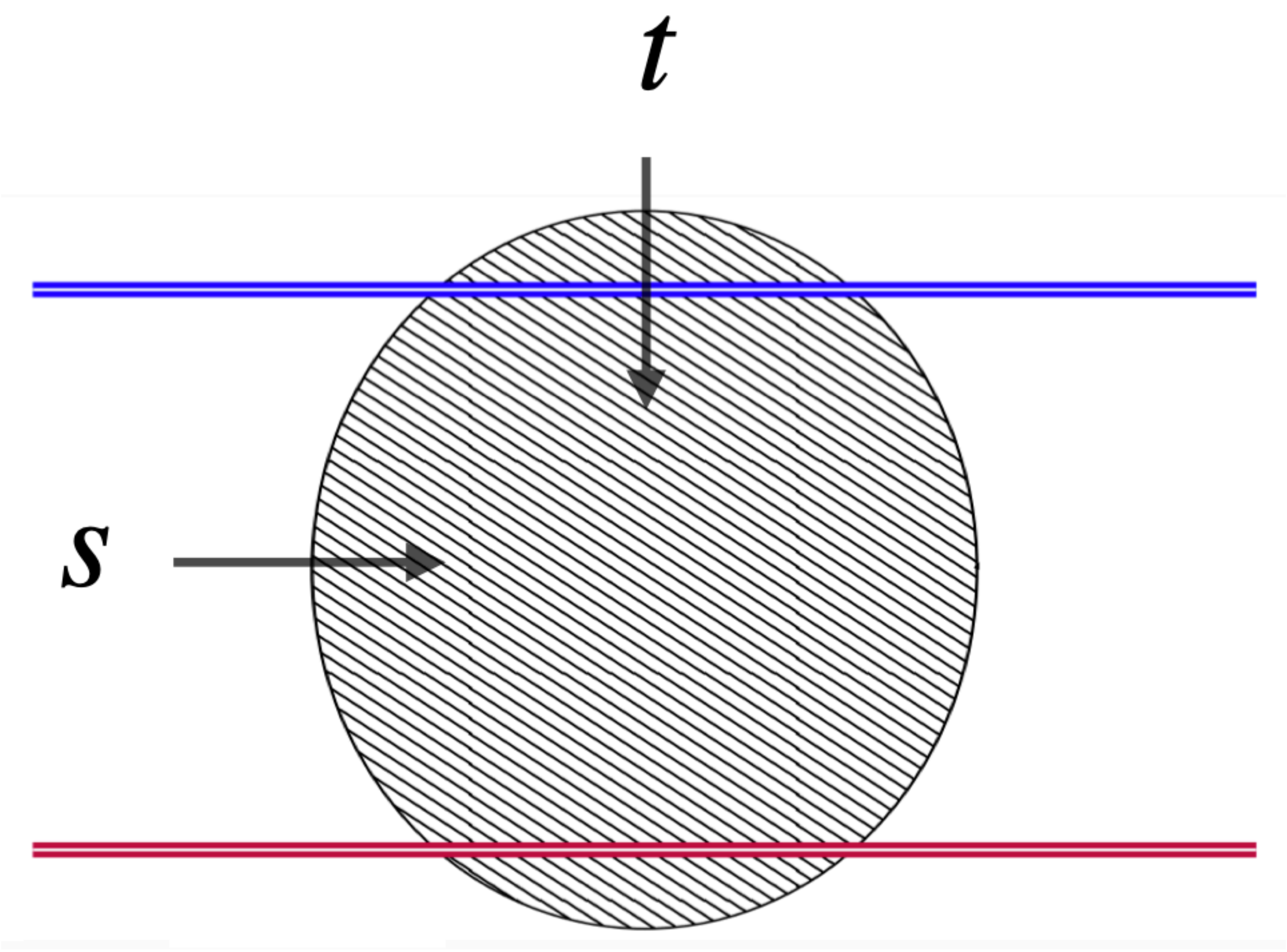
SLAC NATIONAL ACCELERATOR LABORATORY

UNIVERSALITY OF QCD AMPLITUDES AT HIGH ENERGY

With: F. Buccioni, F. Caola, G. Gambuti, JHEP 03 (2025) 129

High-energy limit aka Regge limit



$$s \sim |u| \gg -t \qquad x = -t/s$$

Large logarithms: $\log \dfrac{s}{-t}$

$$\mathscr{A} \propto 1 + \ldots \alpha_s L + \ldots (\alpha_s L)^2 + \ldots (\alpha_s L)^3 + \ldots$$

"Correct" power counting

$$\mathscr{A} \propto \mathscr{A}^{LL} + \mathscr{A}^{NLL} + \mathscr{A}^{NNLL} + \ldots$$

