

Perturbative renormalizability

- How to deal with the UV divergences in perturbative QFT ?
  - Assume that all quantities in the starting Lagrangian are bare and divergent
  - Then, apply renormalization theory which consists of two main steps :
    (i) implement a regularization method to isolate the divergent part of a diagram/integral
    (ii) absorb the divergences into a redefinition of the quantities in the Lagrangian (i.e. masses, interaction couplings, fields).

[Theorem : if the starting bare $\mathcal{L}$ is local $\Rightarrow$ UV divergences are local $\Rightarrow$
$\Rightarrow$ counterterms are local]

Renormalizable QFTs are defined such that only a finite number of terms in $\mathcal{L}$ is needed to renormalize all physical quantities at any loop order in perturbation theory

Superficial degree of divergence

$\mathcal{G} \equiv$ diagram ; L-loop integral : $I^{(L)}(\mathcal{G}) = \int d^4k_1 \cdots d^4k_L \; f(\{k_i\})$

$$\lim_{\lambda \to \infty} \lambda^{4L} f(\{\lambda k_i\}) \sim \lambda^{\delta(\mathcal{G})}$$

$\delta(\mathcal{G}) \equiv$ superficial degree of divergence ;

$\begin{cases} \delta(\mathcal{G}) < 0 : \; I^{(L)} \text{ is superf. convergent} \\ \delta(\mathcal{G}) = 0 : \; \log \text{ divergence} \\ \delta(\mathcal{G}) > 0 : \; \text{power-law divergence} \end{cases}$

- **Examples :**

- Tadpole diagram : $I^{(1)}\left(\bigcirc\!\!-\right) = \int d^4k \, \frac{1}{k^2+m^2}$; $\lim_{\lambda\to\infty} \frac{\lambda^4}{\lambda^2 k^2 + m^2} \sim \lambda^2 \Rightarrow \delta(\mathcal{G}) = 2$

- Bubble diagram : $I^{(1)}\left(-\bigcirc-\right) = \int d^4k \, \frac{1}{k^2+m^2} \frac{1}{(k-p)^2+m^2}$; $\lim_{\lambda\to\infty} \frac{\lambda^4}{(\lambda^2 k^2)^2} \sim \lambda^0 \Rightarrow \delta(\mathcal{G}) = 0$

Useful formula for the maximum value of $\delta(\mathcal{G})$

$E_f \equiv$ # external legs of type f

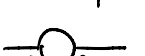
$V_i \equiv$ # vertices of type i

$F_f \equiv [\phi_f] \equiv$ mass dimension of f-th ϕ_f

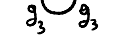
$\Delta_i \equiv [g_i] \equiv$ mass dimension of i-th coupling g_i relevant in the UV

$$\Rightarrow \quad \boxed{\delta(\mathcal{G}) = 4 - \sum_f E_f F_f - \sum_i V_i \Delta_i}$$

Examples: $\mathcal{L} = \frac{1}{2}\phi(\Box - m^2)\phi - \frac{g_3}{3!}\phi^3 - \frac{g_4}{4!}\phi^4$



: $\delta(\mathcal{G}) = 4 - EF - V\Delta$; $\{E=2,\ F=[\phi]=1,\ \Delta=[g_4]=0\} \Rightarrow \delta(\mathcal{G}) = 4-2 = 2$



: $\delta(\mathcal{G}) = 4 - EF - V\Delta$; $\{E=2,\ F=1,\ \Delta=[g_3]=1,\ V=2\} \Rightarrow \delta(\mathcal{G}) = 4-2-2 = 0$

$$\delta(\mathcal{G}) = 4 - \sum_f E_f F_f - \sum_i V_i \Delta_i$$

Classification:

- (Perturbatively) non-renormalizable QFT:

  If $\exists\, i$ such that $\Delta_i < 0 \Rightarrow$ $\delta(\mathcal{G})$ increases with $V_i \Rightarrow$

  $\Rightarrow$ UV divergences become worse and worse, and an infinite # of counterterms is needed

- (Perturbatively) renormalizable QFT:

  If $\Delta_i \geq 0 \;\; \forall\, i \Rightarrow$ $\delta(\mathcal{G})$ decreases with V_i or remains constant $\Rightarrow$

  $\Rightarrow$ UV divergences do NOT become worse and worse, and a finite # of counterterms is needed.

Sub-classification of renormalizable QFT ($\Delta_i \geq 0 \ \forall i$)

- Super-renormalizable QFT:

  if $\Delta_i > 0 \ \forall i \Rightarrow$ loop integrals become superficially finite for sufficiently large values of V_i

- Strictly renormalizable QFT:

  if $\Delta_i \geq 0 \ \forall i$ and $\exists\, i$ such that $\Delta_i = 0 \Rightarrow$

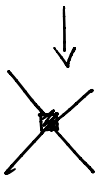
  $\Rightarrow \delta(G)$ does not decrease with V_i for any i, and divergences need to be renormalized at every loop order.

