Microscopic Giant Dipole Response and Nuclear Level Densities within Covariant Energy Density Functional Theory

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CNDC CHINA NUCLEAR DATA CENTER

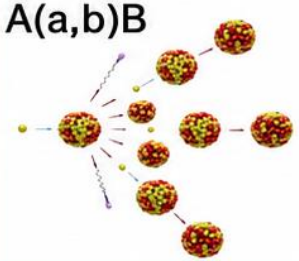
- Introduction
- Giant dipole resonance
- Nuclear Level Densities
- Conclusion and outlook

INTRODUCTION

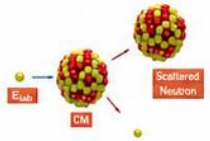
High-Precision Integrated Theoretical Platform for Neutron-Induced Reaction Data Calculations

Key Reaction Channels

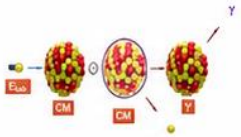
A(a,b)B



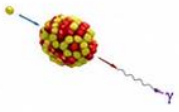
Elastic Scattering
(n, elastic)



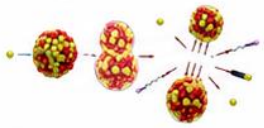
Inelastic Scattering
(n, inelastic)



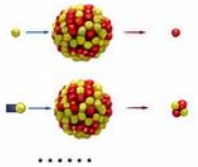
Radiative Capture
(n, γ)



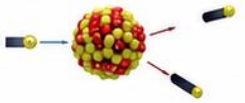
Fission Reaction
(n, fission)



Charged-Particle Emission
(n, p) (n, d) (n, α) ···

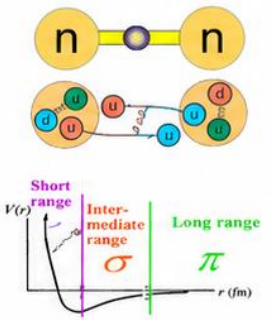


Secondary-Particle Emission
(n, 2n) (n, np) ···

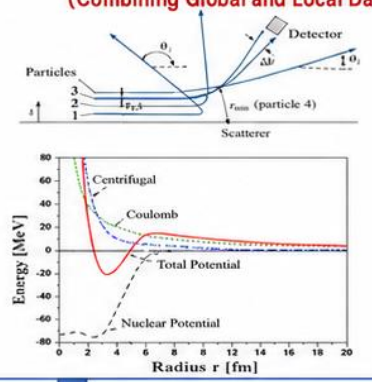


Key Reaction Mechanisms and Theories

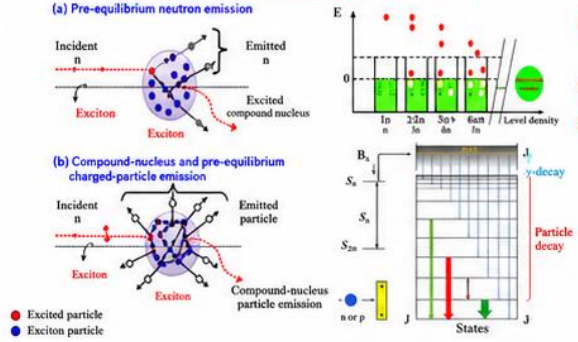
High-Precision Realistic Nuclear-Force Scattering Calculations
(CD-Bonn, Nijmegen, etc.)



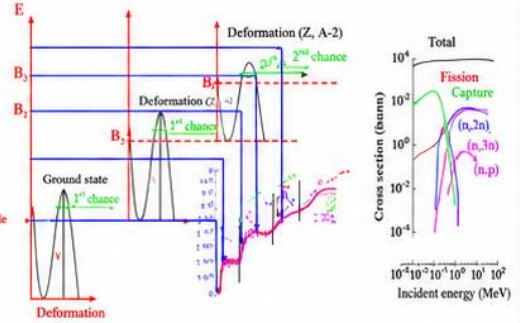
High-Precision Optical Potential
(Combining Global and Local Data)



High-Precision Compound-Nucleus Emission Mechanism
(Unified Hauser-Feshbach and Exciton Model)



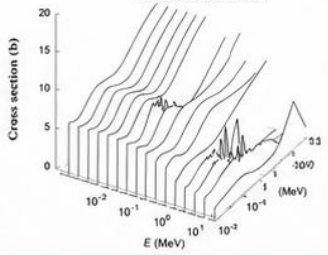
Accurate Fission Reaction System
(Multi-Humped Fission Barriers, Phenomenological and Microscopic)



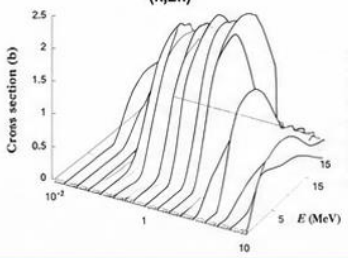
Comprehensive Data Information Files

- Reaction cross sections
- Angular distributions
- Double-differential particle-emission cross sections
- Photon-production cross sections
- Photon angular distributions
- …

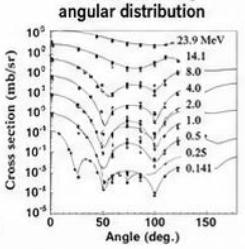
Total cross section



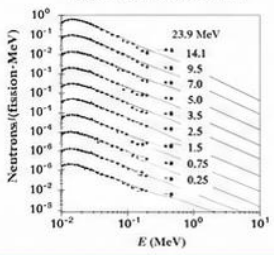
(n,2n)



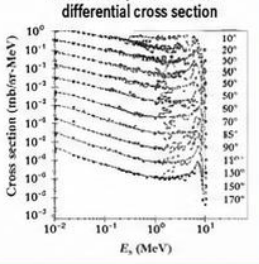
Elastic-scattering angular distribution



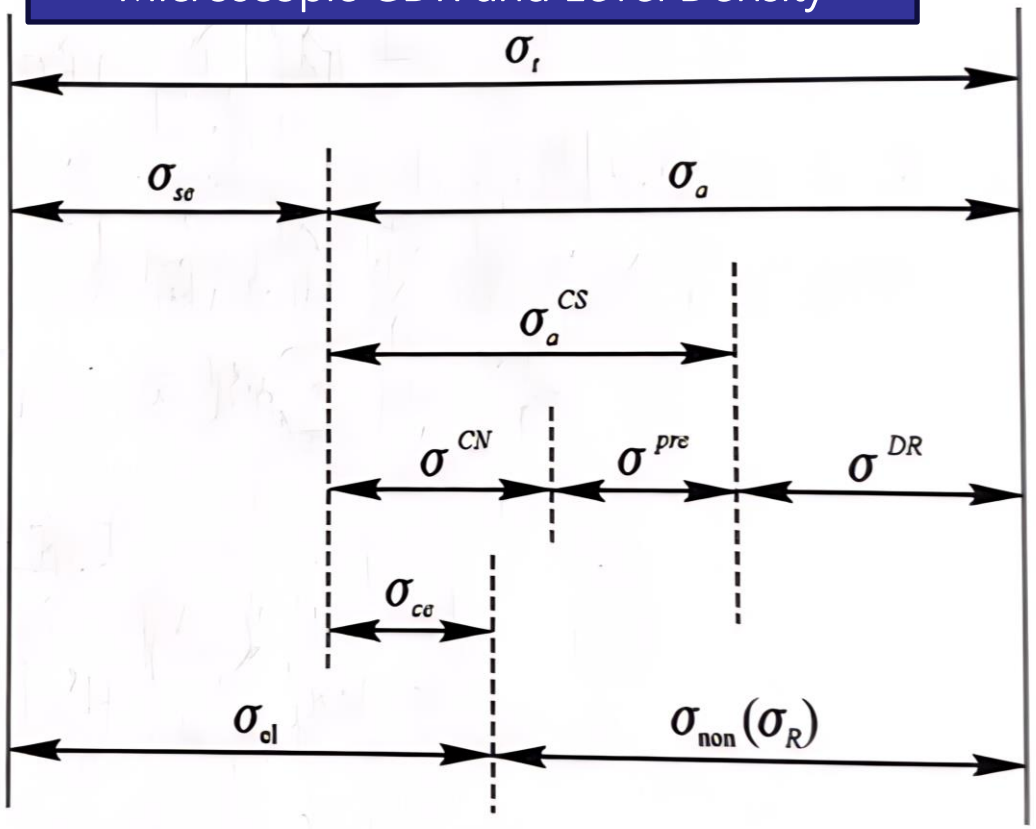
Fission neutron spectrum

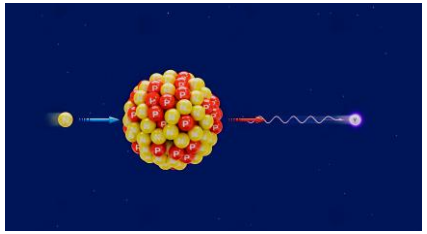


Secondary-particle double-differential cross section

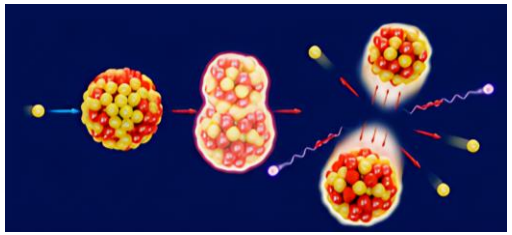


Microscopic GDR and Level Density

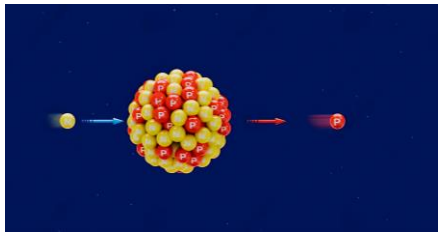




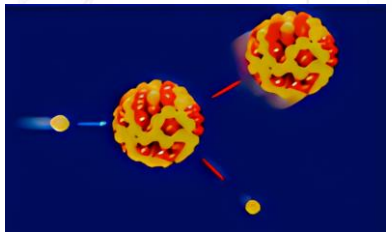
Radiative Capture Reaction
Photon Strength Function (RQRPA)
Nuclear level density(RHB)



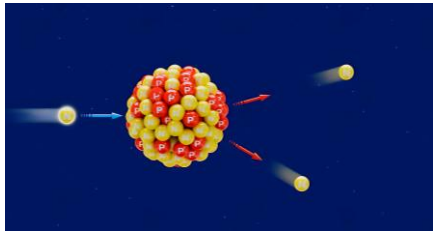
Fission Reaction
Multi-humped Fission Barrier (RHB)
Nuclear level density (RHB)



One-particle emission reactions
unified Hauser-Feshbach and exciton model
Nuclear level density (RHB)



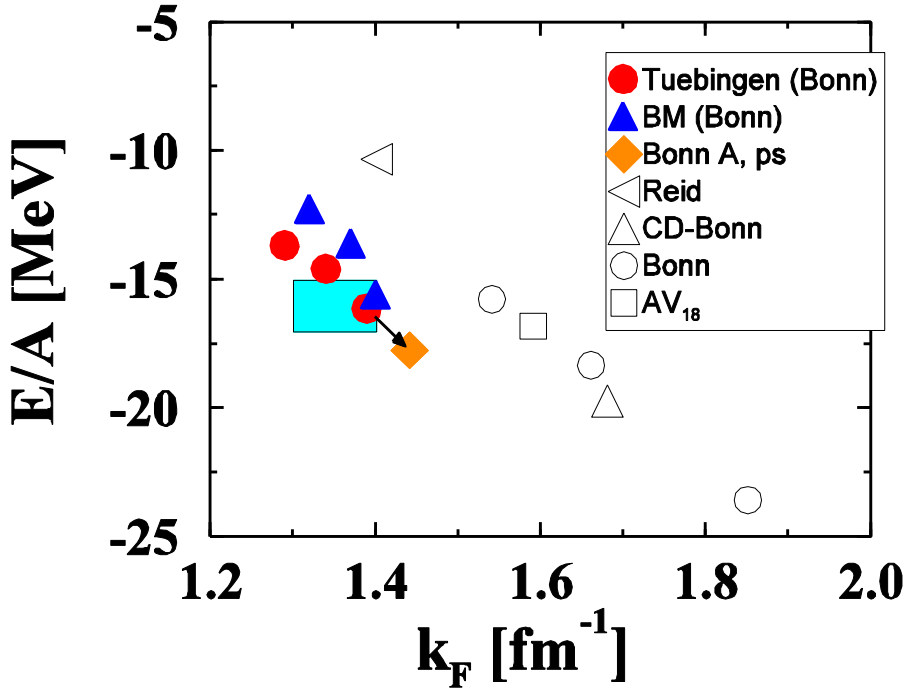
Elastic Reaction
Optical potential model
(CTOM or RHB)



Multi-particle emission reactions
Hauser-Feshbach
Nuclear level density (RHB)

Covariant density functional theory:

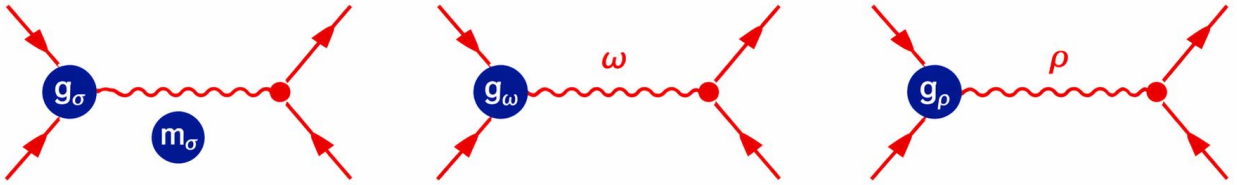
Why covariant?