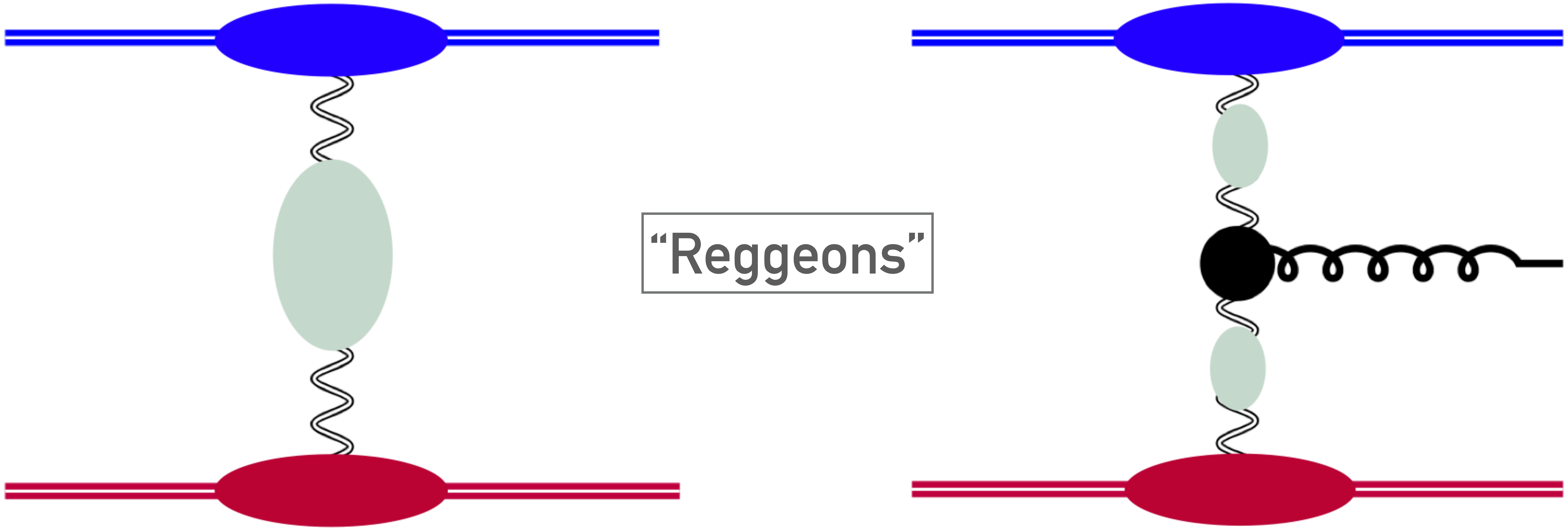
Leading log

Next-to-Leading log

Resummation required

Large separation of scales

Factorization properties

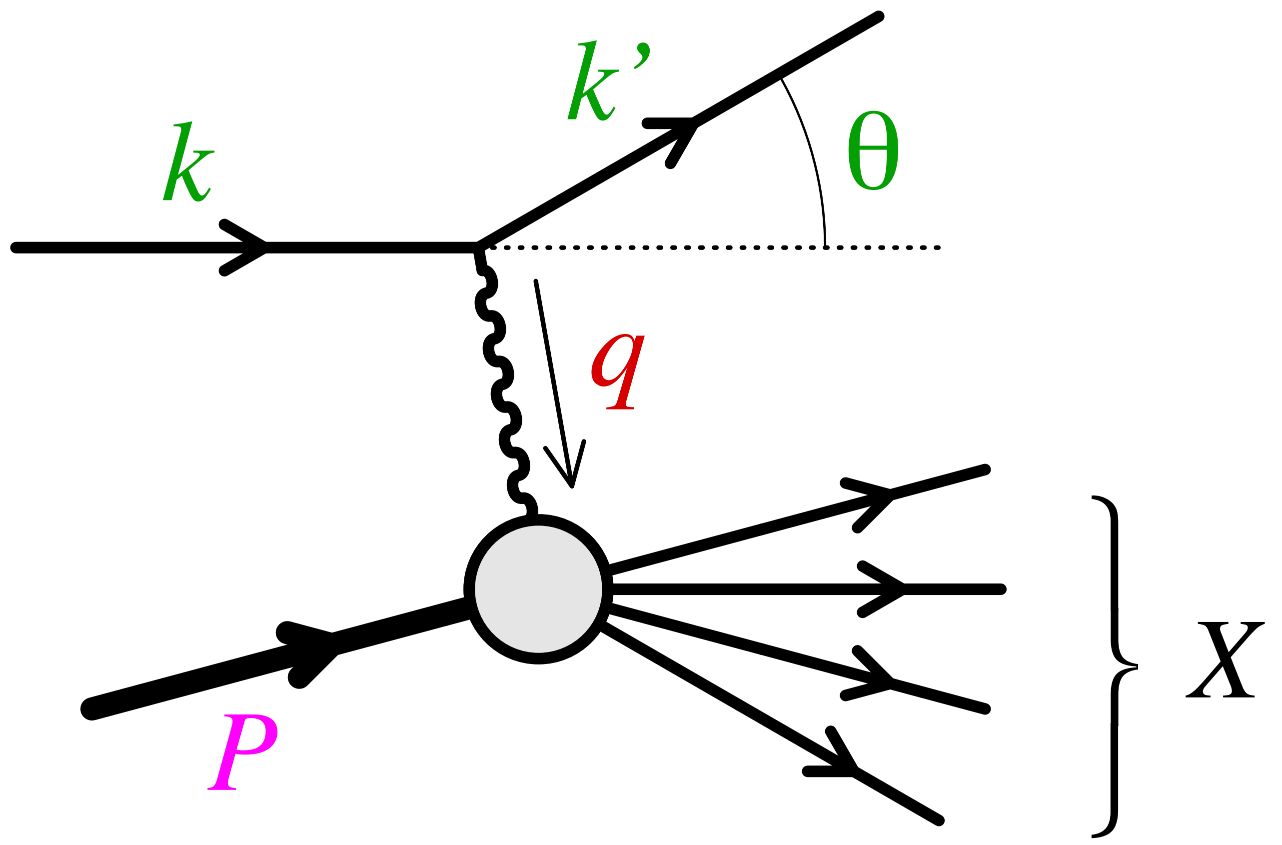


Amplitude breaks down in simpler universal "building blocks"

Thanks to Giulio Gambuti for all the artwork used throughout these slides

MOTIVATION: QCD IN DIFFERENT REGIMES

Deep Inelastic Scattering

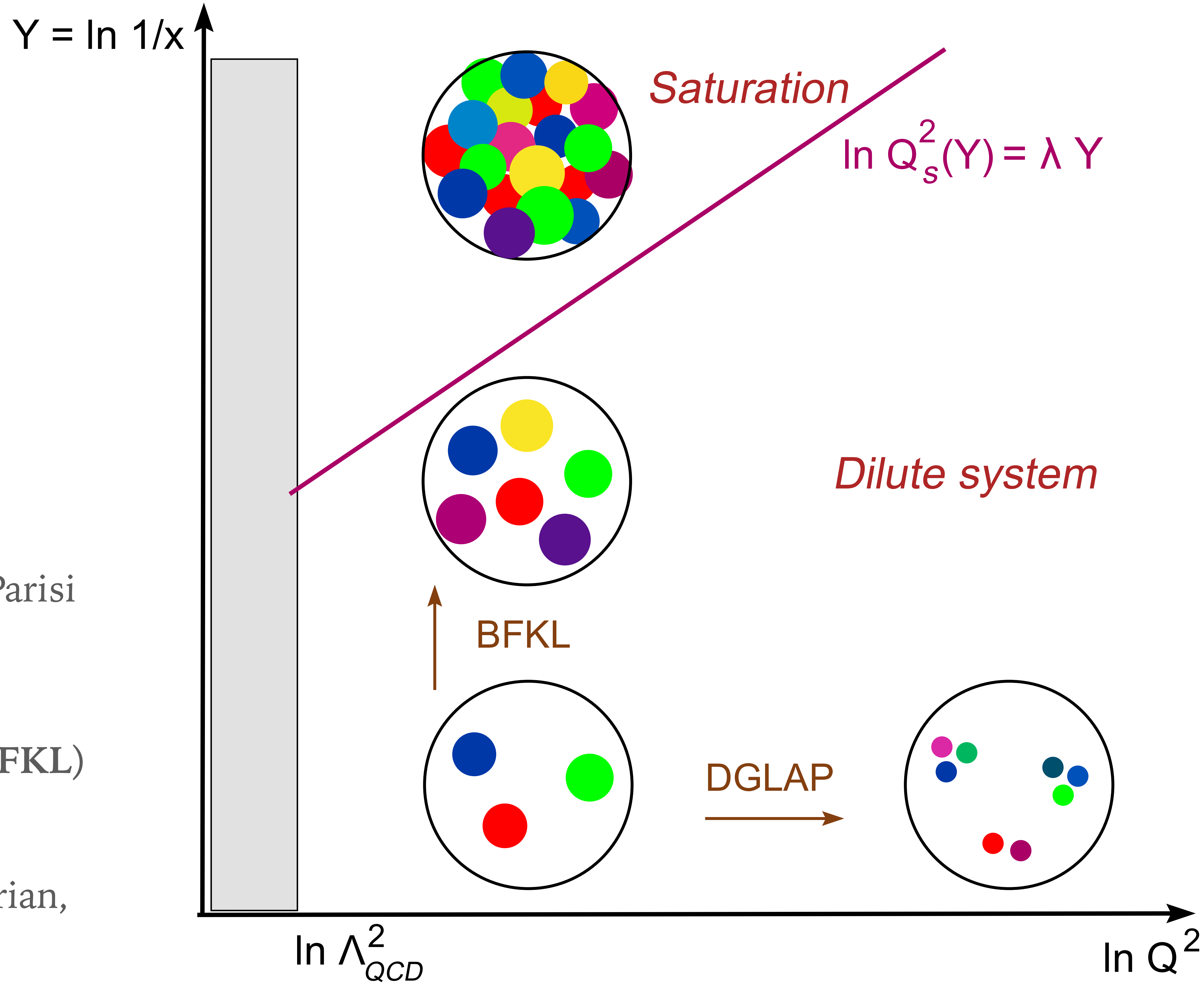


$$Q^2 \equiv -q^2$$

$$x = \frac{Q^2}{2P \cdot q}$$

$$s = (P + k)^2$$

[Gelis et al 1002.0333]



- fixed x and Q^2, $s \to \infty$: Dokshitzer–Gribov–Lipatov-Altarelli–Parisi (**DGLAP**)
- fixed Q^2 and $x \to 0$, $s \to \infty$: Balitsky-Fadin-Kuraev-Lipatov (**BFKL**)
- saturation/dense regime: **Balitsky** (+ Kovchegov)/Jalilian-Marian, Iancu, McLerran, Weigert, Leonidov and Kovner (**JIMWLK**)