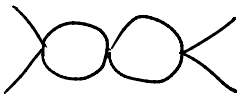
Exemples : $\mathcal{L} = \frac{1}{2}\phi(\Box - m^2)\phi - V(\phi)$; $D = 4$; $F = [\phi] = 1$

- $V(\phi) = \frac{g_3}{3!}\phi^3 + \frac{g_4}{4!}\phi^4$; $[g_3] = 1$, $[g_4] = 0 \Rightarrow$ strictly renormalizable.
- $V(\phi) = \frac{g_3}{3!}\phi^3$; $[g_3] = 1 \Rightarrow \delta(G) = 4 - E - V \Rightarrow$ super-renormalizable
- $V(\phi) = \frac{g_4}{4!}\phi^4$; $[g_4] = 0 \Rightarrow \delta(G) = 4 - E \Rightarrow$ strictly renormalizable
- $V(\phi) = \frac{g_n}{n!}\phi^n$, $n \geq 5$; $\Delta_n = [g_n] = 4 - n < 0 \Rightarrow$ non-renormalizable

Example: $V(\phi) = \frac{g_4}{4!}\phi^4$


$= \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2+m^2} \frac{1}{(k-p)^2+m^2} \sim \int d^4k \frac{1}{k^4} \sim k^0 \rightarrow$ log divergence

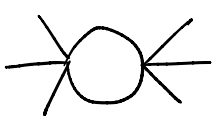

$=$ 1-loop counterterm $\sim \mathcal{O}(g_4^2)$


$\sim \int dk^4\, d^4k \frac{1}{k^8} \sim k^0$

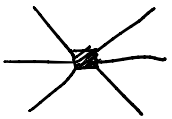
If $E = 4 \Rightarrow \delta(G) = 4 - 4 = 0$

Only the vertex $g_4 \phi^4$ is needed to renormalize the theory.

Example : $V(\phi) = \frac{g_5}{5!}\phi^5$



$\sim \int d^4k \, \frac{1}{k^4} \sim k^0$



$\equiv$ 1-loop counterterm is a 6 legs contact vertex $\sim \Theta(g_5^2)$

we need $g_6\phi^6$ but it is not present in the starting $\mathcal{L}$

REMARK : All interactions in the Standard Model of particle physics are strictly renormalizable !

$e\bar{\Psi}\gamma^\mu A_\mu \Psi$ | $[e] = 0$

$\lambda |H|^4$ | $[\lambda] = 0$

$y\bar{\Psi} H \Psi$ | $[y] = 0$

$g(\partial A)AA$, $gAAAA$ | $[g] = 0$

Elements of classical GR (Section 2.1 - arXiv: 2412.08690)

- **Einstein - Hilbert action** : $S_{EH}[g] = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\,(R-2\Lambda)$; $\kappa \equiv \sqrt{8\pi G_N} = \frac{1}{M_P}$

- **Einstein's equations**: $G_{\mu\nu} + \Lambda\, g_{\mu\nu} = \kappa^2 T_{\mu\nu}$; $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$; $T_{\mu\nu} \equiv \frac{-2}{\sqrt{-g}} \frac{\delta S_m}{\delta g^{\mu\nu}}$

- **Bianchi identity & conservation law**: $\nabla_\mu G^{\mu\nu} = 0$; $\nabla_\mu T^{\mu\nu} = 0$.

- **Diffeomorphism invariance**: $S_{EH} + S_m$ is invariant under $x^\mu \to x'^\mu(x) = x^\mu + \xi^\mu(x)$

$$\Rightarrow \quad \delta_\xi g_{\mu\nu}(x) = -\nabla_\mu \xi_\nu(x) - \nabla_\nu \xi_\mu(x)$$

$$\Rightarrow \quad 0 = \delta_\xi S_{EH}[g] = \frac{1}{2\kappa^2}\int d^4x\, \frac{\delta\left(\sqrt{-g}\,(R-2\Lambda)\right)}{\delta g_{\mu\nu}}\,\delta_\xi g_{\mu\nu} = -\frac{1}{\kappa^2}\int d^4x\sqrt{-g}\,\left(\nabla_\mu G^{\mu\nu}\right)\xi_\nu$$

- **Metric fluctuations & action expansion**

$$g_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x) + 2\kappa\, h_{\mu\nu}(x) \quad ; \quad \kappa\,|h_{\mu\nu}| \ll 1$$

($\bar{g}_{\mu\nu}$: background; $h_{\mu\nu}$: metric field fluctuation; "in some coordinate system")

$$\Rightarrow S_{EH}[g] = S_{EH}[\bar{g} + 2\kappa h] = S^{(0)}_{EH}[\bar{g}] + S^{(1)}_{EH}[\bar{g}, h] + S^{(2)}_{EH}[\bar{g}, h] + \cdots + S^{(m)}_{EH}[\bar{g}, h] + \cdots$$

$$S^{(0)}_{EH}[\bar{g}] = \frac{1}{2\kappa^2}\int d^4x\,\sqrt{-\bar{g}}\,\left(\bar{R} - 2\Lambda\right), \qquad S^{(1)}_{EH}[\bar{g}, h] = 2\kappa \int d^4y\, \frac{\delta S_{EH}}{\delta g_{\mu\nu}(y)}\bigg|_{g=\bar{g}} h_{\mu\nu}(y),$$