C. Fuchs, LNP (2004)

Meson Exchange Interactions in Covariant Density Functional Theory

Nucleon–nucleon interaction mediated by meson exchange



σ meson (scalar–isoscalar) | ω meson (vector–isoscalar) | ρ meson (vector–isovector)

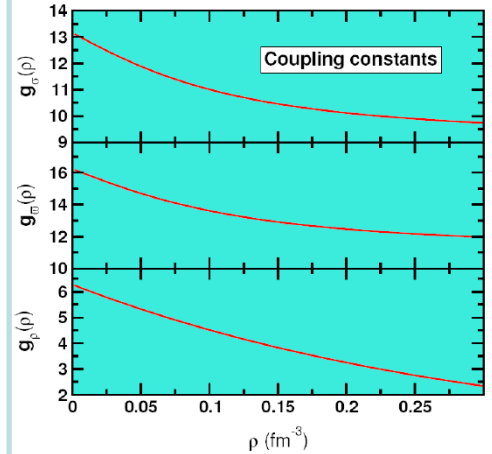
1) Large spin-orbit splitting in nuclei
2) Large fields V≈350 MeV , S≈-400 MeV
3) Success of Relativistic Brueckner
4) Success of intermediate energy proton scatt.
5) relativistic saturation mechanism
6) consistent treatment of time-odd fields
7) Pseudo-spin Symmetry
8) Connection to underlying theories ?
9) As many symmetries as possible

Taken from Prof. Peter Ring's talk

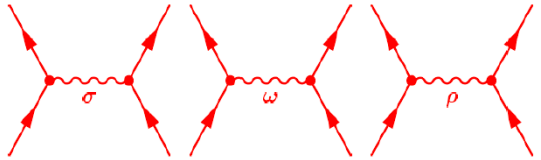
Density Dependent coupling constants:

Typel, Wolter, NPA **656**, 331 (1999)
Niksic, Vretenar, Finelli, P.R., PRC **66**, 024306 (2002): **DD-ME1**
Lalazissis, Niksic, Vretenar, P.R., PRC 78, 034318 (2008):**DD-ME2**

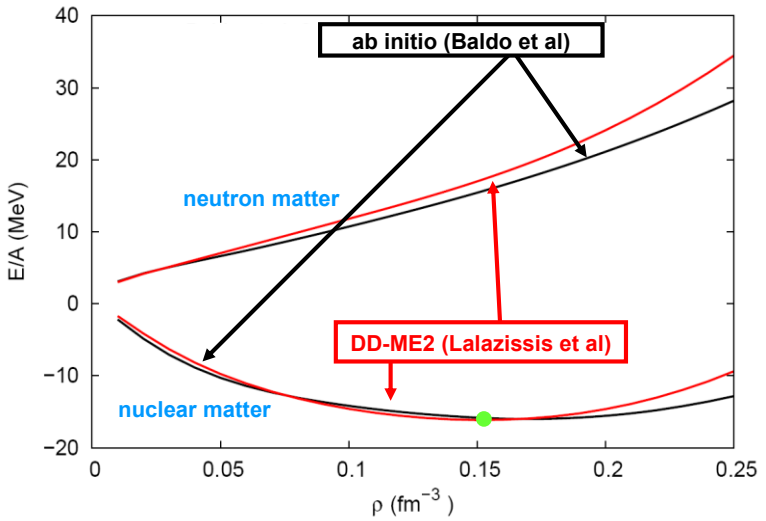
	DD-ME2	DD-ME1	TW-99	NL3	NL3*
ρ_o (fm^{-3})	0.152	0.152	0.153	0.149	0.150
E/A (MeV)	-16.14	-16.20	-16.25	-16.25	-16.31
K (MeV)	250.89	244.5	240.0	271.8	258.5
J (MeV)	32.3	33.1	32.5	37.9	38.3
m^*/m	0.572	0.578	0.556	0.60	0.595

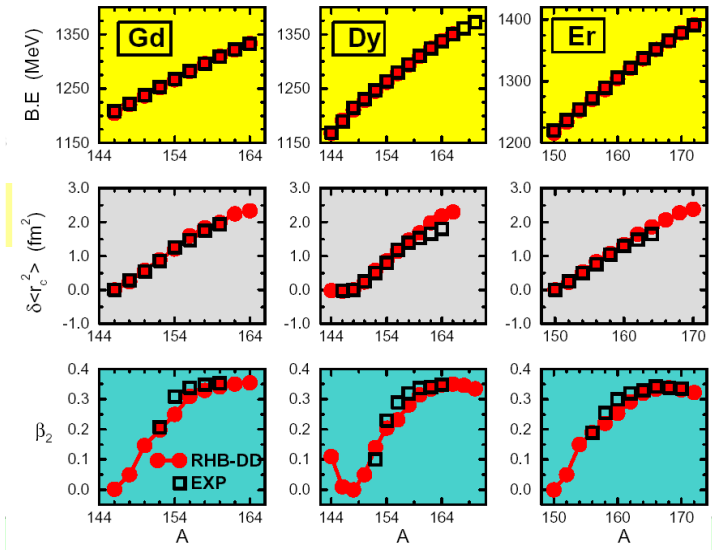


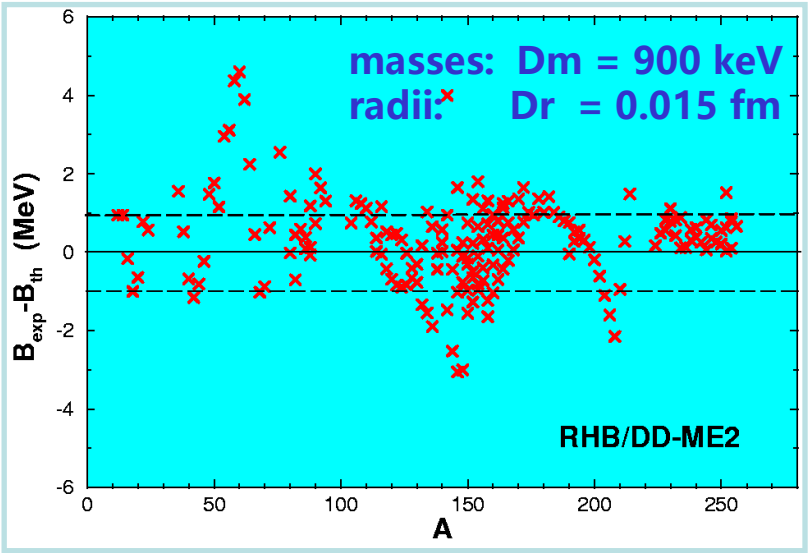
Effective interactions with medium-dependent couplings:



COUPLING PARAMETERS:
$g_\sigma(\rho)$, $g_\omega(\rho)$, $g_\rho(\rho)$



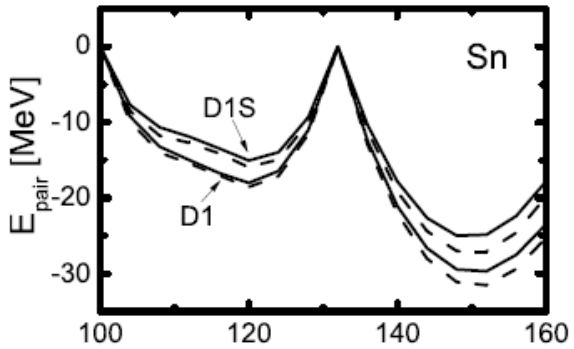


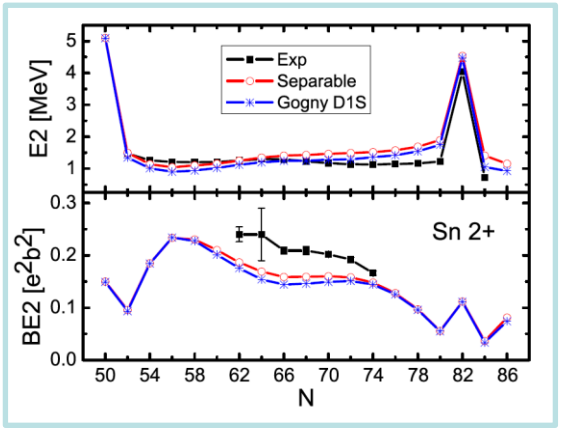


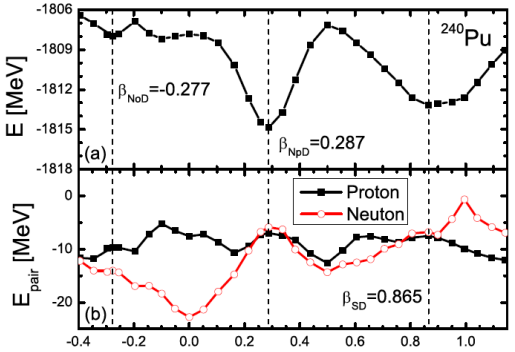
Separable pairing interaction (TMR)

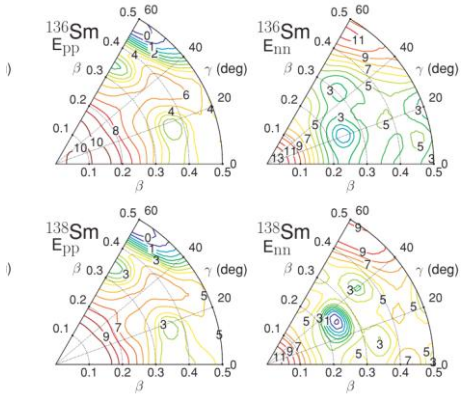
$$V_{pp}(\boldsymbol{r}_1\boldsymbol{r}_2,\boldsymbol{r}'_1,\boldsymbol{r}'_2) = -G\,\delta(\boldsymbol{R}-\boldsymbol{R}')e^{-(\alpha r)^2}e^{-(\alpha r')^2} \qquad V^0_{121'2'} = -G\sum_N V^N_{12}V^N_{1'2'}$$

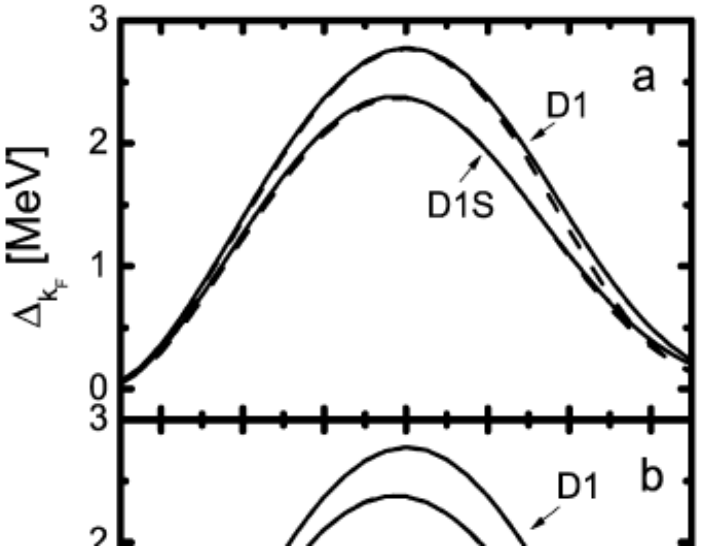
where $\boldsymbol{R} = \frac{1}{2}(\boldsymbol{r}_1+\boldsymbol{r}_2), \quad \boldsymbol{r} = \boldsymbol{r}_1-\boldsymbol{r}_2$ $\qquad V^N_{12} = M^{N0n0}_{n_1ln_2l}\frac{\hat{j}}{\hat{s}\hat{l}}\int_0^\infty R_{n0}(r,b_r)P(r)r^2dr$











Work	Contribution	Citations
Tian, Ma & Ring PLB **676**, 44 (2009)	Finite-range separable pairing force for nuclear DFT	**400+**
Tian, Ma & Ring PRC **80**, 024313 (2009)	Axially deformed RHB calculations	**100+**
Tian, Ma & Ring PRC **79**, 064301 (2009)	Relativistic QRPA extension	**80+**
Nikšić, Vretenar, Tian, Ma & Ring PRC **81**, 054318 (2010)	3D/triaxial RHB calculations	**120+**