High energy evolution: State of the art

Planar N=4 sYM → Resummation to all log orders

QCD & Yang-Mills → Resummation up to NLL

New results

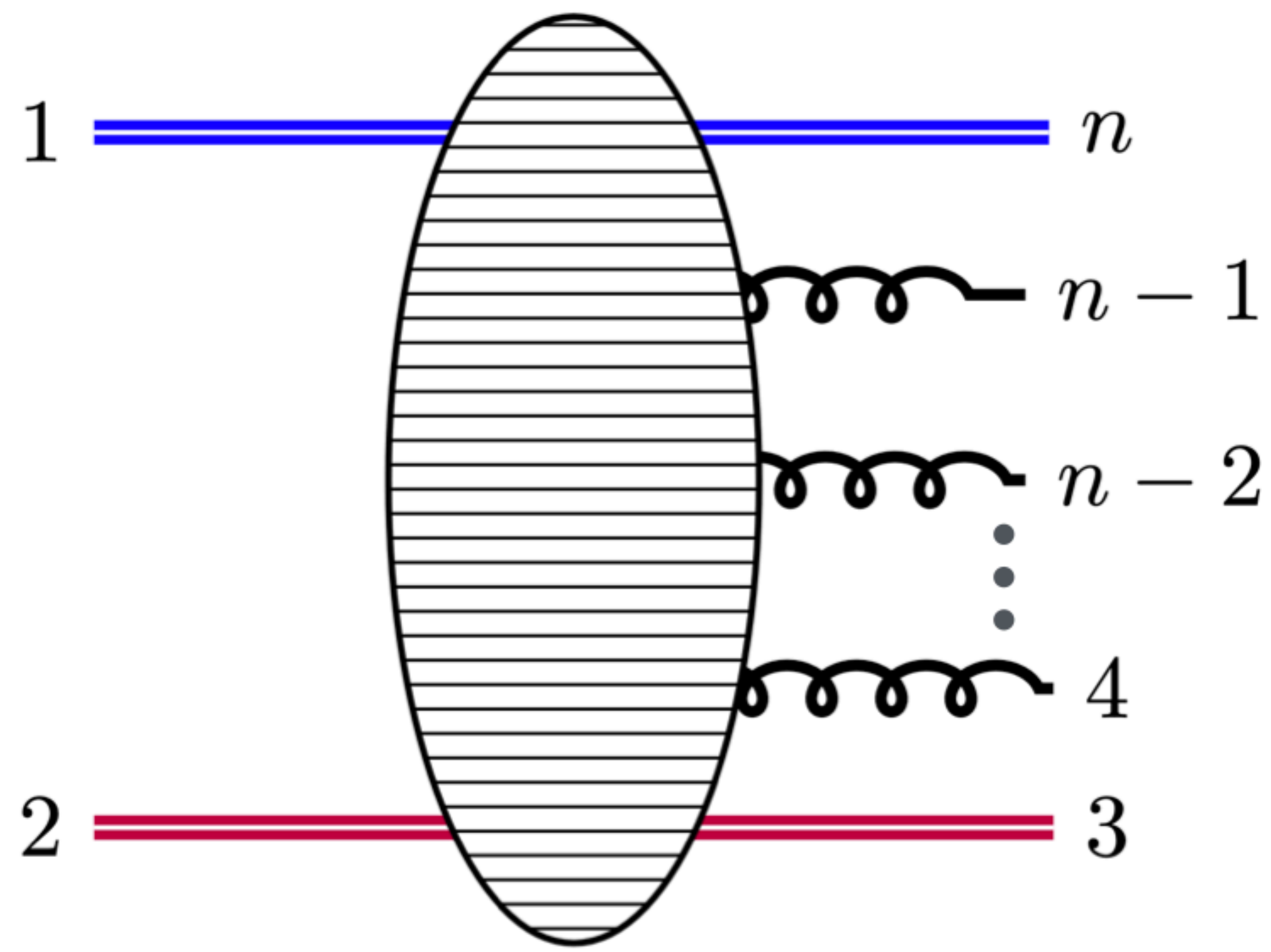
Extraction of the 2-loop Lipatov vertex in QCD, first steps towards extending BFKL at NNLL

Factorization breaking effects starting at NNLL

Outline

1. Multi-Regge kinematics of $2 \to N$ scattering amplitudes
2. Amplitudes from shockwave formalism
3. Factorization and universality of scattering amplitudes in Regge limit
4. Extraction on new NNLL building block: 2-loop Lipatov vertex

Multi-Regge kinematics (MRK)