$$S^{(2)}_{EH}[\bar{g}, h] = \frac{1}{2}(2\kappa)^2 \int d^4y_1\, d^4y_2\, \frac{\delta S_{EH}}{\delta g_{\mu_1\nu_1}(y_1)\,\delta g_{\mu_2\nu_2}(y_2)}\bigg|_{g=\bar{g}} h_{\mu_1\nu_1}(y_1)\, h_{\mu_2\nu_2}(y_2),$$

$$\vdots$$

$$S^{(m)}_{EH}[\bar{g}, h] = \frac{1}{m!}(2\kappa)^m \int d^4y_1 \cdots d^4y_m\, \frac{\delta S_{EH}}{\delta g_{\mu_1\nu_1}(y_1)\cdots\delta g_{\mu_m\nu_m}(y_m)}\bigg|_{g=\bar{g}} h_{\mu_1\nu_1}(y_1)\cdots h_{\mu_m\nu_m}(y_m),$$

$$\vdots$$

$$S_{EH}^{(1)}[\bar{g},h] = \frac{1}{\kappa}\int d^4y \left[\frac{\delta}{\delta g_{\mu\nu}(y)}\int d^4x\,\sqrt{-g(x)}\,\left(R(x)-2\Lambda\right)\right]\Big|_{g=\bar{g}} h_{\mu\nu}(y)$$

$$= -\frac{1}{\kappa}\int d^4y\,\sqrt{-g(y)}\,\left[\bar{R}^{\mu\nu}(y) - \frac{1}{2}\bar{g}^{\mu\nu}(y) + \bar{g}^{\mu\nu}(y)\,\Lambda\right] h_{\mu\nu}(y) = 0$$

Kinetic term:

$$S^{(2)}[\bar{g},h] = \frac{1}{2!}\,\delta^{(2)} S_{EH}[\bar{g},h]$$

$$= \frac{1}{4\kappa^2}\int d^4x\,\left\{\delta\left[\delta(\sqrt{-g})R + \sqrt{-g}\,\delta R\right]\right\}\Big|_{g=\bar{g}}$$

$$= \frac{1}{4\kappa^2}\int d^4x\,\left\{\delta^2(\sqrt{-g})\,R + 2\,\delta(\sqrt{-g})\,\delta R + \sqrt{-g}\,\delta^{(2)}R\right]\Big|_{g=\bar{g}}$$

$$= \int d^4x\,\sqrt{-\bar{g}}\,\left[-\frac{1}{2}\bar{\nabla}_\rho h_{\mu\nu}\bar{\nabla}^\rho h^{\mu\nu} + \bar{\nabla}_\rho h^\rho{}_\mu \bar{\nabla}_\sigma h^{\sigma\mu} - \bar{\nabla}_\mu h\,\bar{\nabla}_\nu h^{\mu\nu} + \frac{1}{2}\bar{\nabla}_\rho h\,\bar{\nabla}^\rho h\right.$$

$$\left. - \frac{1}{2}\left(\bar{R}-2\Lambda\right)\left(h_{\mu\nu}h^{\mu\nu} - \frac{1}{2}h^2\right) + \left(h^{\mu\rho}h_\rho{}^\nu - h\,h^{\mu\nu}\right)\bar{R}_{\mu\nu} + h^{\mu\rho}h^{\nu\sigma}\bar{R}_{\mu\nu\rho\sigma}\right]$$

Interactions terms :

$$S_{EH}^{(n)}[\bar{g},h] \sim \mathcal{O}\left(\kappa^{n-2} h^n\right) \quad , \quad n \geq 3 \quad , \quad \kappa^{n-2} = \left(\frac{1}{M_P}\right)^{n-2}$$

AIM : Quantize $h_{\mu\nu}(x)$ in the framework of perturbative QFT !

[classical $\longrightarrow$ quantum]

$$g_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x) + 2\kappa h_{\mu\nu}(x) \longrightarrow \hat{g}_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x)\,\mathbb{1} + 2\kappa\,\hat{h}_{\mu\nu}(x)$$

Working assumptions : $\Lambda \simeq 0$; $\bar{g}_{\mu\nu} = \eta_{\mu\nu} \Rightarrow \sqrt{-\bar{g}} = 1$, $\bar{\nabla}_\mu \longrightarrow \partial_\mu$,

Minkowski background $\quad \bar{R}_{\mu\nu\rho\sigma}(\eta) = 0$, $h = \eta^{\mu\nu} h_{\mu\nu}$.