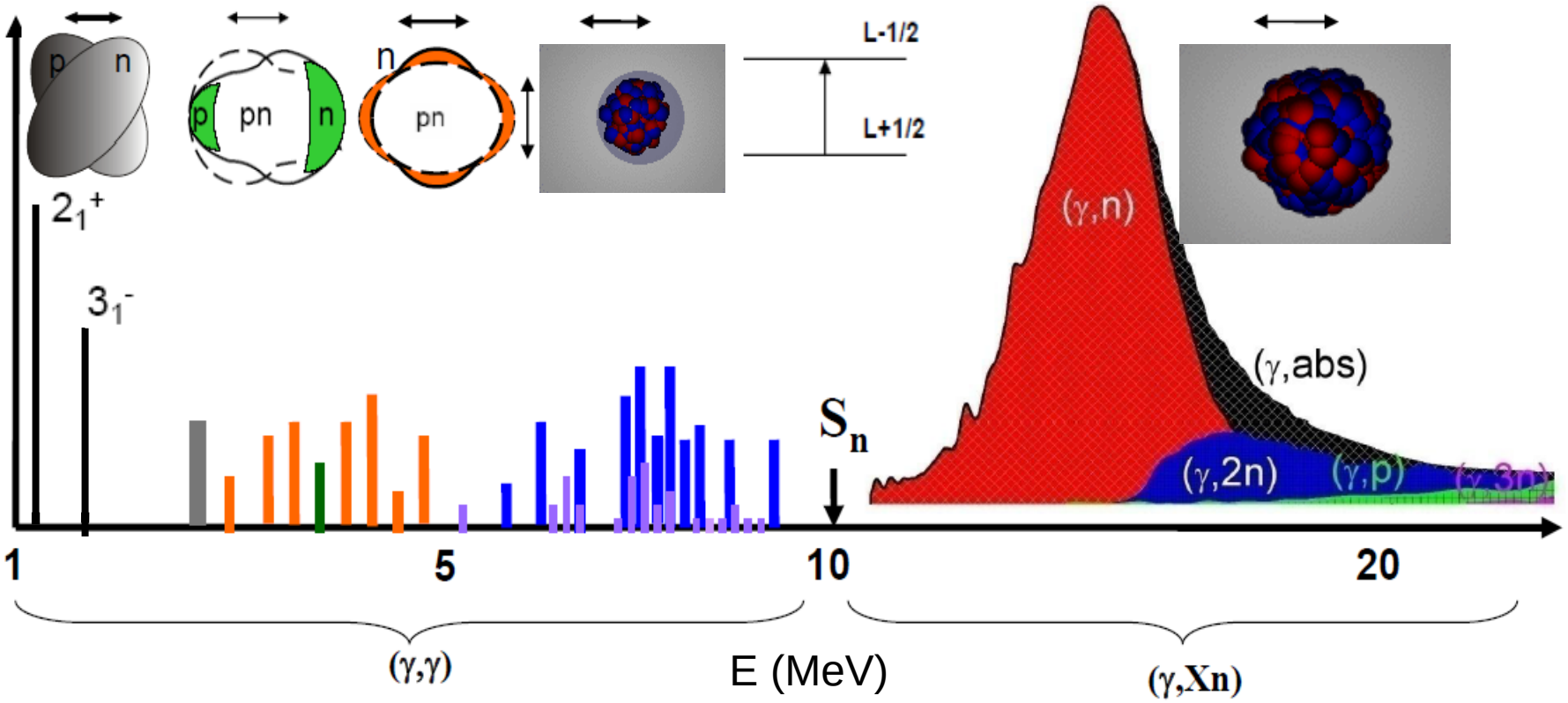
Giant Dipole Resonance

Radiative Capture Reaction

Scissors mode | Two-phonon excitation | Pygmy-quadrupole resonance | Pygmy-dipole resonance | Giant M1 resonance | Giant Dipole Resonance

M1 $E1_{(2_1^+ \times 3_1^-)_1}$ E2(PQR) E1(PDR) Spin-flip M1

E1 (GDR)



Moderate and Heavy nuclei

Theoretical prediction of Pygmy Quadrupole Resonance: N. Tsoneva, H. Lenske, Phys. Lett. B 695 (2011) 174.

N.Tsoneva, ERICE14

RQRPA: using the diagonalization approach

$$\begin{pmatrix} A^J & B^J \\ B^{*J} & A^{*J} \end{pmatrix} \begin{pmatrix} X^{\nu,JM} \\ Y^{\nu,JM} \end{pmatrix} = \omega_\nu \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} X^{\nu,JM} \\ Y_{\nu,JM} \end{pmatrix}$$

Small Matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

Small dimension, can be diagonalized directly

Direct Diagonalization

$$Ax = \lambda x$$

Computational cost ~ $O(N^3)$

Memory cost ~ $O(N^2)$

Large Matrix

$N \times N$

Huge dimension, memory and computation time grow rapidly

Memory Bottleneck

Memory cost ~ $O(N^2)$

Memory demand quickly exceeds available capacity

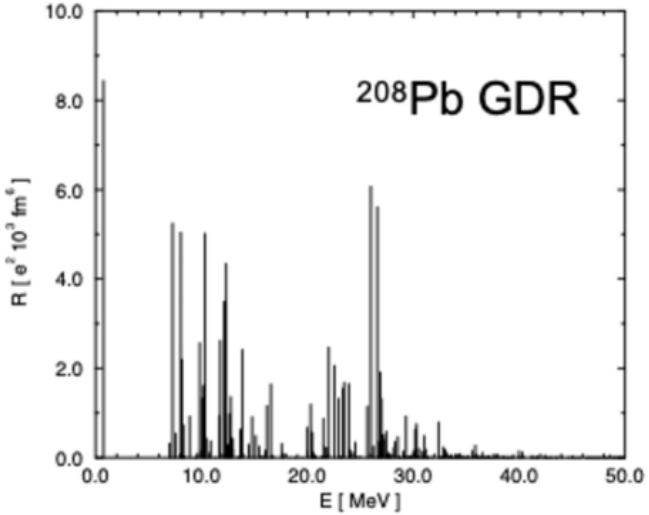
Time Bottleneck

Computational cost ~ $O(N^3)$

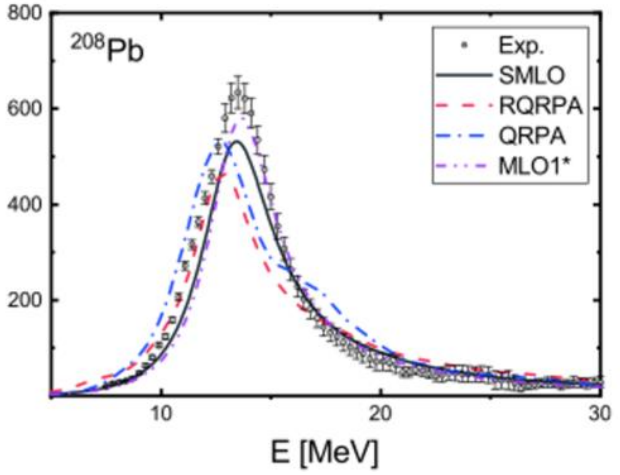
Runtime increases dramatically making completion within a reasonable time difficult

When the matrix dimension becomes very large, direct diagonalization becomes extremely difficult

Example: spherical QRPA ~ 3×10^3, manageable; deformed QRPA ~ 10^5, direct diagonalization is extremely expensive



requires the introduction of a width to obtain the excitation curve



RQRPA: for the spherical nuclei

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Giant dipole resonance parameters from photoabsorption cross-sections*

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Ruirui Xu(续瑞瑞)[1;2)] Zhigang Ge(葛智刚)[1]
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[2]Guangxi Normal University, Guilin, Guangxi Province, 541004, China

$$\sigma_{E1}(\epsilon_\gamma) = \frac{16\pi^3}{9}\frac{e^2}{\hbar c}G\epsilon_\gamma S_{E1}(\epsilon_\gamma)$$

$$S_{E1}(\epsilon_\gamma) = \sum_\nu \frac{B(1,\omega_\nu)}{\pi}\frac{\Gamma(\epsilon_\gamma)/2}{(\epsilon_\gamma-\omega_\nu)^2+(\Gamma(\epsilon_\gamma)/2)^2}$$

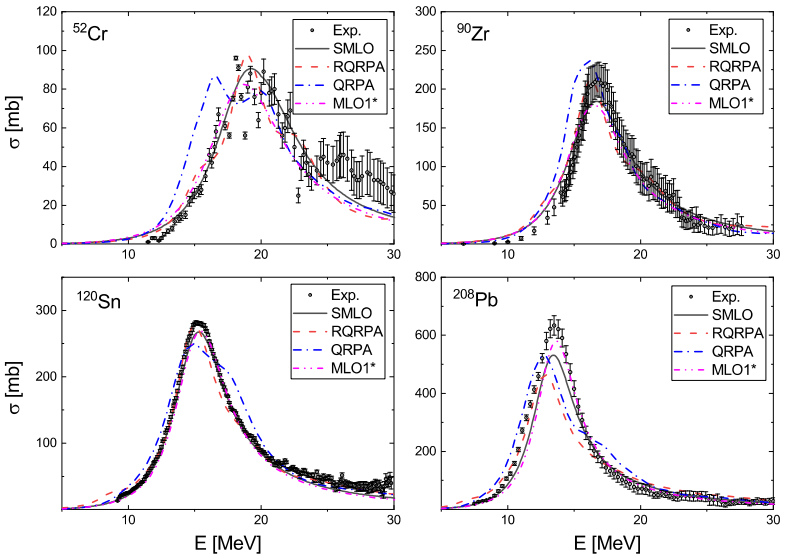
$$\Gamma(\epsilon_\gamma) = \Gamma\sqrt{\epsilon_\gamma/E_{E1}}$$

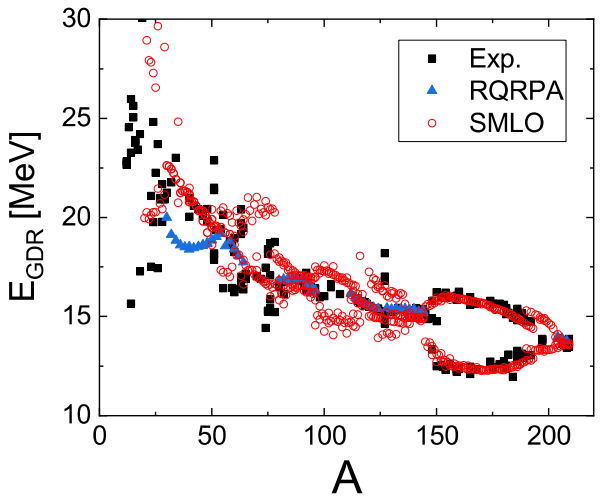
TABLE III. The parameters and χ^2_{tot} of microscopic GDR parameters within RQRPA. C means constant width of the Lorentzian distribution., ED stands for energy-dependent width.

	Γ	G	χ^2_{tot}
RQRPA	2.0 (C)	1.0	270.0
RQRPA(fitted)	1.59 (ED)	0.65	21.0

TABLE I. Experimental value of the spherical nuclei we use to fit the systematic GDR parameters.