Large rapidity gaps

$$\eta_2 \sim \eta_3 \ll \eta_4 \ll \ldots \ll \eta_n \sim \eta_1$$

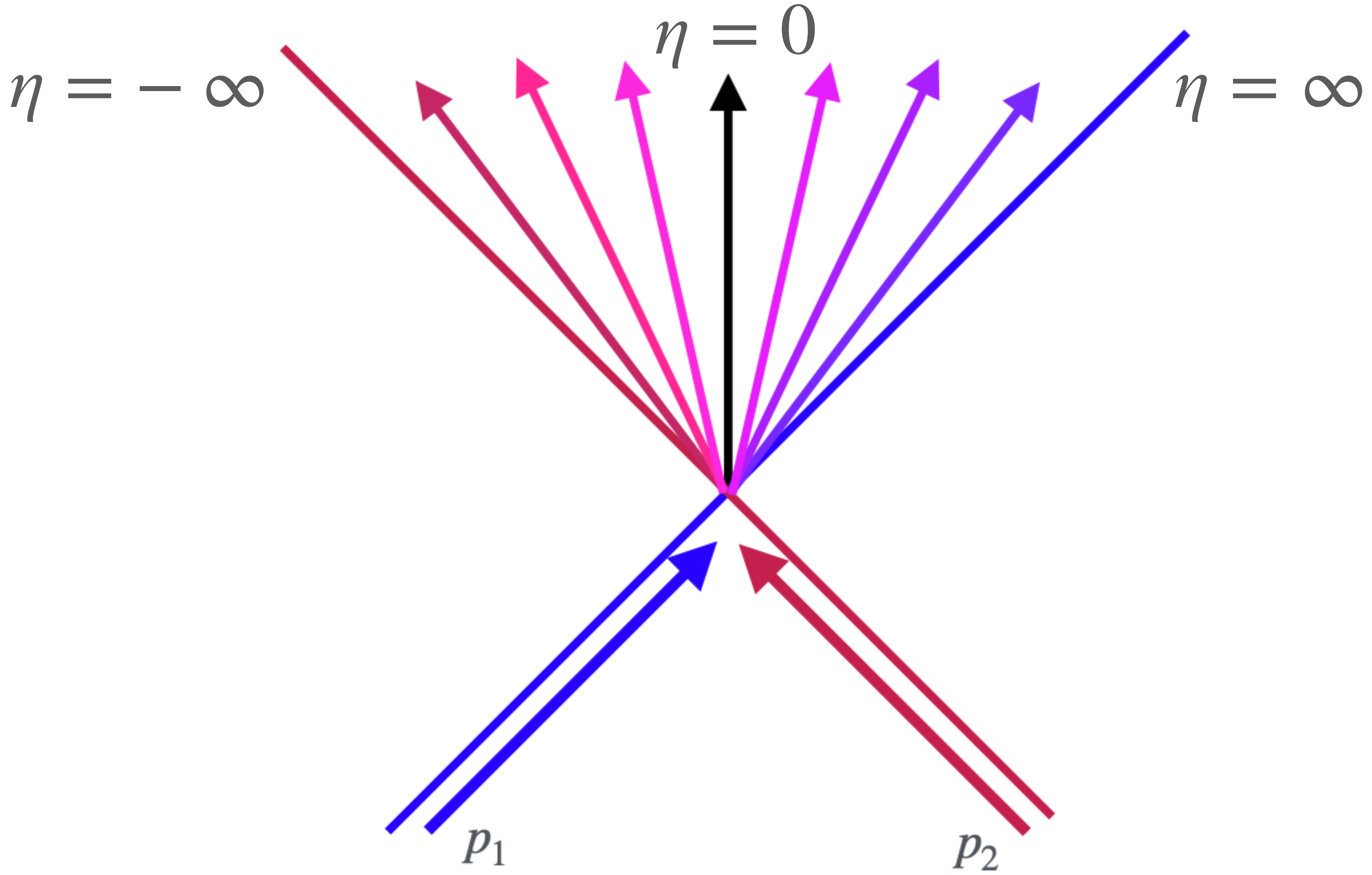
No hierarchy in transverse momenta

Lightcone coordinates

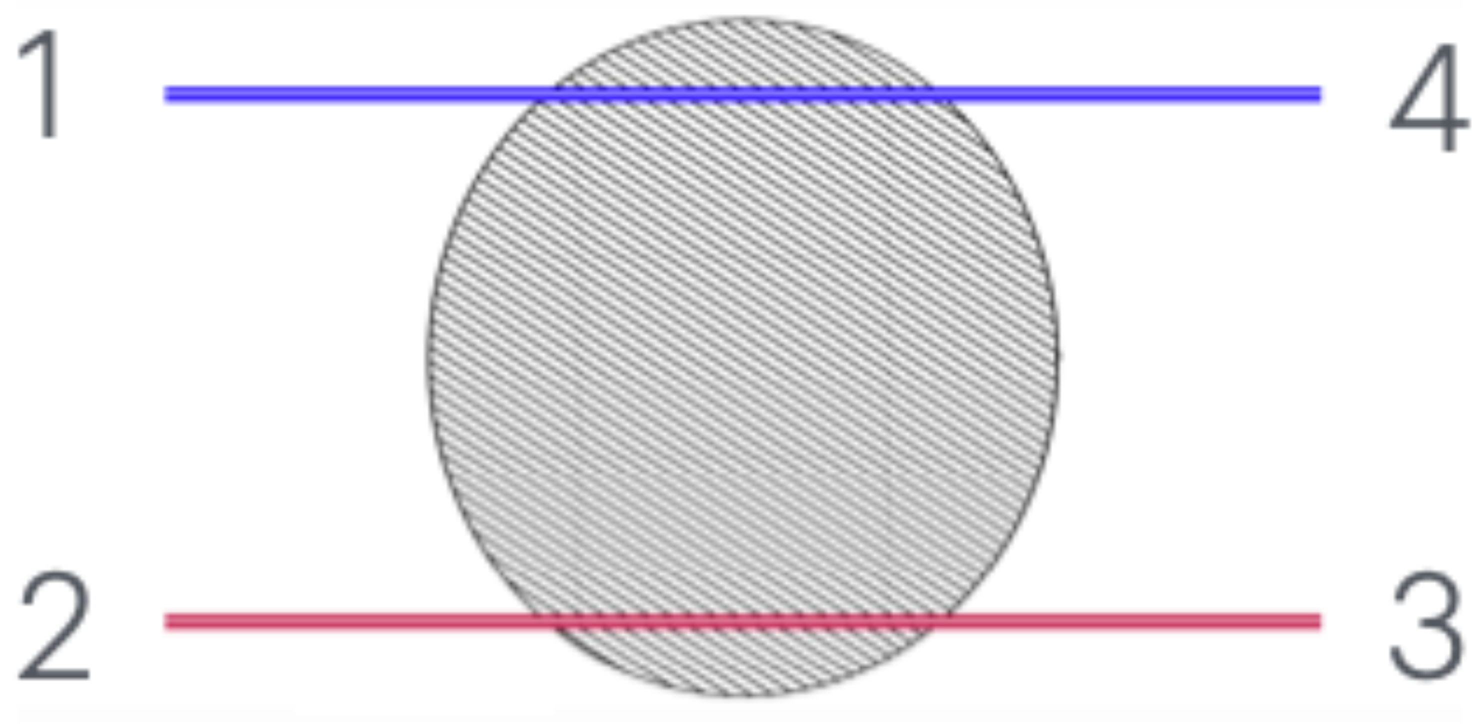
$$p^{\pm} = p^0 \pm p^3$$

$$\mathbf{p} = (p^1, p^2)$$

$$\eta_i = \frac{1}{2} \log \frac{p_i^+}{p_i^-}$$



$2 \to 2$ SCATTERING AT HIGH ENERGY

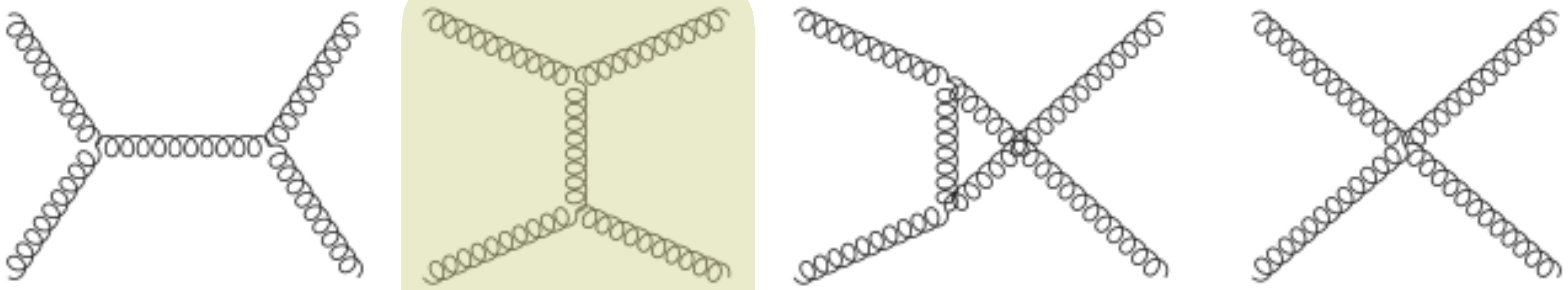


$$s \sim |u| \gg -t \qquad x = -t/s$$

$$\eta_2 \sim \eta_3 \ll \eta_4 \sim \eta_1$$

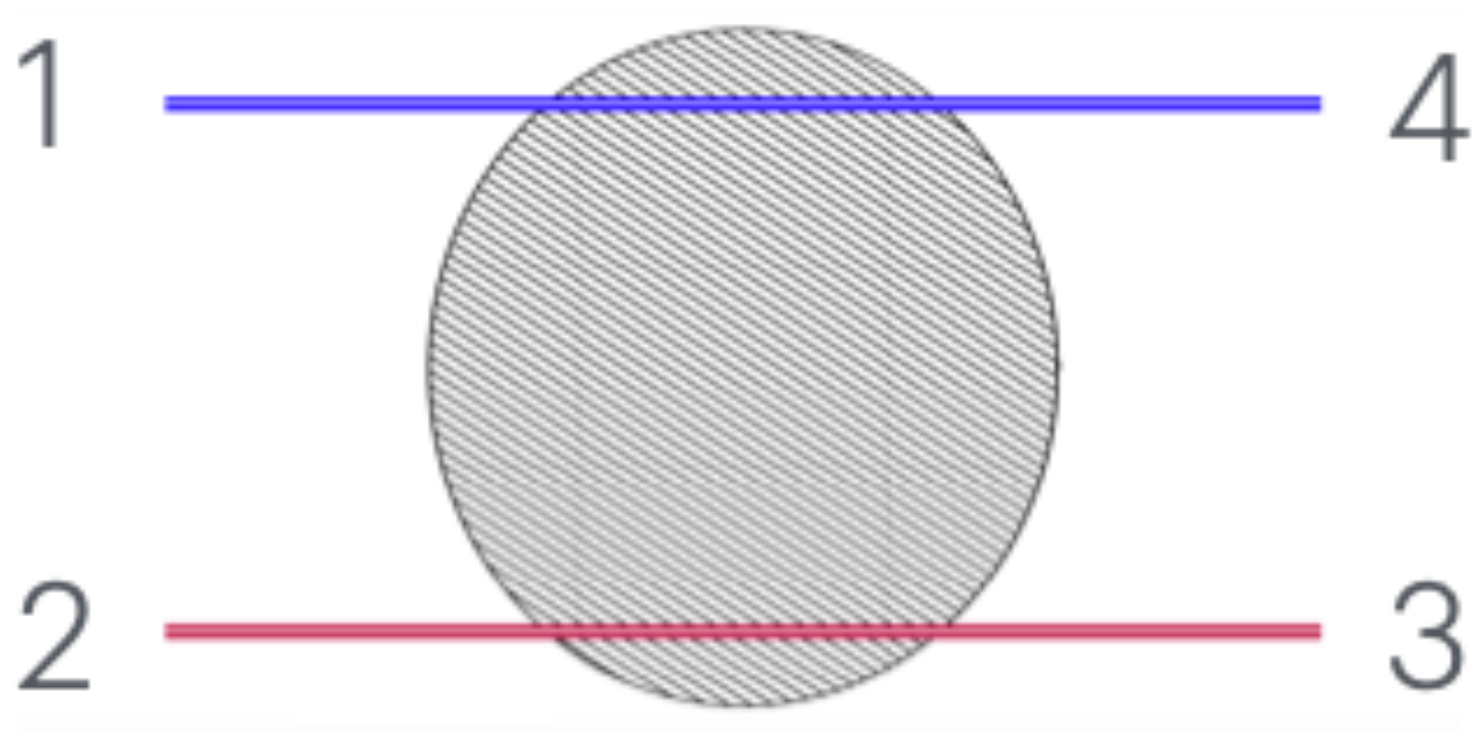
$$\Delta\eta_{43} = \log \frac{1}{x} \to \infty$$