LECTURE 1.2 : GR as a QFT - Free theory (Section 2.2 - arXiv: 2412.08690)

Quadratic-order action:

$$S^{(2)}_{EH}[\eta, h] = \int d^4x \left[-\frac{1}{2}\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu} + \partial_\rho h^\rho{}_\mu \partial_\sigma h^{\sigma\mu} - \partial_\mu h \partial_\nu h^{\mu\nu} + \frac{1}{2}\partial_\rho h \partial^\rho h \right]$$

$$= \int d^4x \; \frac{1}{2} h_{\mu\nu} \mathbb{K}^{\mu\nu\rho\sigma} h_{\rho\sigma}$$

$$\mathbb{K}^{\mu\nu\rho\sigma} \equiv \frac{1}{2}\left(\eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}\right)\Box - \eta^{\mu\nu}\eta^{\rho\sigma}\Box + \eta^{\mu\nu}\partial^\rho\partial^\sigma + \eta^{\rho\sigma}\partial^\mu\partial^\nu$$
$$-\frac{1}{2}\left(\eta^{\mu\rho}\partial^\nu\partial^\sigma + \eta^{\mu\sigma}\partial^\nu\partial^\rho + \eta^{\nu\rho}\partial^\mu\partial^\sigma + \eta^{\nu\sigma}\partial^\mu\partial^\rho\right) ;$$

$$\mathbb{K}^{\mu\nu\rho\sigma} = \mathbb{K}^{\nu\mu\rho\sigma} = \mathbb{K}^{\mu\nu\sigma\rho} = \mathbb{K}^{\rho\sigma\mu\nu}$$

Matter couplings :

$$S_m[\eta + 2\kappa h] = S_m[\eta] + 2\kappa \int d^4x \left.\frac{\delta S_m}{\delta g_{\mu\nu}}\right|_{g=\eta} h_{\mu\nu} + \mathcal{O}(\kappa^2 h^2)$$

$$= S_m[\eta] + \kappa \int d^4x \, T^{\mu\nu} h_{\mu\nu} + \mathcal{O}(\kappa^2 h^2)$$

Linearized diffeomorphism :

Recall : $x'^\mu(x) = x^\mu + \xi(x) \Rightarrow \delta g_{\mu\nu} = -\nabla_\mu \xi_\nu - \nabla_\nu \xi_\mu$.

$$\left\{ \begin{array}{l} g_{\mu\nu} = \eta_{\mu\nu} + 2\kappa h_{\mu\nu} \\ \xi(x) \to -2\kappa \xi(x) \end{array} \right\} \Rightarrow \delta_\xi h_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$$

$$= \partial_\mu \xi_\nu + \partial_\nu \xi_\mu + 2\kappa \left(h_{\mu\rho} \partial_\nu \xi^\rho + h_{\nu\rho} \partial_\mu \xi^\rho + \xi^\rho \partial_\rho h_{\mu\nu} \right)$$

At order $\mathcal{O}(h^2)$, $S^{(2)}[\eta, h]$ is invariant under $\delta_\xi h_{\mu\nu} \overset{(2)}{=} \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$

Linearized Einstein's equations:

$$\mathbb{K}_{\mu\nu}{}^{\rho\sigma}\, h_{\rho\sigma} = -\kappa T_{\mu\nu} \quad \Longleftrightarrow \quad \Box h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\Box h + \eta_{\mu\nu}\partial_\rho\left(\partial_\sigma h^{\rho\sigma} - \frac{1}{2}\partial^\rho h\right)$$
$$- \partial_\mu\left(\partial_\rho h^\rho{}_\nu - \frac{1}{2}\partial_\nu h\right) - \partial_\nu\left(\partial_\rho h^\rho{}_\mu - \frac{1}{2}\partial_\mu h\right) = -\kappa T_{\mu\nu}$$

de Donder gauge: $\partial_\rho h^\rho{}_\nu - \frac{1}{2}\partial_\nu h = 0$, $\nu = 0,1,2,3$ (4 constraints)

$$\Longrightarrow \quad \Box h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\Box h = -\kappa T_{\mu\nu}$$

Residual gauge: $\delta_\xi h_{\mu\nu} = h'_{\mu\nu} - h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$ such that $\delta_\xi\left(\partial^\rho h_{\rho\nu} - \frac{1}{2}\partial_\nu h\right) = 0$

$$\Longleftrightarrow \quad \Box \xi_\mu = 0 \quad , \quad \mu = 0,1,2,3.$$

Dynamical DOFs: $10 - 4 - 4 = 2$ (± 2 helicity graviton)

Graviton propagator:

$$G_{\mu\nu}{}^{\alpha\beta}(p)\, K_{\alpha\beta}{}^{\rho\sigma}(p) \stackrel{?}{=} i\, \mathbb{1}_{\mu\nu}{}^{\rho\sigma} \quad ; \quad \mathbb{1}_{\mu\nu}{}^{\rho\sigma} = \frac{1}{2}\left(\delta_\mu{}^\rho \delta_\nu{}^\sigma + \delta_\nu{}^\rho \delta_\mu{}^\sigma\right)$$

$$\Longleftrightarrow \quad G_{\mu\nu\alpha\beta}(p)\, K^{\mu\nu\rho\sigma}(p) \stackrel{?}{=} i\, \mathbb{1}_{\mu\nu\rho\sigma} \quad ; \quad \mathbb{1}_{\mu\nu\rho\sigma} = \frac{1}{2}\left(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\nu\rho}\eta_{\mu\sigma}\right)$$