Nucleus	points	Energy range [MeV]	Ref.	def. β_2
^{34}S	29	12.0 - 26.0	[28]	0.00
^{40}Ar	61	10.0 - 40.0	[29]	0.00
^{40}Ca	70	11.25 - 28.1	[30]	0.00
^{42}Ca	151	10.0 - 40.0	[31]	0.00
^{44}Ca	58	11.5 - 40.0	[31]	0.00
^{48}Ca	60	10.5 - 40.0	[31]	0.00
^{48}Ti	58	11.5 - 40.0	[31]	0.00
^{51}V	41	10.47 - 32.69	[32]	0.00
^{52}Cr	115	11.5 - 40.0	[29]	0.00
^{90}Zr	46	12.2 - 25.7	[33]	0.035
^{91}Zr	101	10.8 - 30.0	[34]	0.053
^{92}Zr	105	15.8 - 27.8	[35]	0.053
^{94}Zr	92	7.85 - 31.0	[34]	0.063
^{112}Sn	167	10.9 - 27.5	[36]	0.018
^{114}Sn	168	10.4 - 27.1	[36]	0.00
^{116}Sn	200	9.70 - 29.6	[36]	0.00
^{117}Sn	232	7.40 - 30.9	[36]	-0.044
^{118}Sn	211	9.40 - 30.7	[36]	0.00
^{119}Sn	233	7.40 - 31.1	[36]	0.00
^{120}Sn	207	9.20 - 29.8	[36]	0.00
^{122}Sn	184	8.90 - 27.2	[36]	0.00
^{124}Sn	220	8.60 - 30.9	[36]	0.00
^{138}Ba	61	8.48 - 27.1	[37]	0.00
^{208}Pb	101	7.50 - 37.5	[38]	0.00
^{209}Bi	109	8.01 - 26.4	[37]	-0.008





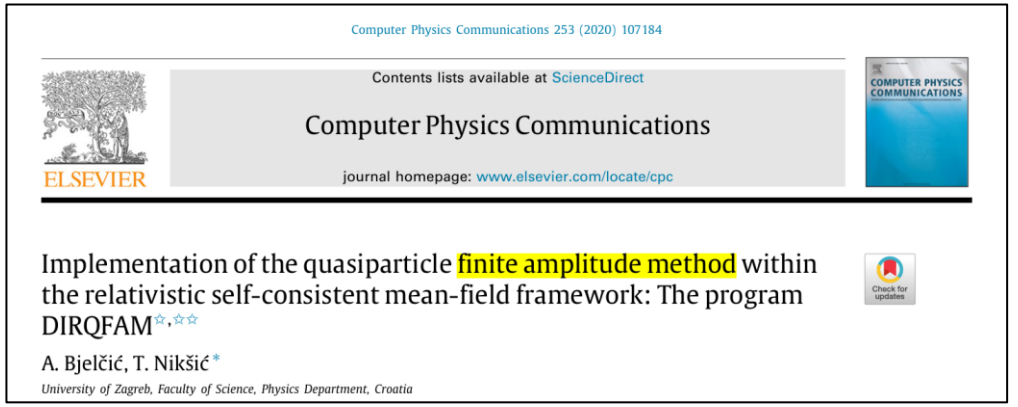
FAM solves the linear response equations through a self-consistent iterative approach

$$(E_\mu + E_\nu - \omega_\gamma) X_{\mu\nu}(\omega_\gamma) = -\left(F^{20}_{\mu\nu}(\omega_\gamma) + \delta H^{20}_{\mu\nu}(\omega_\gamma)\right),$$
$$(E_\mu + E_\nu + \omega_\gamma) Y_{\mu\nu}(\omega_\gamma) = -\left(F^{02}_{\mu\nu}(\omega_\gamma) + \delta H^{02}_{\mu\nu}(\omega_\gamma)\right),$$

$$\delta H^{20}_{\mu\nu}(\omega) = \sum_{\mu'<\nu'} \{A_{\mu\nu,\mu'\nu'} - (E_\mu + E_\nu)\delta_{\mu\mu'}\delta_{\nu\nu'}\} X_{\mu'\nu'}(\omega) + \sum_{\mu'<\nu'} B_{\mu\nu,\mu'\nu'} Y_{\mu'\nu'}(\omega)$$
$$\delta H^{02}_{\mu\nu}(\omega) = \sum_{\mu'<\nu'} \{A^*_{\mu\nu,\mu'\nu'} - (E_\mu + E_\nu)\delta_{\mu\mu'}\delta_{\nu\nu'}\} X_{\mu'\nu'}(\omega) + \sum_{\mu'<\nu'} B^*_{\mu\nu,\mu'\nu'} Y_{\mu'\nu'}(\omega)$$

$$\left[\begin{pmatrix} A & B \\ B^* & A^* \end{pmatrix} - \omega \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] \begin{pmatrix} X(\omega) \\ Y(\omega) \end{pmatrix} = - \begin{pmatrix} F^{20} \\ F^{02} \end{pmatrix}$$

Restricting calculations to fixed-size widths makes it difficult to reproduce experimental data


Computer Physics Communications 253 (2020) 107184

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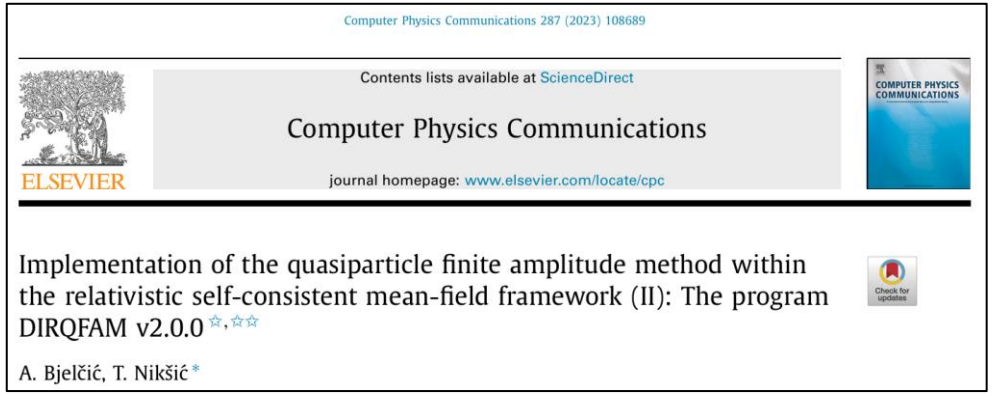
Implementation of the quasiparticle finite amplitude method within the relativistic self-consistent mean-field framework: The program DIRQFAM☆,☆☆

A. Bjelčić, T. Nikšić*

University of Zagreb, Faculty of Science, Physics Department, Croatia


Computer Physics Communications 287 (2023) 108689

Contents lists available at ScienceDirect

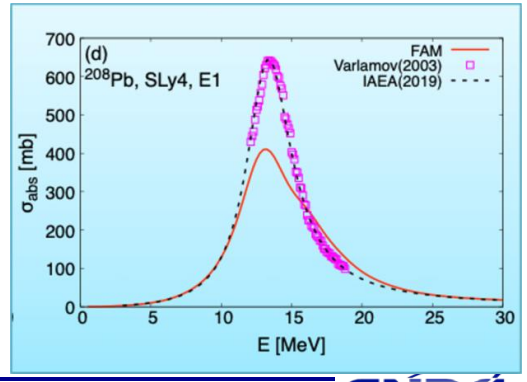
Computer Physics Communications

journal homepage: www.elsevier.com/locate/cpc

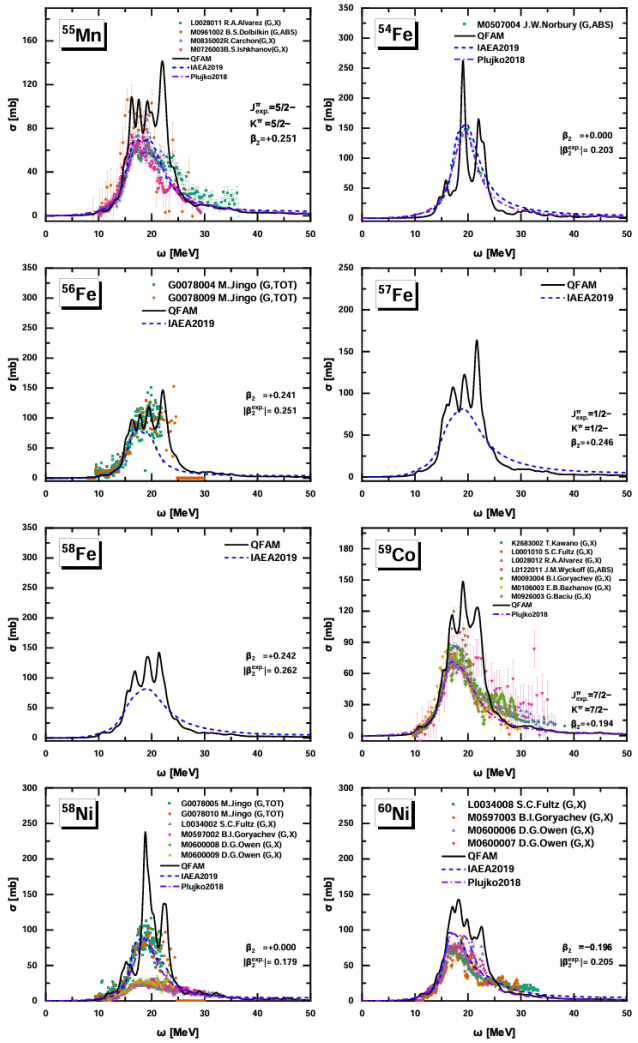
Implementation of the quasiparticle finite amplitude method within the relativistic self-consistent mean-field framework (II): The program DIRQFAM v2.0.0☆,☆☆

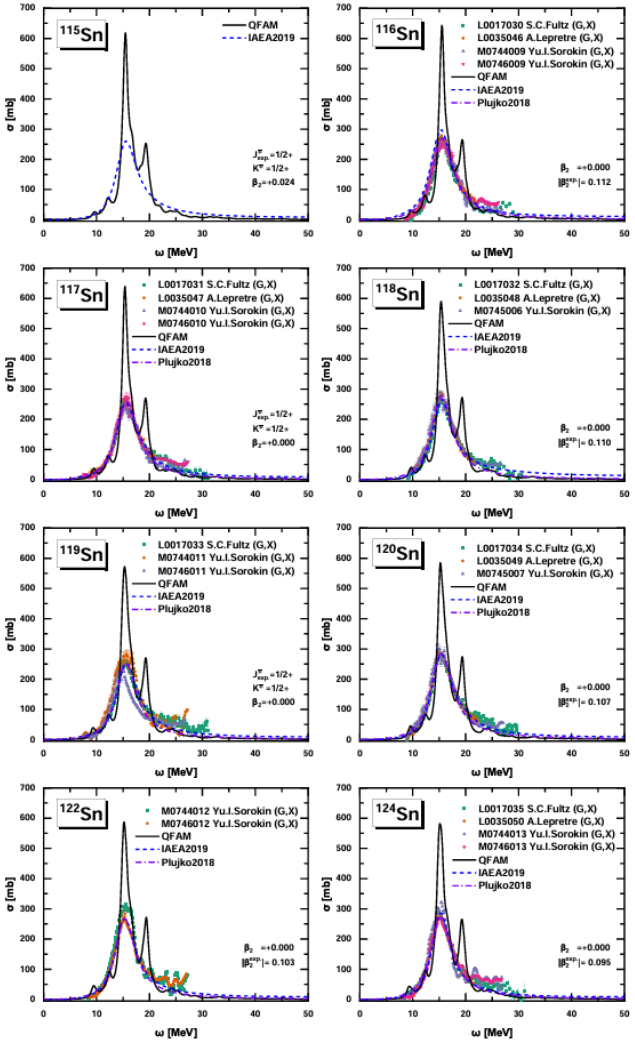
A. Bjelčić, T. Nikšić*

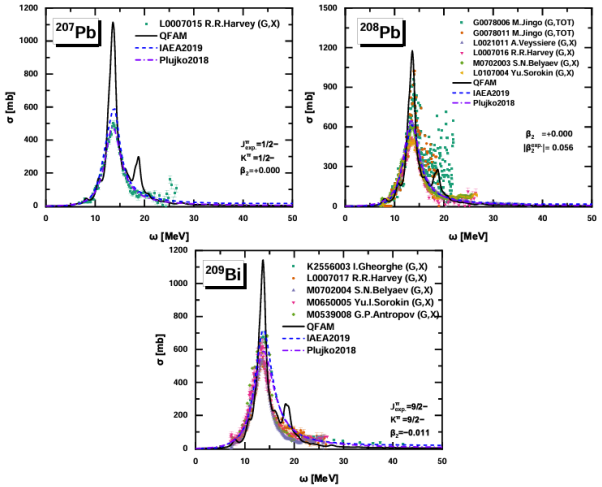
N_{shells}	Memory [GB]	Time [s]
10	0.42 (0.75)	0.24 (0.36)
12	0.68 (1.46)	0.49 (0.82)
14	1.11 (2.75)	0.99 (1.75)
16	1.77 (5.68)	1.82 (3.32)
18	2.83 (8.39)	3.23 (5.71)
20	4.40 (13.9)	5.52 (9.54)
22	6.65 (22.2)	8.97 (15.5)
24	11.1 (34.2)	14.1 (26.0)