$gg \to gg$ @ tree level



$\sim \dfrac{1}{s}$ $\quad \sim \dfrac{1}{t}$ $\quad \sim \dfrac{1}{s+t}$ $\quad$ constant

$2 \to 2$ SCATTERING AT HIGH ENERGY

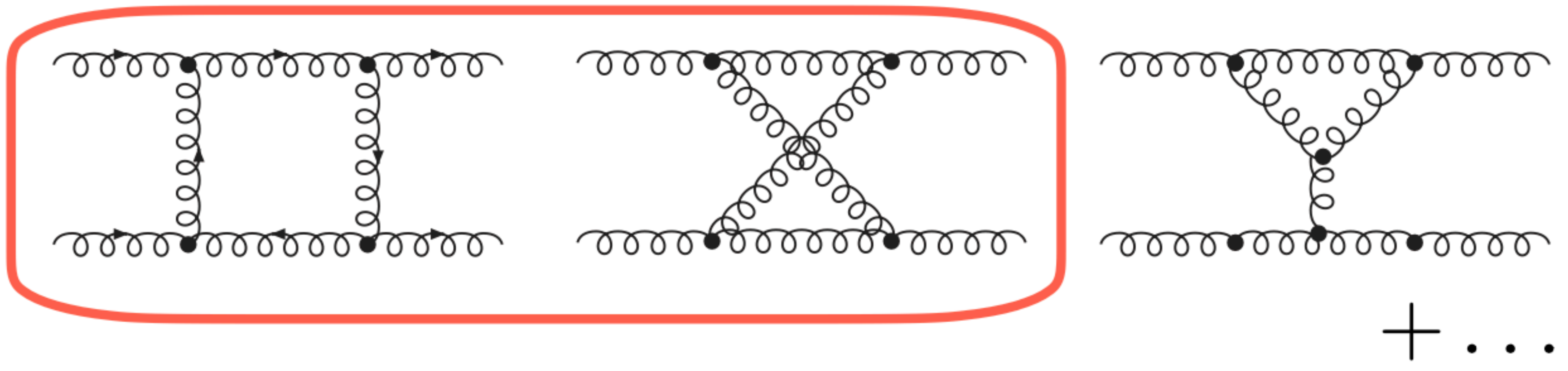


$$s \sim |u| \gg -t \qquad x = -t/s$$

$$\eta_2 \sim \eta_3 \ll \eta_4 \sim \eta_1$$

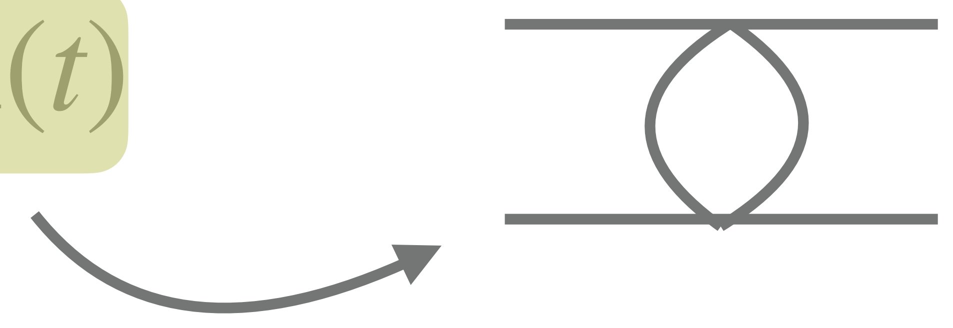
$$\Delta\eta_{43} = \log\frac{1}{x} \to \infty$$

$gg \to gg$ @ one loop

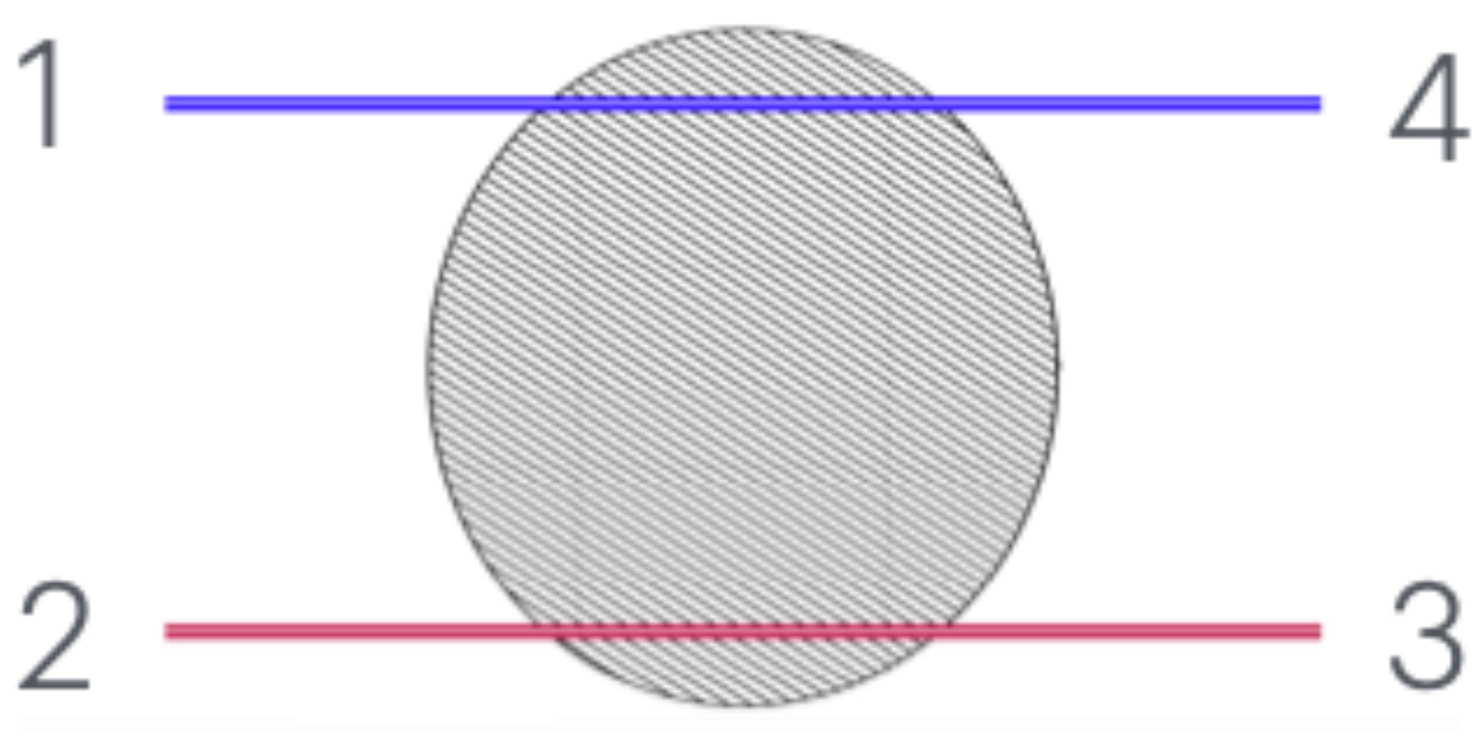


Structure is repeated to all loop orders. Main contributions from (generalized) ladders