However, the kinetic operator $K^{\mu\nu\rho\sigma}$ is <u>NOT</u> invertible because the kernel of $K^{\mu\nu\rho\sigma}$ is <u>NOT</u> empty: $\exists\ V_{\rho\sigma} = \partial_\rho \xi_\sigma + \partial_\sigma \xi_\rho \neq 0$, $K^{\mu\nu\rho\sigma} V_{\rho\sigma} = 0$

Gauge fixing:

$$S_{gf}[\eta, h] = -\frac{1}{\alpha}\int d^4x\ \mathcal{F}_\mu \mathcal{F}^\mu \ , \quad \mathcal{F}_\mu \equiv \partial_\nu h^\nu{}_\mu - \frac{1}{2}\partial_\mu h \ , \quad \alpha = \text{const.}$$

$$\Longleftrightarrow \quad S_{gf}[\eta, h] = \int d^4x\ \frac{1}{2}\, h_{\mu\nu}\, K_{gf}^{\mu\nu\rho\sigma}\, h_{\rho\sigma} \ ; \quad K_{gf}^{\mu\nu\rho\sigma} \equiv \frac{1}{2\alpha}\eta^{\mu\nu}\eta^{\rho\sigma}\Box - \frac{1}{\alpha}\left(\eta^{\mu\nu}\partial^\rho\partial^\sigma + \eta^{\rho\sigma}\partial^\mu\partial^\nu\right)$$
$$+ \frac{1}{2\alpha}\left(\eta^{\mu\rho}\partial^\nu\partial^\sigma + \eta^{\mu\sigma}\partial^\nu\partial^\rho + \eta^{\nu\rho}\partial^\mu\partial^\sigma + \eta^{\nu\sigma}\partial^\mu\partial^\rho\right)$$

$$\Rightarrow \quad \tilde{S}^{(2)}[\eta, h] \equiv S^{(2)}_{EH}[\eta, h] + S_{gf}[\eta, h] = \int d^4x \, \frac{1}{2} h_{\mu\nu} \, \tilde{\mathbb{K}}^{\mu\nu\rho\sigma} \, h_{\rho\sigma} \,,$$

$$\begin{aligned}
\tilde{\mathbb{K}}^{\mu\nu\rho\sigma} \equiv \mathbb{K}^{\mu\nu\rho\sigma} + \mathbb{K}^{\mu\nu\rho\sigma}_{gf} = {} & \frac{1}{2}\left(\eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}\right)\Box - \left(1 - \frac{1}{2\alpha}\right)\eta^{\mu\nu}\eta^{\rho\sigma}\Box \\
& + \left(1 - \frac{1}{\alpha}\right)\left(\eta^{\mu\nu}\partial^\rho\partial^\sigma + \eta^{\rho\sigma}\partial^\mu\partial^\nu\right) \\
& - \frac{1}{2}\left(1 - \frac{1}{\alpha}\right)\left(\eta^{\mu\rho}\partial^\nu\partial^\sigma + \eta^{\mu\sigma}\partial^\nu\partial^\rho + \eta^{\nu\rho}\partial^\mu\partial^\sigma + \eta^{\nu\sigma}\partial^\mu\partial^\rho\right)
\end{aligned}$$

$\tilde{\mathbb{K}}^{\mu\nu\rho\sigma}$ is invertible : $\quad \tilde{\mathbb{K}}^{\mu\nu\rho\sigma} V_{\rho\sigma} \neq 0 \quad \forall \quad V_{\rho\sigma} \neq 0 \,.$

Momentum space :

$$\begin{aligned}
\tilde{\mathbb{K}}^{\mu\nu\rho\sigma} = {} & -\frac{1}{2}\left(\eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}\right)p^2 + \left(1 - \frac{1}{2\alpha}\right)\eta^{\mu\nu}\eta^{\rho\sigma}p^2 \\
& - \left(1 - \frac{1}{\alpha}\right)\left(\eta^{\mu\nu}p^\rho p^\sigma + \eta^{\rho\sigma}p^\mu p^\nu\right) \\
& + \frac{1}{2}\left(1 - \frac{1}{\alpha}\right)\left(\eta^{\mu\rho}p^\nu p^\sigma + \eta^{\mu\sigma}p^\nu p^\rho + \eta^{\nu\rho}p^\mu p^\sigma + \eta^{\nu\sigma}p^\mu p^\rho\right)
\end{aligned}$$

Basis of rank-four symmetric tensors:

$\mathcal{G}_{\mu\rho\sigma}(p)$ can be written as a linear combination of the elements of a basis in the space of rank-four symmetric tensors

Using Lorentz invariance, we can easily find a basis given by five independent elements:

$$
\begin{cases}
B^{(1)}_{\mu\rho\sigma}(p) = \eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} \\
B^{(2)}_{\mu\rho\sigma}(p) = \eta_{\mu\nu}\eta_{\rho\sigma} \\
B^{(3)}_{\mu\rho\sigma}(p) = \frac{1}{p^2}\left(\eta_{\mu\nu}p_\rho p_\sigma + \eta_{\rho\sigma}p_\mu p_\nu\right) \\
B^{(4)}_{\mu\rho\sigma}(p) = \frac{1}{p^2}\left(\eta_{\mu\rho}p_\nu p_\sigma + \eta_{\mu\sigma}p_\nu p_\rho + \eta_{\nu\rho}p_\mu p_\sigma + \eta_{\nu\sigma}p_\mu p_\rho\right) \\
B^{(5)}_{\mu\rho\sigma}(p) = \frac{1}{(p^2)^2}\,p_\mu p_\nu p_\rho p_\sigma
\end{cases}
\Rightarrow \boxed{\mathcal{G}_{\mu\rho\sigma}(p) = \sum_{j=1}^{5} c_j(p)\, B^{(j)}_{\mu\rho\sigma}(p)}
$$

(useful relation for any of the gravitational theories we'll consider.)