Radiative Capture Reaction



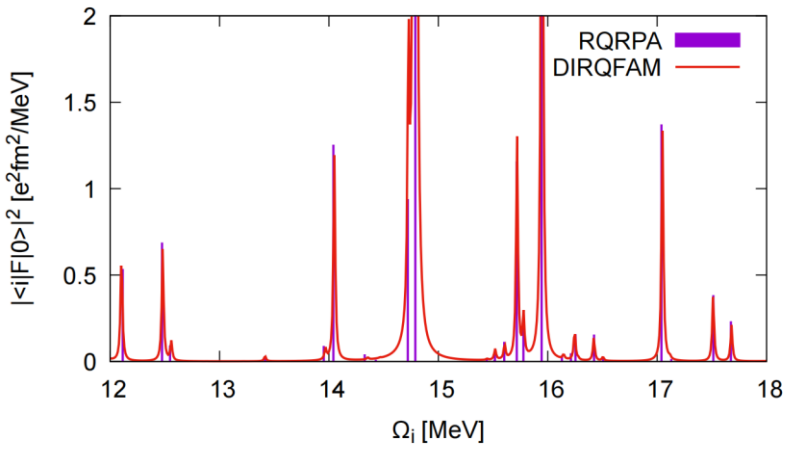


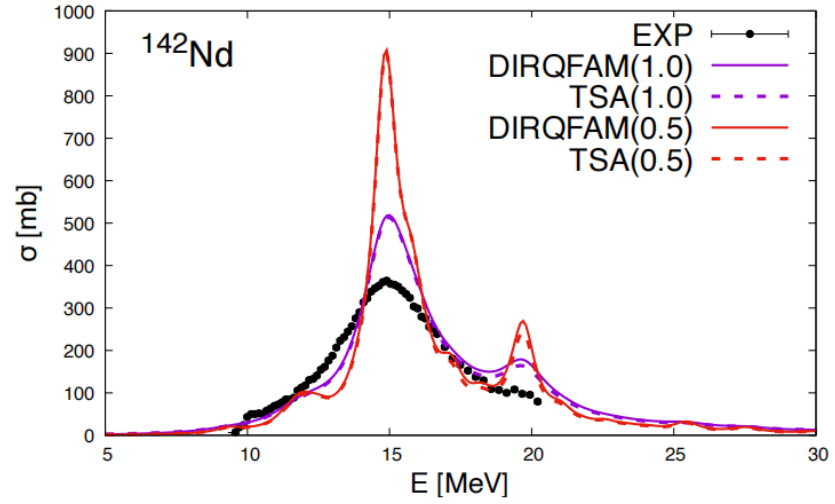


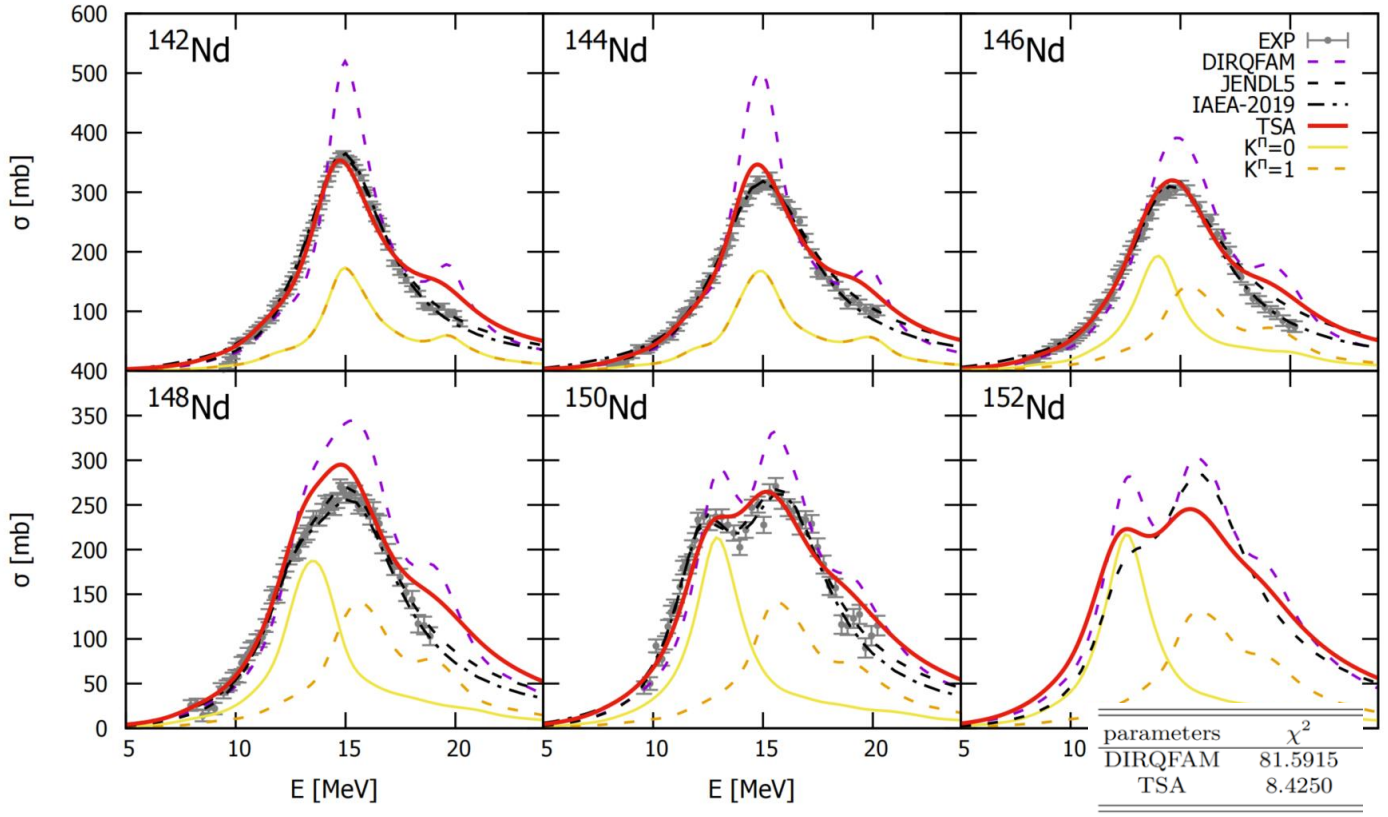
Photoabsorption cross sections for 235 stable nuclei, ranging from ^{40}Ca to ^{209}Bi, were investigated by the quasiparticle finite amplitude method (QFAM) using relativistic poin coupling interaction DD-PC1, with extensions to odd-A nuclei.

Atomic Data and Nuclear Data Tables
Volume 168, February 2026, 101770

Energy-dependent width improves QFAM description of GDR







This paper investigates the photon absorption cross-sections of even-even nuclei within the Nd isotope chain. By fitting energy-dependent widths, the deviation between the original QFAM results and experimental values was reduced by an order of magnitude;

Nuclear Science and Techniques (2025) 36:241

Nuclear Level Densities

Microscopic nuclear level densities based on CDFT

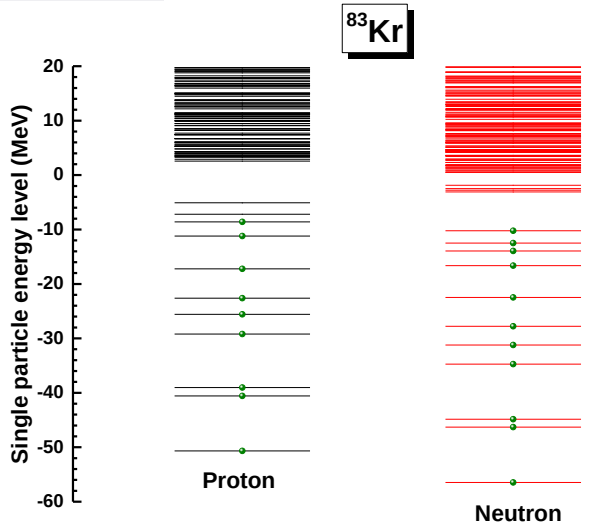
Single particle levels (CDFT)

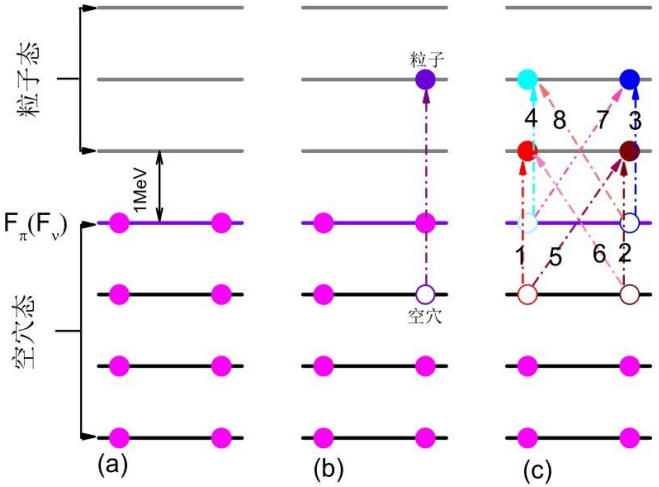
combinatorial method

Particle-hole state densities

rotational and vibrational effects

Nuclear level density





$$\rho_{sph}(U,J,P) = \omega_i(U,M=J,P) - \omega_i(U,M=J+1,P)$$

$$\rho_{\mathrm{def}}(U,J,P) = 1/2[\sum_{K=-J,K\neq 0}^{J} \omega_i\left(U - E_{rot}^{J,K}, K, P\right)] + \delta_{(J_{even})}\delta_{(P=+)}\omega_i\left(U - E_{rot}^{J,0}, 0, P\right)$$

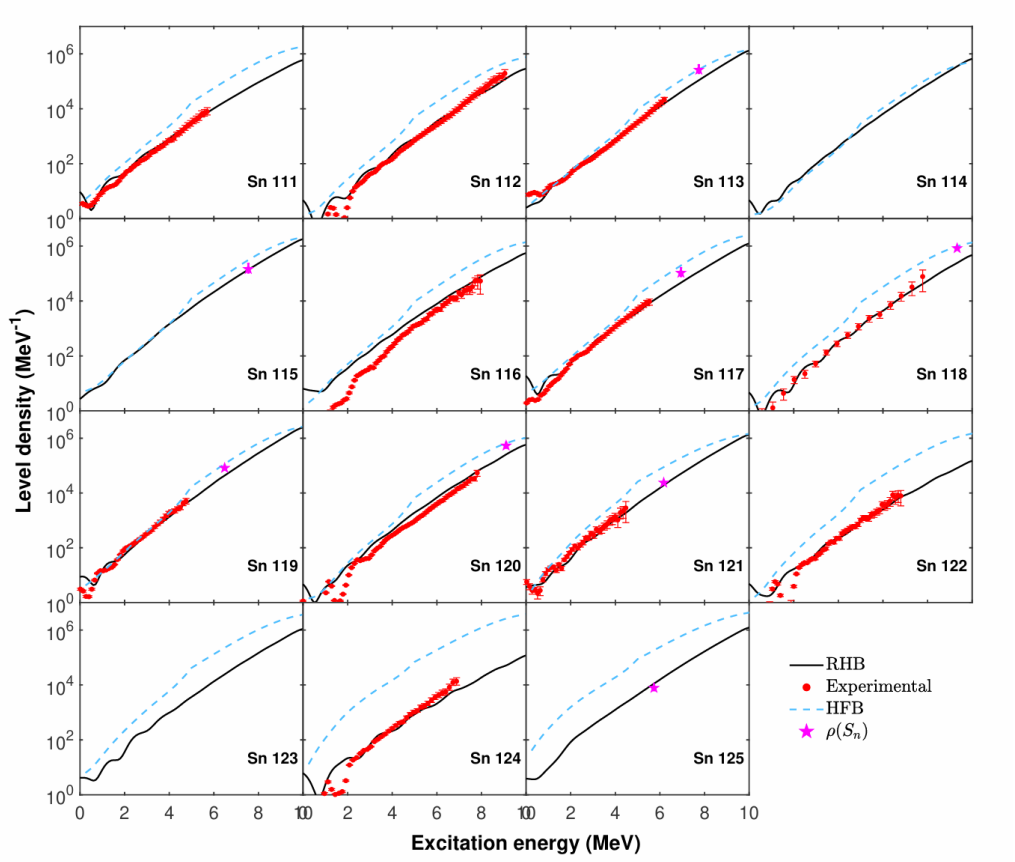
$$+\delta_{(J_{odd})}\delta_{(P=-)}\omega_i\left(U - E_{rot}^{J,0}, 0, P\right)$$

Global microscopic nuclear level densities within the HFB\ plus combinatorial method for practical applications
Hilaire, S. / Goriely, S. 2006 *Nuclear Physics A* , Vol. 779 p. 63 - 81

Chinese Physics C Vol. 50, No. 5 (2026) 054107

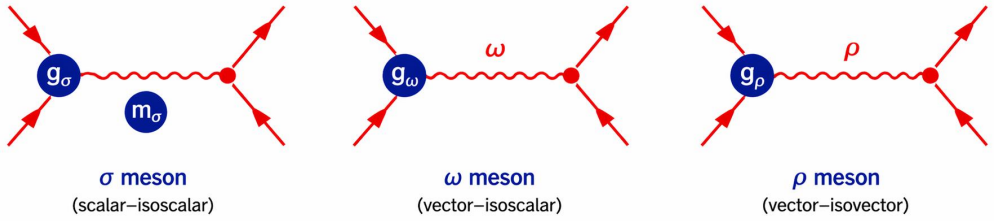
Systematic study of microscopic nuclear level densities of Sn isotopes within a relativistic framework*

Guanbin He (何冠彬)[1] Nuocheng Tang (唐诺程)[1] Yuan Tian (田源)[1†] Ying Cui (崔莹)[1]
Jian Li (李剑)[2] Yi Xu (徐毅)[1,3‡] Ruirui Xu (续瑞瑞)[1§]