$$\sim \frac{\log x}{x}\,\tau_g(t)$$



$2 \to 2$ SCATTERING AT HIGH ENERGY



$$s \sim |u| \gg -t \qquad x = -t/s$$

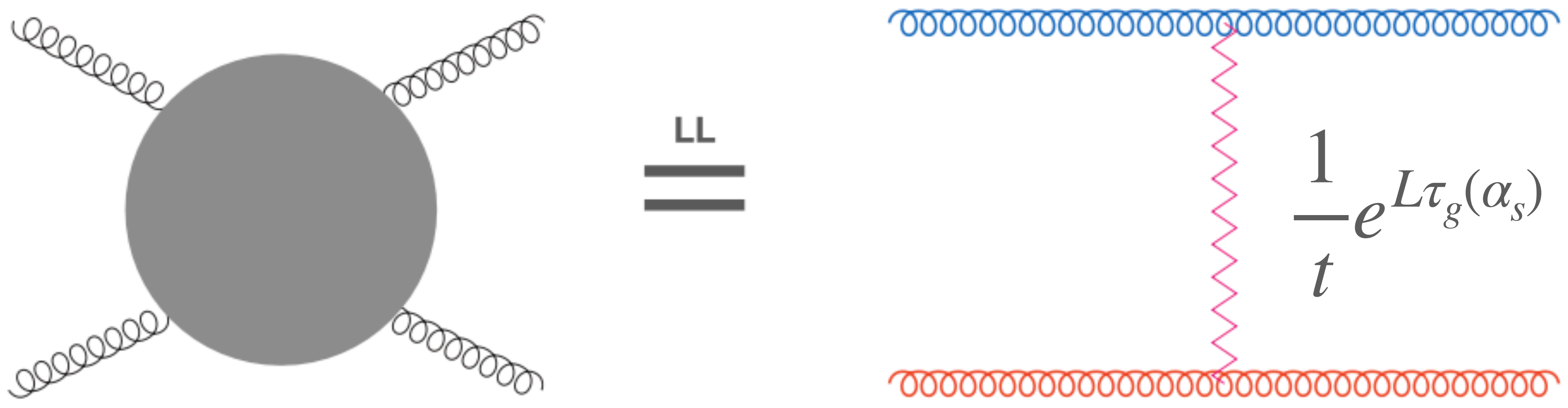
$$\eta_2 \sim \eta_3 \ll \eta_4 \sim \eta_1$$

$$\Delta\eta_{43} = \log\frac{1}{x} \to \infty$$

$gg \to gg$ @ Leading Log (LL)

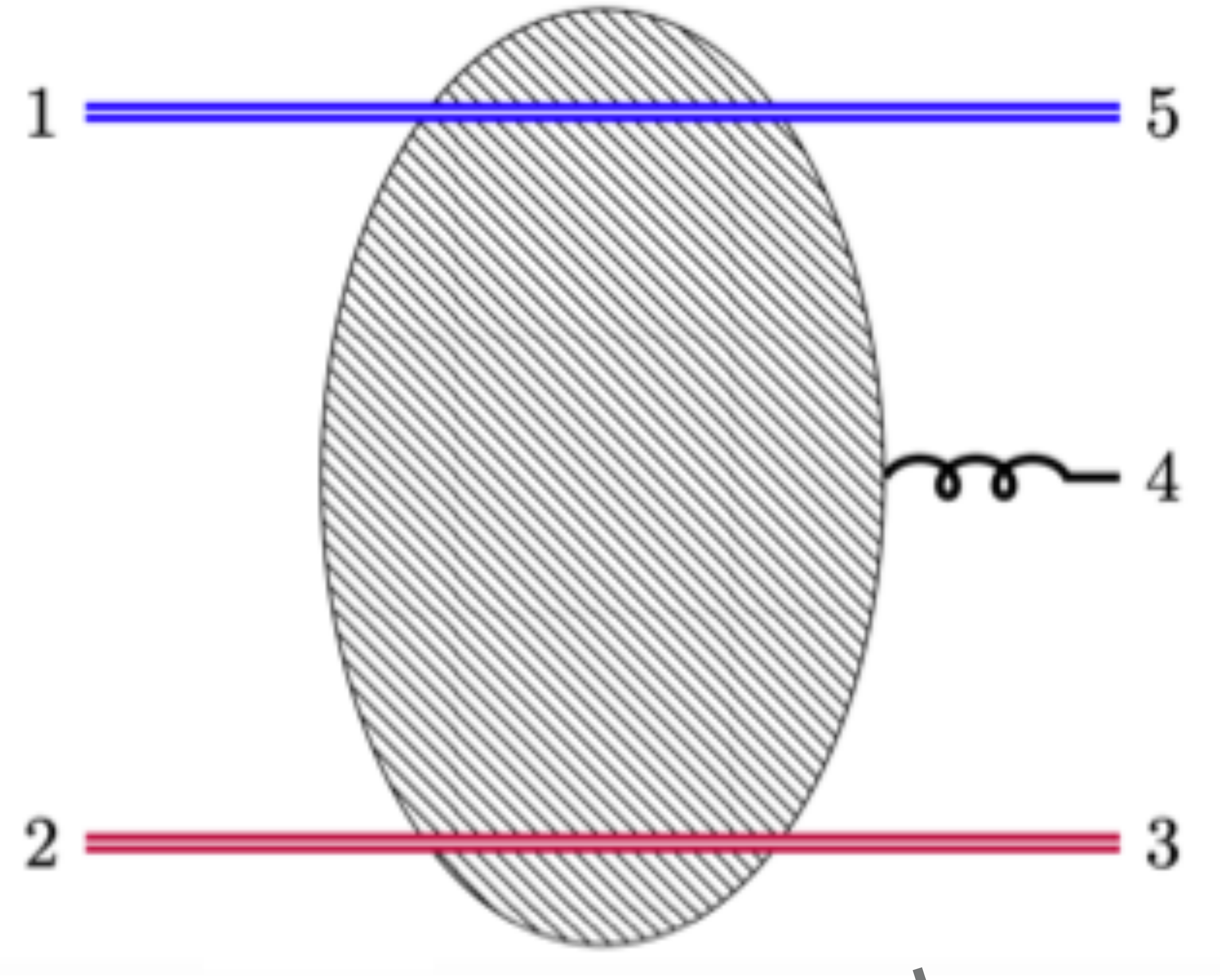
Structure is repeated to all loop orders. Main contributions from (generalized) ladders

$$\mathscr{A} \propto 1 + \ldots\alpha_s L + \ldots(\alpha_s L)^2 + \ldots(\alpha_s L)^3 + \ldots$$