Propagator in the Feynman gauge ($\alpha = 1$):

$$\alpha = 1 \Rightarrow \tilde{K}^{\mu\nu\rho\sigma} = -\frac{1}{2}\left(\eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}\right)p^2 + \frac{1}{2}\eta^{\mu\nu}\eta^{\rho\sigma}p^2$$

Ansatz: $$G^{(\alpha=1)}_{\mu\nu\rho\sigma}(p) = A(p)\left(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}\right) + B(p)\,\eta_{\mu\nu}\eta_{\rho\sigma}$$

Impose $$G^{(\alpha=1)}_{\mu\nu\alpha\beta}\,\tilde{K}^{(\alpha=1)\alpha\beta}{}_{\rho\sigma} = \frac{i}{2}\left(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}\right)$$ and determine $A(p)$, $B(p)$.

$$A(p) = -B(p) = \frac{-i}{2p^2} \Rightarrow G^{(\alpha=1)}_{\mu\nu\rho\sigma}(p) = \frac{1}{2}\frac{-i}{p^2 - i\epsilon}\left(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \eta_{\mu\nu}\eta_{\rho\sigma}\right)$$

We now use the spin-projector formalism to { derive the propagator for any α ; determine its spin components.

Spin-projector formalism (Sections 2.2.6, 2.B – arXiv: 2412.08690)

Warm-up (photon propagator):

$$S_A = \int d^4x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\alpha} \left(\partial_\mu A^\mu\right)^2 \right] = \int d^4x \, A_\mu \, \tilde{K}^{\mu\nu} \, A_\nu \quad ;$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,, \qquad \tilde{K}^{\mu\nu} \equiv \eta^{\mu\nu} \Box - \left(1 - \frac{1}{\alpha}\right) \partial^\mu \partial^\nu \longrightarrow \tilde{K}^{\mu\nu}(p) = -\eta^{\mu\nu} p^2 + \left(1 - \frac{1}{\alpha}\right) p^\mu p^\nu \,.$$

Under SO(3) the 4-vector A_μ can be decomposed into <u>two irreducible representations</u>:

$$A_\mu \in 0 \oplus 1 \qquad (A_0 \to \text{scalar} \,; \; A_i \to \text{vector})$$

- Projection operators: $\theta_{\mu\nu} \equiv \eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}$, $\omega_{\mu\nu} \equiv \frac{p_\mu p_\nu}{p^2}$
- Idempotency & orthogonality: $\theta_{\mu\rho}\theta^\rho{}_\nu = \theta_{\mu\nu}$, $\omega_{\mu\rho}\omega^\rho{}_\nu = \omega_{\mu\nu}$, $\theta_{\mu\rho}\omega^\rho{}_\nu = 0$.
- Completeness: $\theta_{\mu\nu} + \omega_{\mu\nu} = \eta_{\mu\nu} \iff \theta_\mu{}^\nu + \omega_\mu{}^\nu = \delta_\mu{}^\nu$.

(Since the projectors are idempodent, their trace equals their rank.
This means that their trace is equal to the dimension of the corresponding irreducible representation (i.e. $2j+1$).)

- traces: $\eta^{\mu\nu}\theta_{\mu\nu} = 3 = 2(1)+1$, $\eta^{\mu\nu}\omega_{\mu\nu} = 1 = 2(0)+1$

$\Rightarrow$ $\theta_{\mu\nu} \to$ spin-1 , $\omega_{\mu\nu} \to$ spin-0 $\quad \left(A_\mu = \theta_\mu^{\ \nu} A_\nu + \omega_\mu^{\ \nu} A_\nu \right)$

- $\{\theta, \omega\}$ forms a basis in the space of rank-two symmetric tensors:

$\Rightarrow$ $\tilde{K}^{\mu\nu} = -p^2 \left[\theta^{\mu\nu} + \frac{1}{\alpha}\omega^{\mu\nu} \right]$

$\Rightarrow$ Propagator: $G_{\mu\nu}(p) = A(p)\,\theta_{\mu\nu} + B(p)\,\omega_{\mu\nu}$