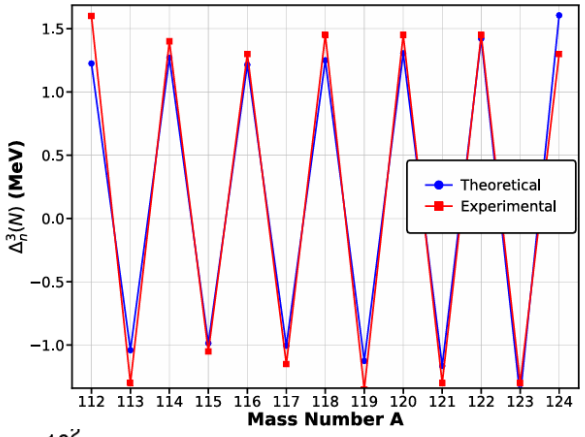


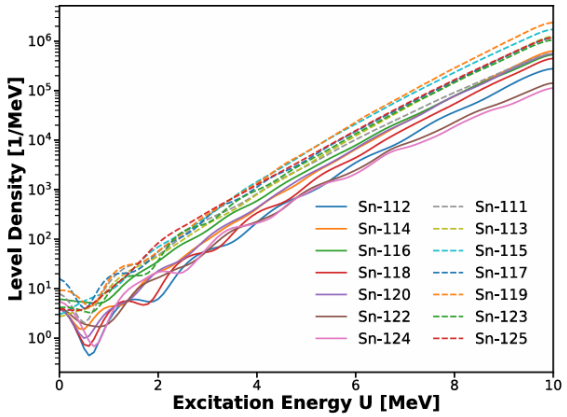
Meson Exchange Interactions in Covariant Density Functional Theory

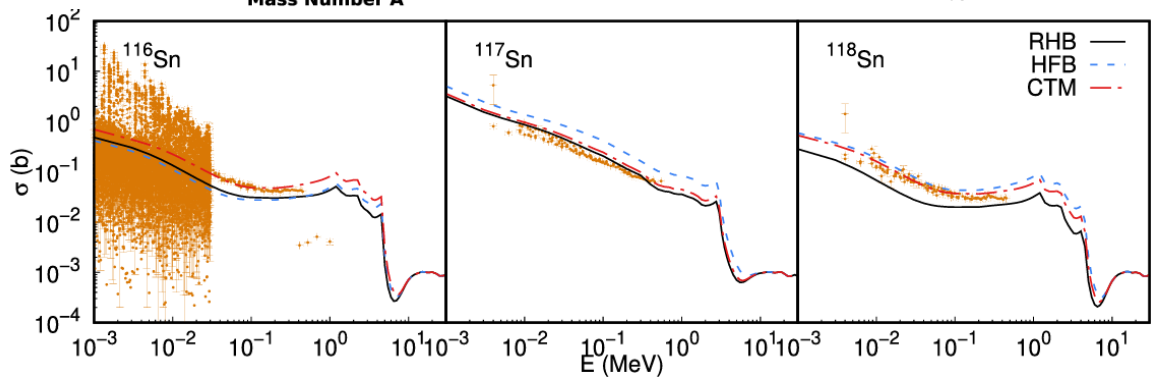
Nucleon–nucleon interaction mediated by meson exchange



σ meson (scalar–isoscalar) ω meson (vector–isoscalar) ρ meson (vector–isovector)

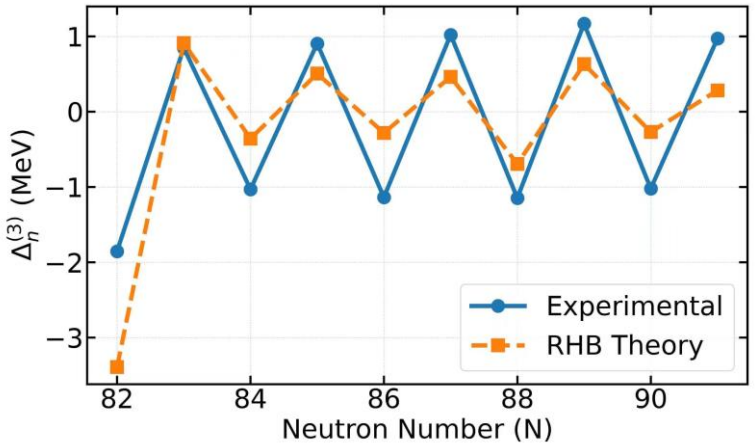


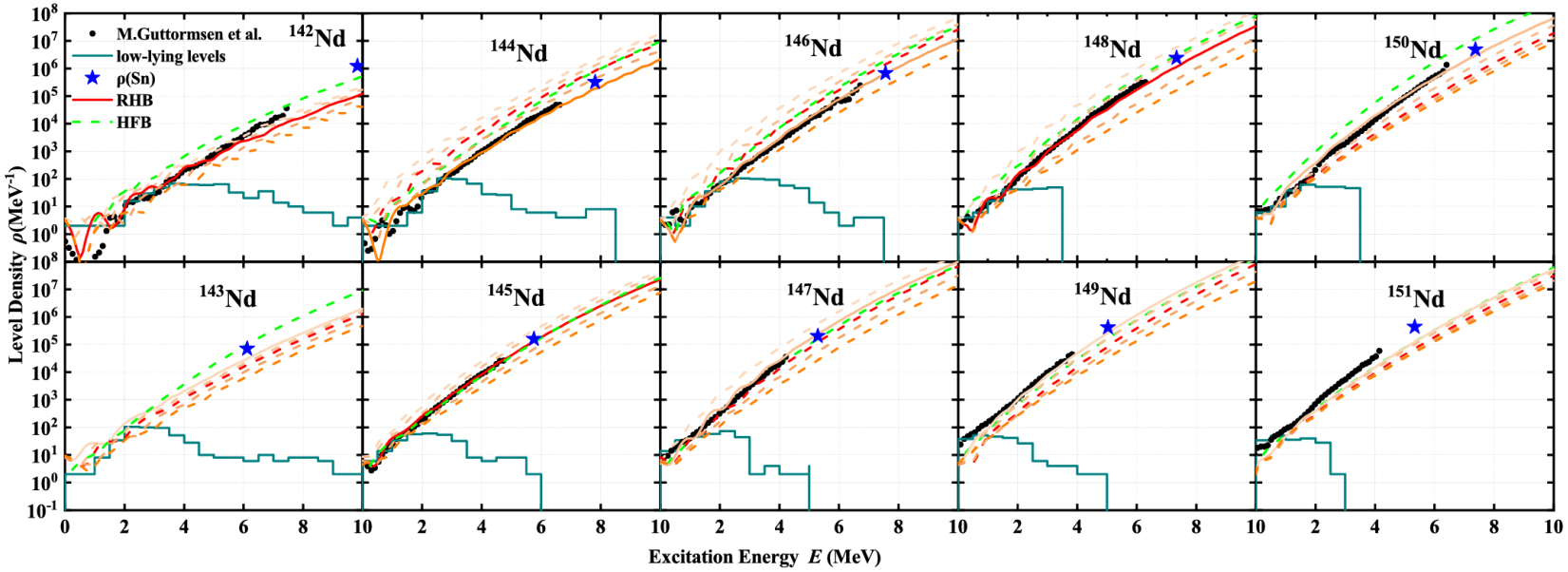




Pairing effect on the Microscopic Level Density

Pairing sensitivity:
The default pairing strength tends to underestimate pairing correlations in Nd isotopes. By scaling the pairing strength from 0.8 to 1.2, we find that microscopic level densities are strongly correlated with pairing correlations, especially in the low-excitation-energy region. This indicates that pairing optimization is essential for reliable microscopic NLD predictions.





In preparation

FIG. 3. Microscopic level densities of $^{142-151}$Nd calculated with different values of the pairing scaling factor λ. The recommended result selected from the D_0 constraint and auxiliary Oslo comparison is highlighted. The low-lying discrete levels, available Oslo-type level densities, and neutron-resonance constraint at S_n are shown for comparison.

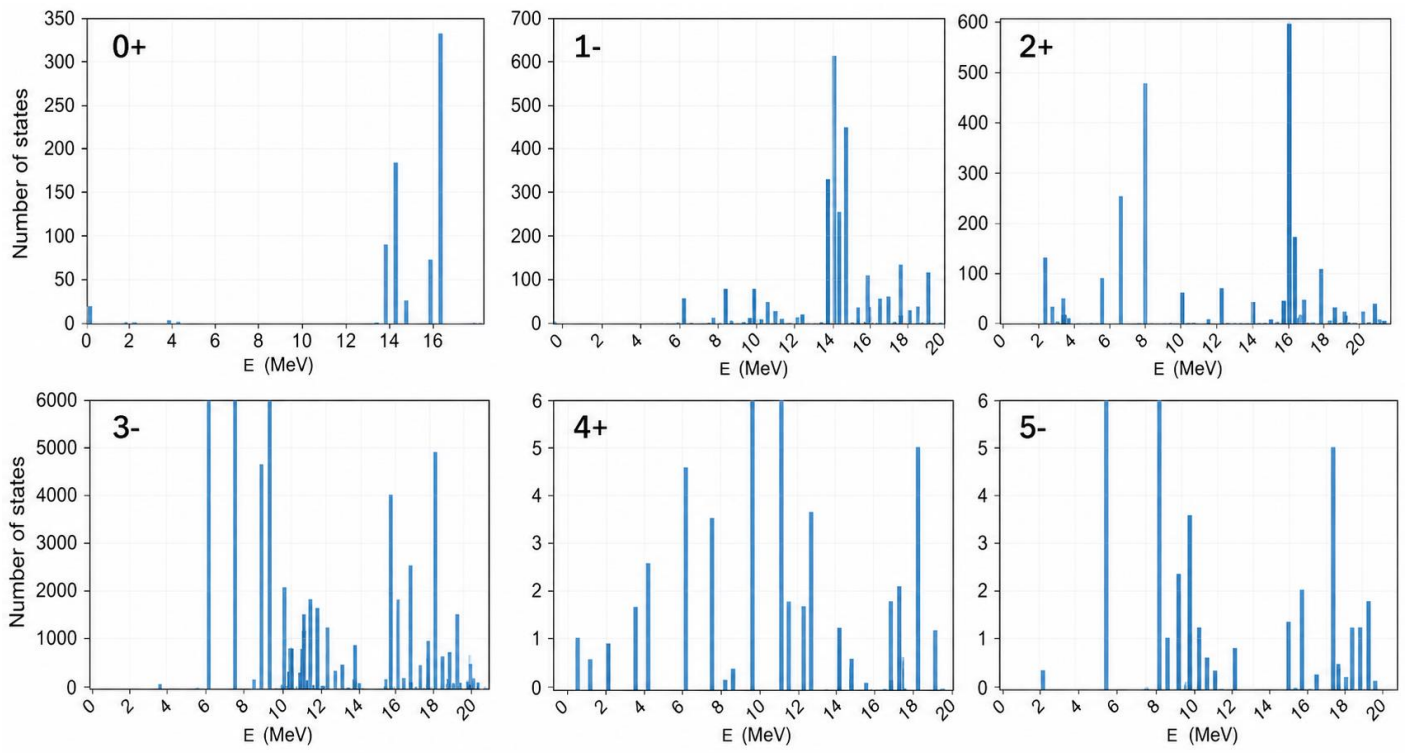
RQRPA + Boson expansion

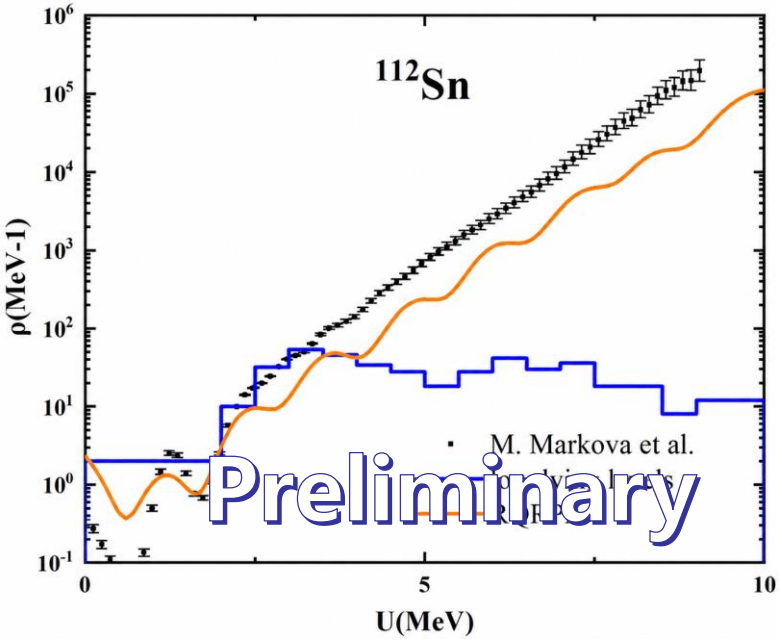
A new approach to nuclear level densities: The QRPA plus boson expansion

S. Hilaire [a,b,*], S. Goriely [c], S. Péru [a,b], G. Gosselin [a,b]

[a] CEA, DAM, DIF, F-91297 Arpajon, France
[b] Université Paris-Saclay, CEA, LMCE, 91680 Bruyères-le-Châtel, France
[c] Institut d'Astronomie et d'Astrophysique, Université Libre de Bruxelles, Campus de la Plaine CP 226, 1050 Brussels, Belgium

Physics Letters B 843 (2023) 137989





Current status and next step

- The present boson-expansion space is constructed from electric RQRPA phonons.
- Magnetic phonons, are not included. 1^+M1 excitations, are included by experiment data.
- Deformed nuclei need to be calculated with DIRQFAM.