"effective" DOF reggeized gluon

$2 \to 3$ SCATTERING AT HIGH ENERGY



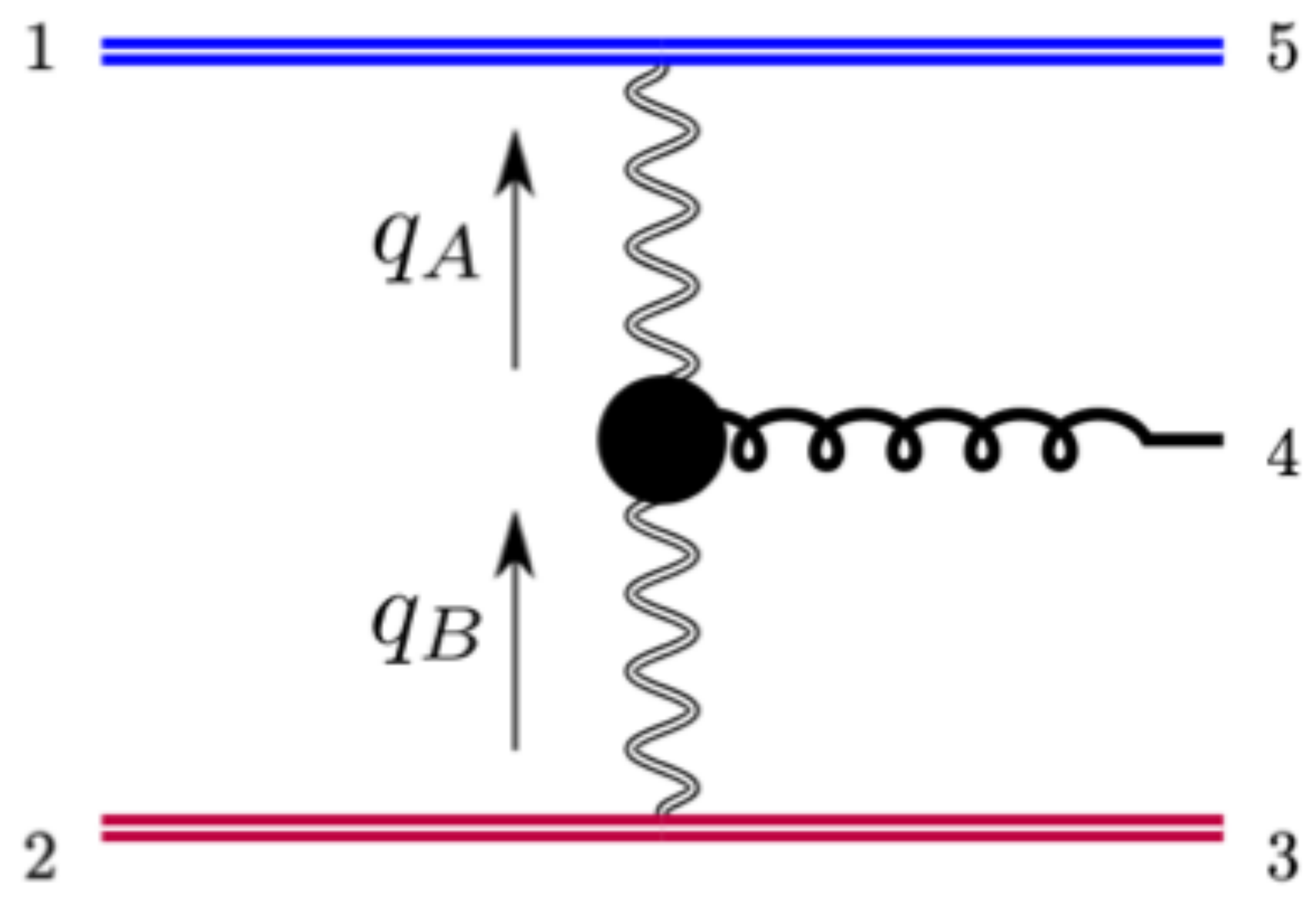
$$\Delta\eta_{54} \sim \log\left(\frac{s_{45}}{-s_{51}}\right)$$

$$\Delta\eta_{43} \sim \log\left(\frac{s_{34}}{-s_{23}}\right)$$

$$p_5^+ \gg p_4^+ \gg p_3^+\,, \quad p_3^- \gg p_4^- \gg p_5^-$$

$$p_4^\pm \sim |p_{3,\perp}| \sim |p_{4,\perp}| \sim |p_{5,\perp}|$$

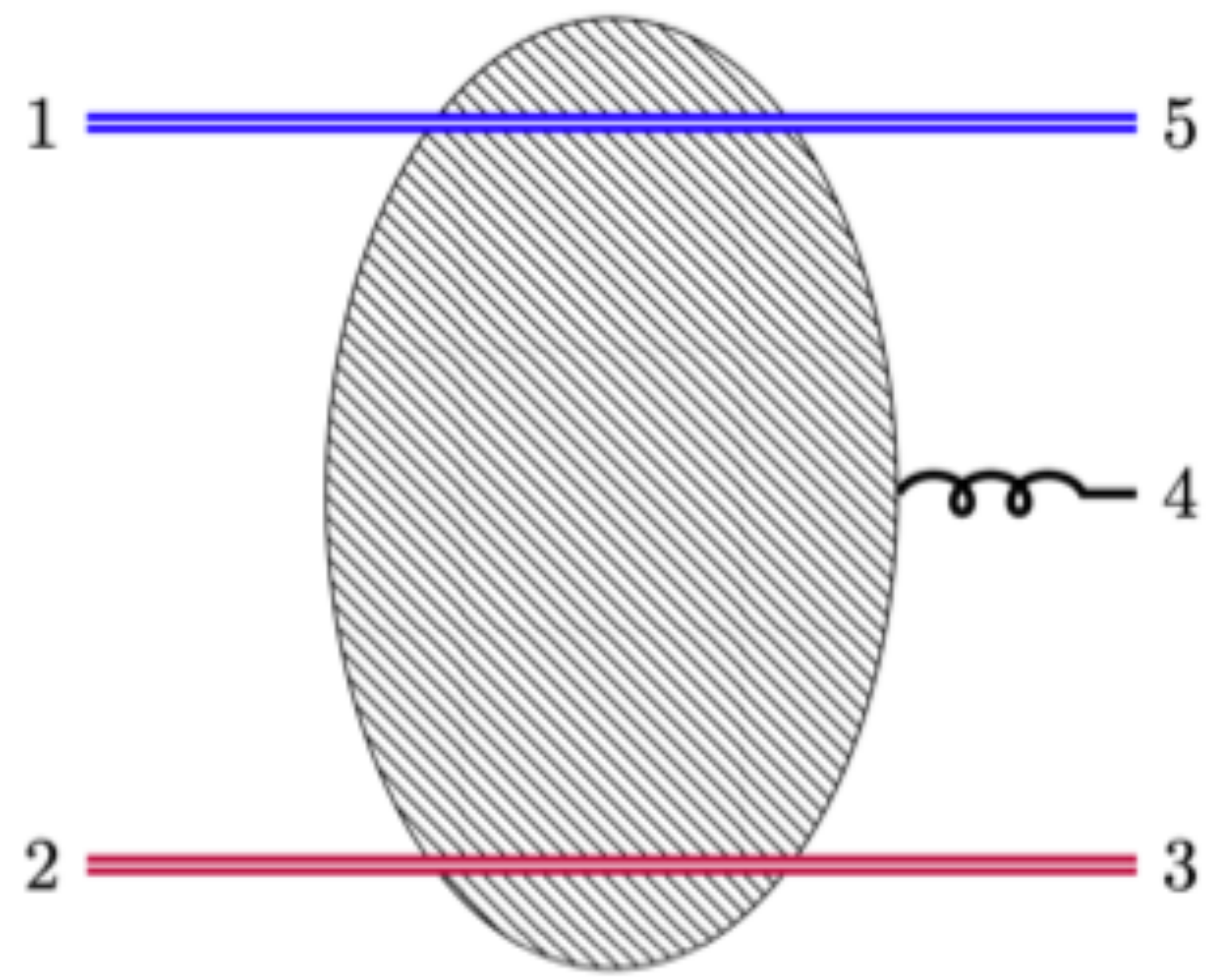
LL



$$\frac{1}{s_{51}} e^{L_{45}\tau_g}$$

$$\frac{1}{s_{23}} e^{L_{34}\tau_g}$$

$2 \to 3$ SCATTERING AT HIGH ENERGY



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$$\Delta\eta_{43} \sim \log\left(\frac{s_{34}}{-s_{23}}\right)$$