Imposing $G_{\mu\rho}(p)\,\tilde{K}^{\rho}_{\ \nu} = i\,\eta_{\mu\nu}$ we determine $A(p) = -\frac{i}{p^2}$, $B(p) = -\alpha\frac{i}{p^2}$ $\Rightarrow$

$$\Rightarrow \quad G_{\mu\rho}(p) = -\frac{i}{p^2}\left(\theta_{\mu\nu} + \alpha\,\omega_{\mu\nu}\right) = -i\left[\frac{\eta_{\mu\nu}}{p^2} + (\alpha-1)\frac{p_\mu p_\nu}{(p^2)^2}\right]$$

Graviton propagator:

Under $SO(3)$ $h_{\mu\nu}$ can be decomposed into: $\begin{cases} \text{two scalars} & \delta_{ij}h^{ij}\,,\ h_{00} \\ \text{vector} & h_{0i} \\ \text{traceless spin-two} & h_{ij} \end{cases}$

$$\Rightarrow \qquad h_{\mu\nu} \in 0 \oplus 0 \oplus 1 \oplus 2 \qquad (1+1+3+5=10)$$

Four spin-projectors:

$$\begin{cases} \mathcal{P}^{(2)}_{\mu\nu\rho\sigma} = \frac{1}{2}\left(\theta_{\mu\rho}\theta_{\nu\sigma} + \theta_{\mu\sigma}\theta_{\nu\rho}\right) - \frac{1}{3}\theta_{\mu\nu}\theta_{\rho\sigma} \\ \mathcal{P}^{(1)}_{\mu\nu\rho\sigma} = \frac{1}{2}\left(\theta_{\mu\rho}\omega_{\nu\sigma} + \theta_{\mu\sigma}\omega_{\nu\rho} + \theta_{\nu\rho}\omega_{\mu\sigma} + \theta_{\nu\sigma}\omega_{\mu\rho}\right) \\ \mathcal{P}^{(0,s)}_{\mu\nu\rho\sigma} = \frac{1}{3}\theta_{\mu\nu}\theta_{\rho\sigma}\,, \qquad \mathcal{P}^{(0,w)}_{\mu\nu\rho\sigma} = \omega_{\mu\nu}\omega_{\rho\sigma} \end{cases}$$

- Idempotency and orthogonality: $\mathcal{P}^{(i,a)}_{\mu\nu\ \alpha\beta}\,\mathcal{P}^{(j,b)}_{\alpha\beta\ \rho\sigma} = \delta^{ij}\delta^{ab}\,\mathcal{P}^{(i,a)}_{\mu\nu\ \rho\sigma}$

- Completeness: $\mathcal{P}^{(2)}_{\mu\nu\rho\sigma} + \mathcal{P}^{(1)}_{\mu\nu\rho\sigma} + \mathcal{P}^{(0,s)}_{\mu\nu\rho\sigma} + \mathcal{P}^{(0,w)}_{\mu\nu\rho\sigma} = \mathbb{1}_{\mu\nu\rho\sigma}$

- Traces:

$$\begin{cases} \mathbb{1}^{\mu\nu\rho\sigma}\,\mathcal{P}^{(2)}_{\mu\nu\rho\sigma} = 5 = 2(2)+1 \longrightarrow \text{spin-2} \\ \mathbb{1}^{\mu\nu\rho\sigma}\,\mathcal{P}^{(1)}_{\mu\nu\rho\sigma} = 3 = 2(1)+1 \longrightarrow \text{spin-1} \\ \mathbb{1}^{\mu\nu\rho\sigma}\,\mathcal{P}^{(0,s)}_{\mu\nu\rho\sigma} = 1 = 2(0)+1 \longrightarrow \text{spin-0} \\ \mathbb{1}^{\mu\nu\rho\sigma}\,\mathcal{P}^{(0,w)}_{\mu\nu\rho\sigma} = 1 = 2(0)+1 \longrightarrow \text{spin-0} \end{cases}$$

$$\Rightarrow \quad h_{\mu\nu} = \mathcal{P}^{(2)}_{\mu\nu}{}^{\rho\sigma}\, h_{\rho\sigma} + \mathcal{P}^{(1)}_{\mu\nu}{}^{\rho\sigma}\, h_{\rho\sigma} + \mathcal{P}^{(0,s)}_{\mu\nu}{}^{\rho\sigma}\, h_{\rho\sigma} + \mathcal{P}^{(0,w)}_{\mu\nu}{}^{\rho\sigma}\, h_{\rho\sigma}$$

- Unlike the photon case, $\{\mathcal{P}^{(2)}, \mathcal{P}^{(1)}, \mathcal{P}^{(0,s)}, \mathcal{P}^{(0,w)}\}$ do <u>NOT</u> form a basis in the space of symmetric rank-four tensors.

Indeed, $B^{(3)}{}_{\mu\nu\rho\sigma}$ cannot be obtained from the 4 spin projectors.

To close the basis we need to add a fifth operator :

$$\mathcal{P}^{(0,X)} = \mathcal{P}^{(0,SW)} + \mathcal{P}^{(0,WS)} \;;\; \mathcal{P}^{(0,SW)} = \frac{1}{\sqrt{3}}\,\theta_{\mu\nu}\,\omega_{\rho\sigma} \;,\; \mathcal{P}^{(0,WS)} = \frac{1}{\sqrt{3}}\,\omega_{\mu\nu}\,\theta_{\rho\sigma} \,.$$

$\Rightarrow$ $\{\mathcal{P}^{(2)}, \mathcal{P}^{(1)}, \mathcal{P}^{(0,S)}, \mathcal{P}^{(0,W)}, \mathcal{P}^{(0,X)}\}$ forms a basis.