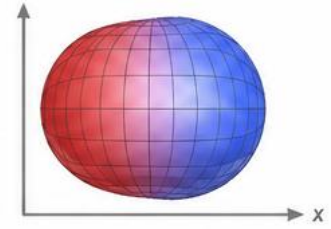
CONCLUSION AND OUTLOOK

A Unified Microscopic Framework:
Structure → Response → Reaction

From covariant energy density functional theory to nuclear observables

1. STRUCTURE (RHB)

Covariant DFT with pairing
Self-consistent ground state

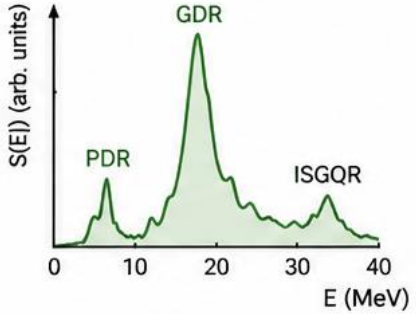


Outputs
- Mean fields (σ, ω, ρ, γ)
- Single-particle spectrum
- Pairing gaps Δ_n, Δ_p
- Deformation (β_2, γ)

Microscopic quasiparticle basis

2. RESPONSE (RQRPA)

Small-amplitude excitations
Electromagnetic response

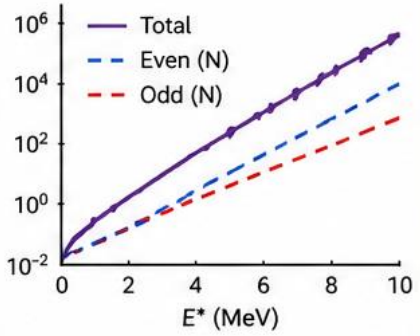


Outputs
- Photoabsorption cross section $\sigma_\gamma(E)$
- E1 strength function $f_{E1}(E)$
- GDR parameters (E_{GDR}, Γ, S)

γ-ray strength function

3. LEVEL DENSITY (NLD)

Combinatorial approach
from quasiparticle spectrum

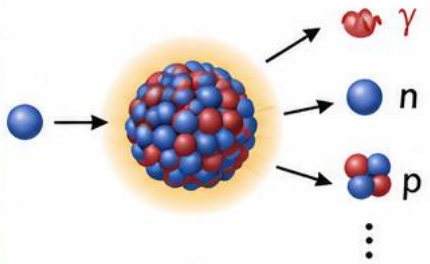


Outputs
- Total level density $\rho(E^*)$
- Even–odd staggering
- Shell, pairing and deformation effects

Nuclear level density

4. REACTION (Hauser–Feshbach)

Statistical reaction model
with microscopic inputs

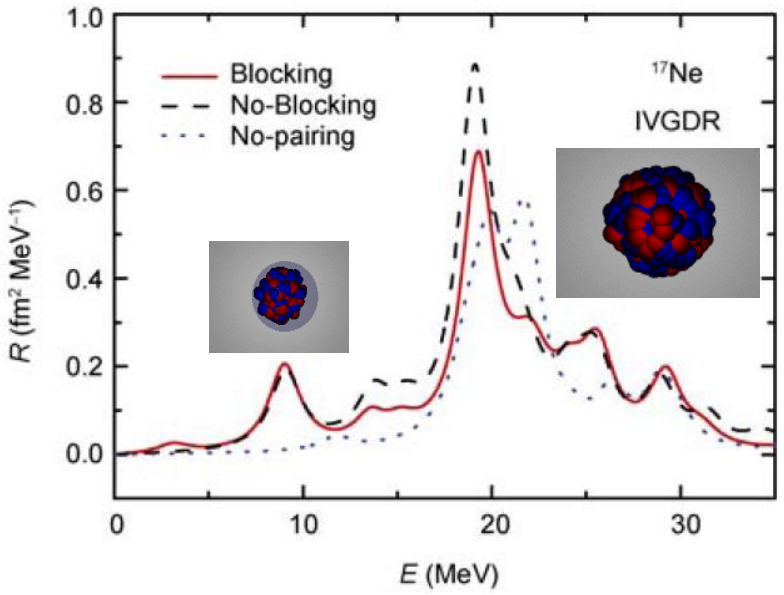


Outputs
- Reaction cross sections
- (n, γ), (p, γ), (n, f), ...
- Reaction rates

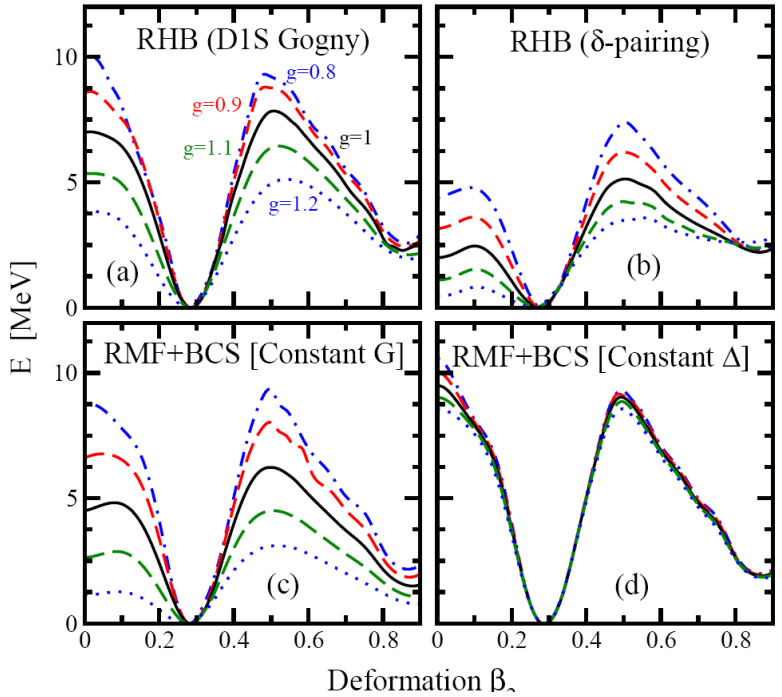
Reaction observables

$f_{E1}(E)$ $\rho(E^*)$ E_{fiss}, B_f, TKE, ...

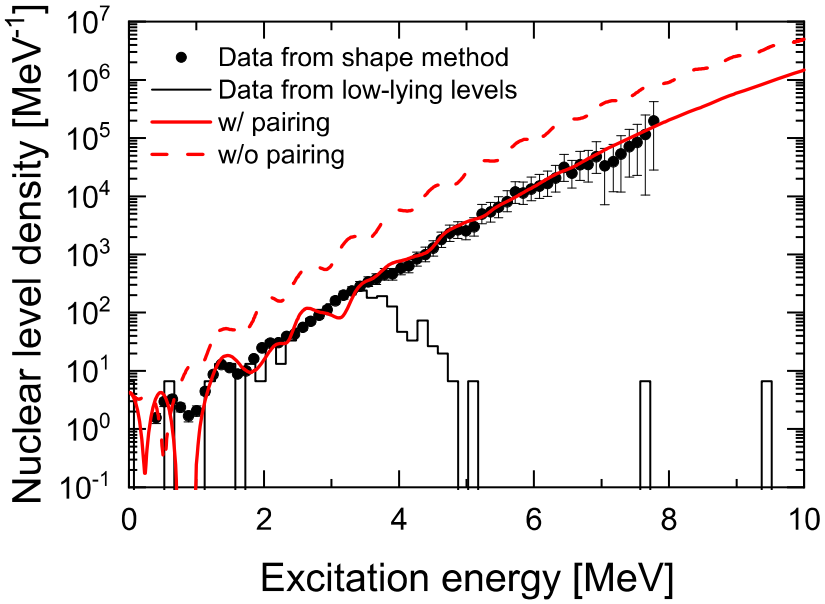
Consistent microscopic inputs with no phenomenological adjustment



SCIENCE CHINA
Physics, Mechanics & Astronomy
• Research Paper •
Radioactive Nuclear Beam Physics and Nuclear Astrophysics
August 2011 Vol.54 Suppl. 1: s49–s52
doi: 10.1007/s11433-011-4416-8



Physics Letters B 689 (2010) 72–8



Phys. Lett. B 849 (2024) 138448

A self-consistent microscopic treatment of pairing correlations in finite nuclei is essential for improving predictions of giant dipole resonances, fission barriers, and nuclear level densities.

Thank you for your attention !

Physics Letters B 753 (2016) 13–17

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Physics Letters B

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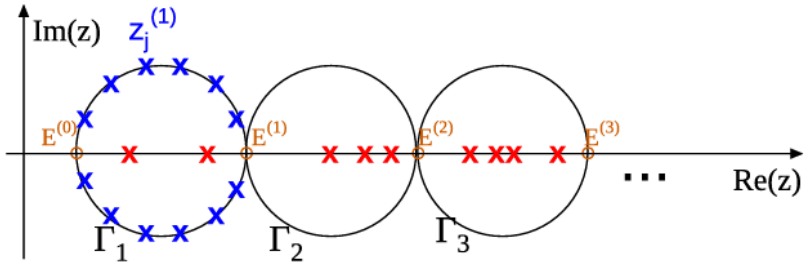
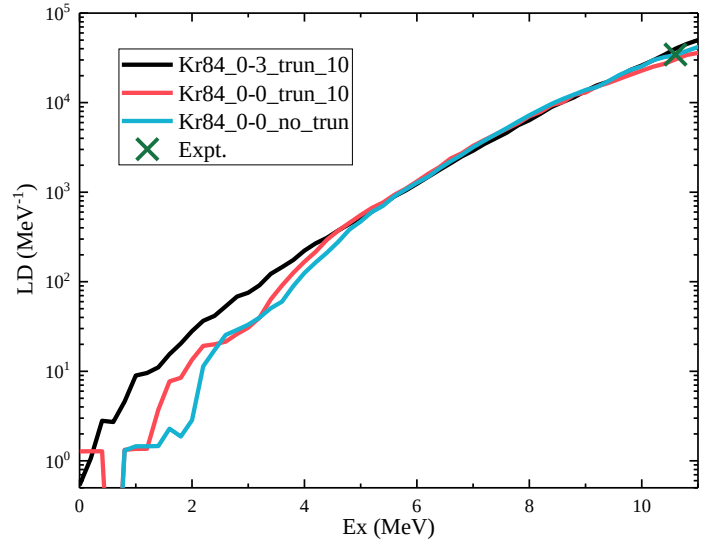
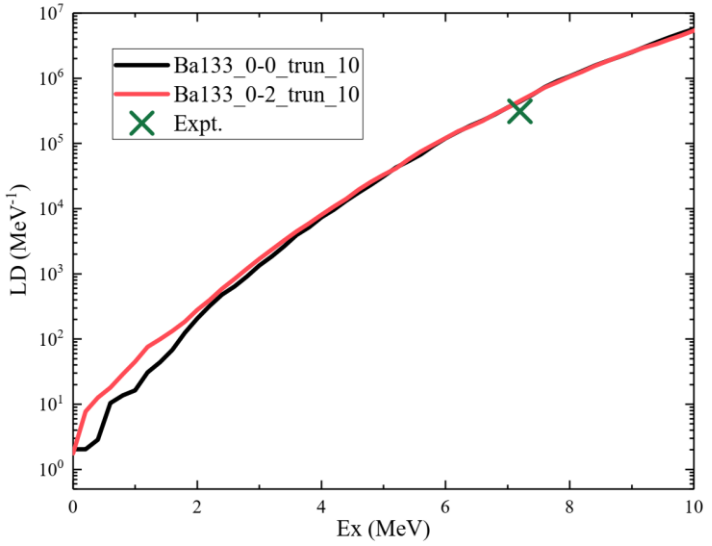
Stochastic estimation of nuclear level density in the nuclear shell model: An application to parity-dependent level density in ^{58}Ni

Noritaka Shimizu [a,*], Yutaka Utsuno [a,b], Yasunori Futamura [c], Tetsuya Sakurai [c,d], Takahiro Mizusaki [e], Takaharu Otsuka [a,f,g,h]

[a] Center for Nuclear Study, the University of Tokyo, Hongo, Tokyo, 113-0033, Japan
[b] Advanced Science Research Center, Japan Atomic Energy Agency, Tokai, Ibaraki 319-1195, Japan
[c] Faculty of Engineering, Information and Systems, University of Tsukuba, Tsukuba 305-8573, Japan
[d] CREST, Japan Science and Technology Agency, Kawaguchi 332-0012, Japan
[e] Institute of Natural Sciences, Senshu University, Tokyo 101-8425, Japan
[f] Department of Physics, the University of Tokyo, Hongo, Tokyo 113-0033, Japan
[g] National Superconducting Cyclotron Laboratory, Michigan State University, East Lansing, MI 48824, USA
[h] Instituut voor Kern- en Stralingsfysica, Katholieke Universiteit Leuven, B-3001 Leuven, Belgium