- Extended product relations : $\mathcal{P}^{(i,ab)}{}_{\mu\nu}{}^{\alpha\beta}\,\mathcal{P}^{(j,cd)}{}_{\alpha\beta}{}^{\rho\sigma} = \delta^{ij}\,\delta^{bc}\,\mathcal{P}^{(i,ad)}{}_{\mu\nu}{}^{\rho\sigma}$

where $i,j \in \{2,1,0\}$ and $a,b,c,d \in \{s,w\}$; $\begin{cases} \mathcal{P}^{(2,ab)} = \mathcal{P}^{(2,a)} = \mathcal{P}^{(2)} \\ \mathcal{P}^{(i,aa)} = \mathcal{P}^{(i,a)} \end{cases}$, same for $\mathcal{P}^{(1)}$

$\Rightarrow$

$$\tilde{K}^{\mu\nu\rho\sigma}(p) = \sum_{j\in\{2,1,0s,0w,x\}} a^{(j)}(p)\;\mathcal{P}^{(j)\,\mu\nu\rho\sigma}(p)$$

$$\mathcal{G}_{\mu\nu\rho\sigma}(p) = \sum_{j\in\{2,1,0s,0w,x\}} b^{(j)}(p)\;\mathcal{P}^{(j)}_{\mu\nu\rho\sigma}(p)$$

(useful relations for any gravitational theory we'll consider !)

Rewriting $\tilde{K}^{\mu\nu\rho\sigma}(p)$ in terms of the spin-projectors we get:

$$\tilde{K}^{\mu\nu\rho\sigma}(p) = -p^2\Big[\mathcal{P}^{(2)\mu\nu\rho\sigma} + \frac{1}{\alpha}\mathcal{P}^{(1)\mu\nu\rho\sigma} + \Big(\frac{3}{2\alpha} - 1\Big)\mathcal{P}^{(0,s)\mu\nu\rho\sigma} + \frac{1}{2\alpha}\mathcal{P}^{(0,w)\mu\nu\rho\sigma} - \frac{\sqrt{3}}{2\alpha}\mathcal{P}^{(0,x)\mu\nu\rho\sigma}\Big]$$

Then, imposing $G_{\mu\nu\alpha\beta}(p)\,\tilde{K}^{\alpha\beta}{}_{\rho\sigma}(p) = i\,\mathbb{1}_{\mu\nu\rho\sigma}$ we can determine the coefficients $b^{(i)}(p)$:

$$G_{\mu\nu\rho\sigma}(p) = -\frac{i}{p^2}\Big[\mathcal{P}^{(2)}_{\mu\nu\rho\sigma} - \frac{1}{2}\mathcal{P}^{(0,s)}_{\mu\nu\rho\sigma} + \alpha\,\mathcal{P}^{(1)}_{\mu\nu\rho\sigma} + \frac{4\alpha-3}{2}\mathcal{P}^{(0,w)}_{\mu\nu\rho\sigma} - \frac{\sqrt{3}}{2}\mathcal{P}^{(0,x)}_{\mu\nu\rho\sigma}\Big]$$

- Components that survive after contraction with a <u>conserved current</u> $T_{\mu\nu}$ ($p^\mu T_{\mu\nu} = 0$):

$$T^{\mu\nu}\,\mathcal{P}^{(1)}_{\mu\nu\rho\sigma} = 0\,, \qquad T^{\mu\nu}\,\mathcal{P}^{(0,w)}_{\mu\nu\rho\sigma} = 0\,, \qquad T^{\mu\nu}\,\mathcal{P}^{(0,x)}_{\mu\nu\rho\sigma} = 0$$

The only terms that survive are $T^{\mu\nu}\,\mathcal{P}^{(2)}_{\mu\nu\rho\sigma} \neq 0$, $T^{\mu\nu}\,\mathcal{P}^{(0,s)}_{\mu\nu\rho\sigma} \neq 0 \Rightarrow$

$$\mathcal{G}_{\mu\nu\rho\sigma}(p) \to \frac{-i}{p^2}\left[\mathcal{P}^{(2)}_{\mu\nu\rho\sigma} - \frac{1}{2}\,\mathcal{P}^{(0,s)}_{\mu\nu\rho\sigma}\right];$$

- off-shell we have $5+1=6$ DOFs: $(-2\,-1\;\;0\;\;+1\,+2)$ and (0)
- on-shell we get $6-4=2$ DOFs: $(-2,+2)$

[What happens is that on-shell there is still a residual gauge that kills the helicities $\pm 1, 0$ (of the spin-2) and the spin-0 component.]

[The scalar component $\mathcal{P}^{(0,s)}$ is *gauge-like* and <u>NOT</u> dynamical. On-shell it cancels the 0-helicity component of $\mathcal{P}^{(2)}$ when computing a physical amplitude.]