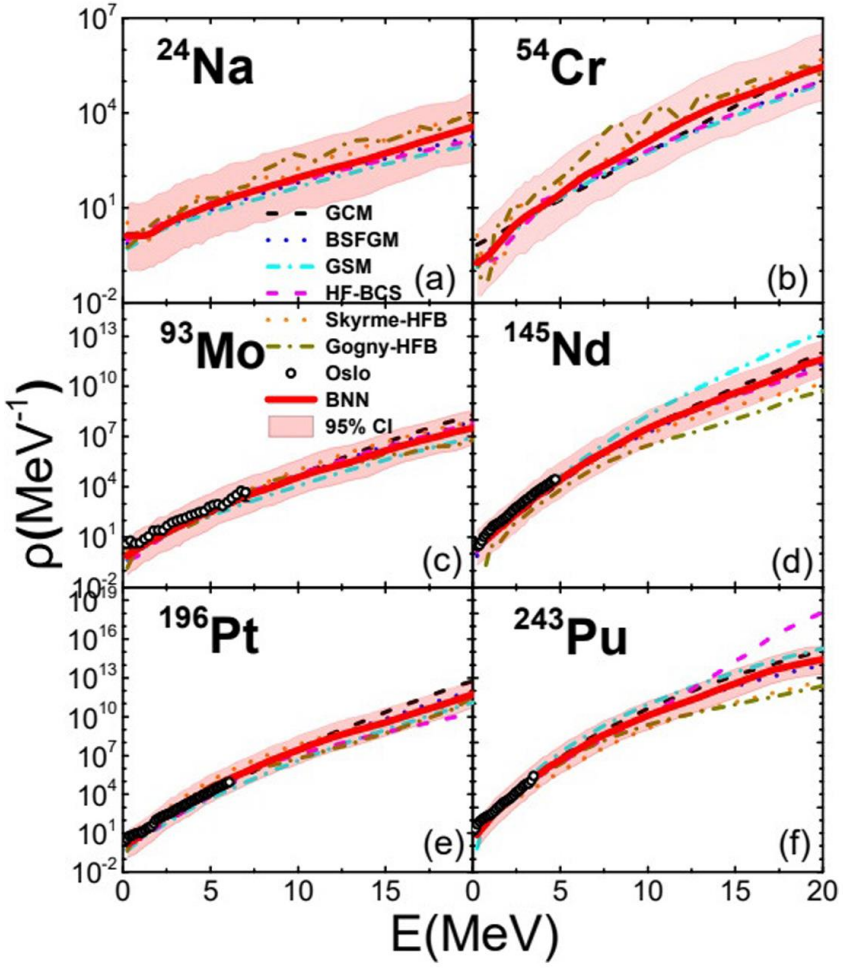


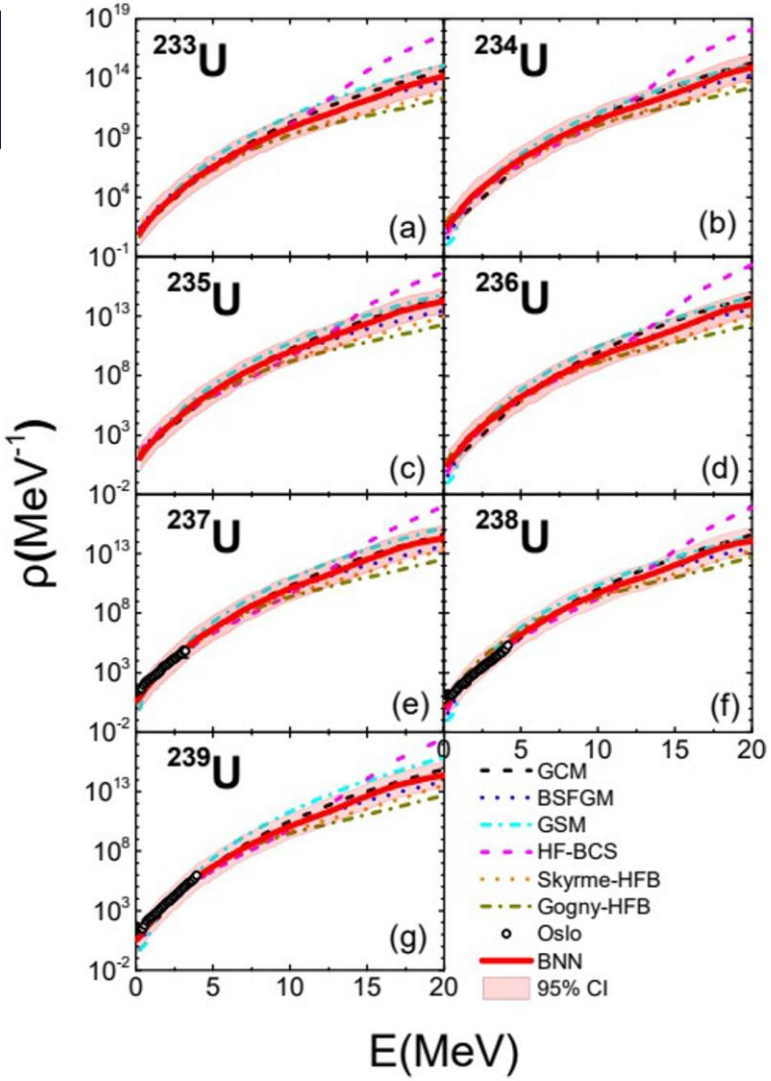


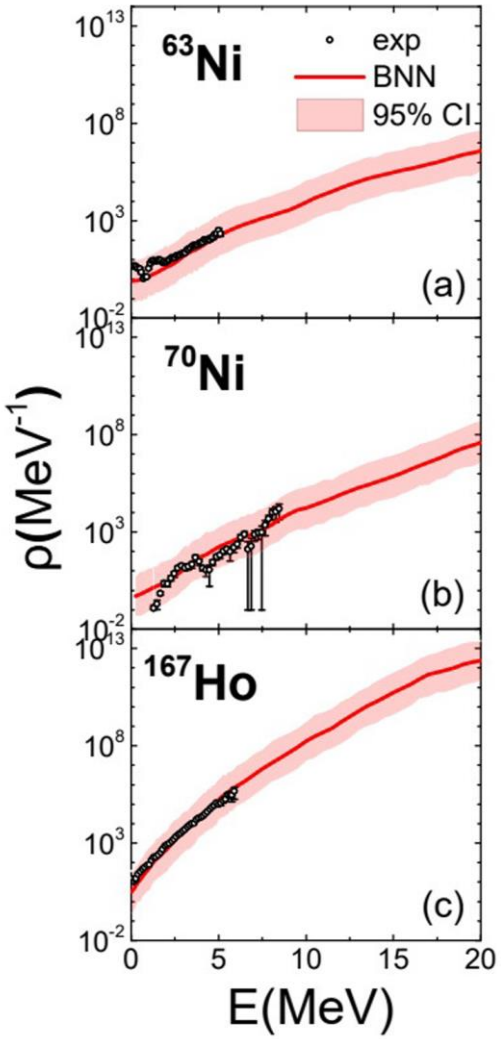


Collaborate with Prof. Yuan from ZhongShan University

Uncertainties of nuclear level density estimated using Bayesian neural networks

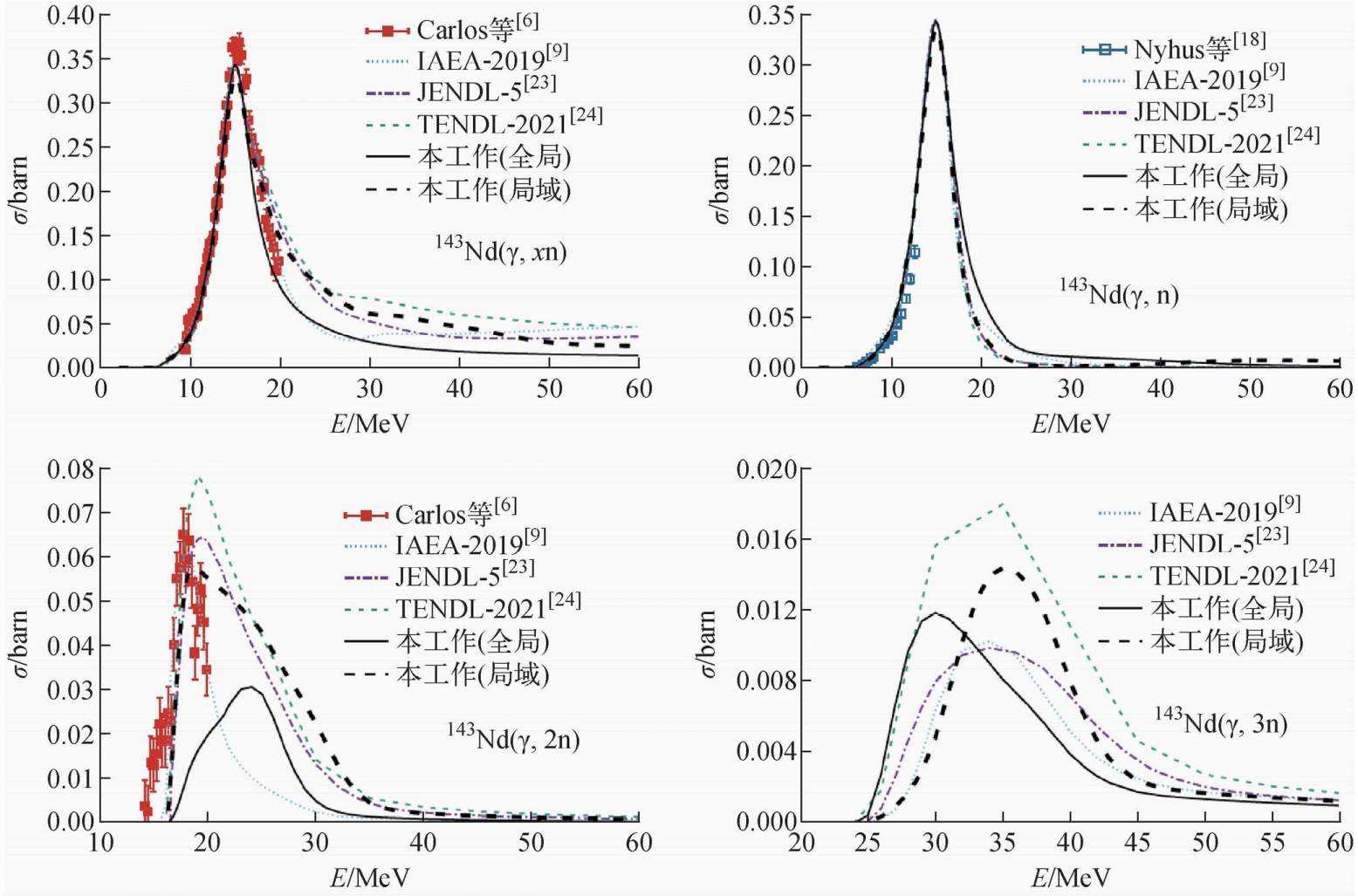






Chinese Physics C Vol. 48, No. 8 (2024) 084105

Application for Nd isotope



3.1 The GCCI (Gilbert-Cameron-Cook-Ignatyuk) level density parameter[37]

$$a = a_c \frac{1-e^{-u}}{u} + a_1\left(1 - \frac{1-e^{-u}}{u}\right), \quad (19)$$

where $u = \mathrm{fued}(U-\Delta)$

$$a_c = \begin{cases} A/8.0, & \text{if } Z<9 \text{ or } N<9, \\ A(A_{ss}\cdot S + Q_b), & \text{for other nuclei}, \end{cases} \quad (20)$$

$$a_1 = (aif1 - aif2\cdot A)A, \quad (21)$$

$$\text{or} \quad a_1 = (aif1\cdot A + aif2\cdot A^{2/3}), \quad (22)$$

Z, N and A are the number of charge, neutron and mass of the compound nucleus, respectively; U, Δ and S are the excited energy, pair energy and shell correction, respectively. $Fued$, $aif1$, $aif2$ can be taken as adjustable parameters in some range, usually $fued = 0.05$; for Eq. (21), $aif1 = 0.1375$, $aif2 = 8.36\times10^{-5}$[37]; for Eq. (22), $aif1 = 0.073$, $aif2 = 0.115$[40], or $aif1 = 0.0666$, $aif2 = 0.2587$[41].

$$Q_b = \begin{cases} 0.142 & \text{for spherical nuclei} \\ 0.12 & \text{for deformed nuclei} \end{cases}, \quad (23)$$

$$A_{ss} = \begin{cases} 0.00917, & \text{for Cameron-Cook}[38] \\ 0.0088, & \text{for Su Zongdi}[39] \end{cases}. \quad (24)$$