$$p_5^+ \gg p_4^+ \gg p_3^+\,, \quad p_3^- \gg p_4^- \gg p_5^-$$

$$p_4^\pm \sim |p_{3,\perp}| \sim |p_{4,\perp}| \sim |p_{5,\perp}|$$

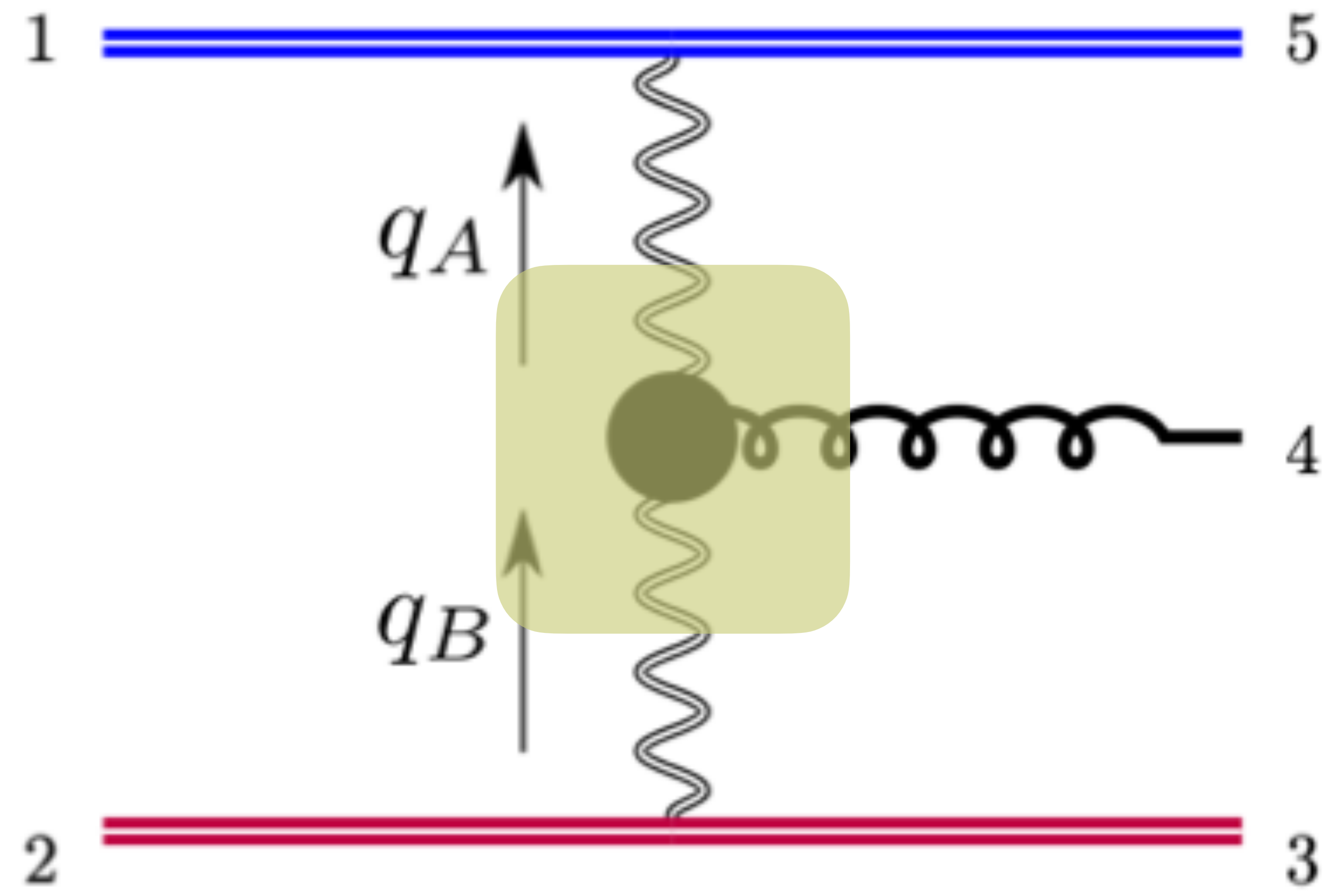
LL

Lipatov vertex

$$V_+(\mathbf{q}_A, \mathbf{p}_4) = \frac{\bar{q}_{A,\perp}\, q_{B,\perp}}{p_{4,\perp}},$$

$$V_-(\mathbf{q}_A, \mathbf{p}_4) = \frac{q_{A,\perp}\, \bar{q}_{B,\perp}}{\bar{p}_{4,\perp}}$$

2 loops: this talk!



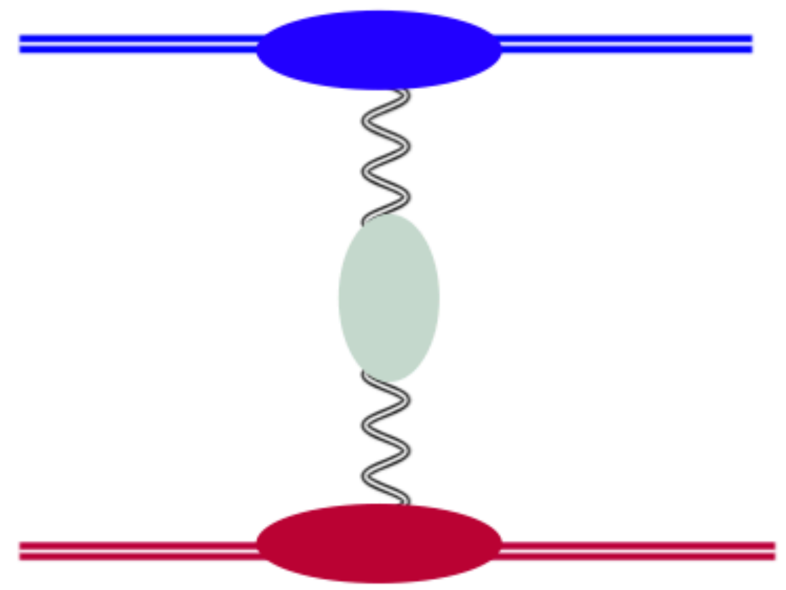
$$\frac{1}{s_{51}} e^{L_{45}\tau_g}$$

$$\frac{1}{s_{23}} e^{L_{34}\tau_g}$$

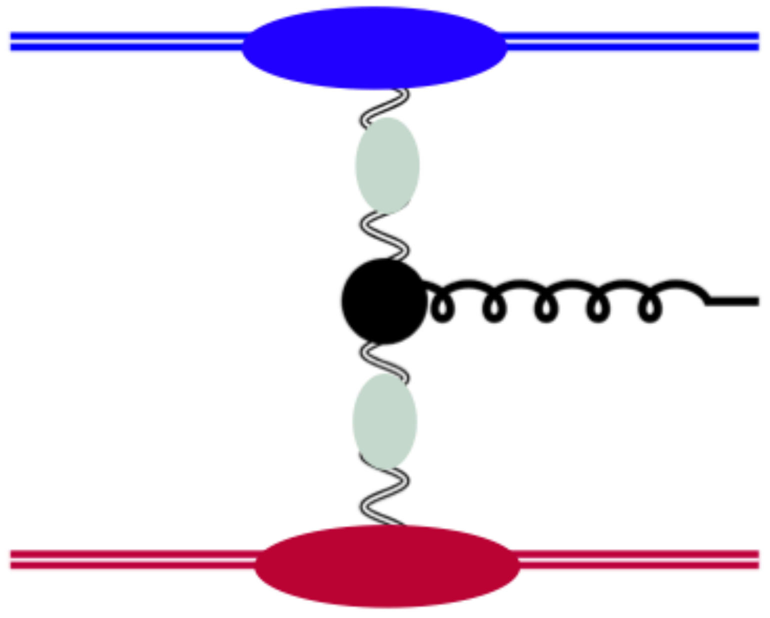
1 loop: [Del Duca, Schmidt: hep-ph/9810215][Fadin, Fucilla, Papa: 2302:09868]

LL AND BEYOND

$$\text{LL} \sim \left(\frac{\alpha_s}{2\pi}\log x\right)^n \qquad \text{NLL} \sim \frac{\alpha_s}{2\pi}\left(\frac{\alpha_s}{2\pi}\log x\right)^n \qquad \text{NNLL} \sim \left(\frac{\alpha_s}{2\pi}\right)^2\left(\frac{\alpha_s}{2\pi}\log x\right)^n \qquad \ldots$$

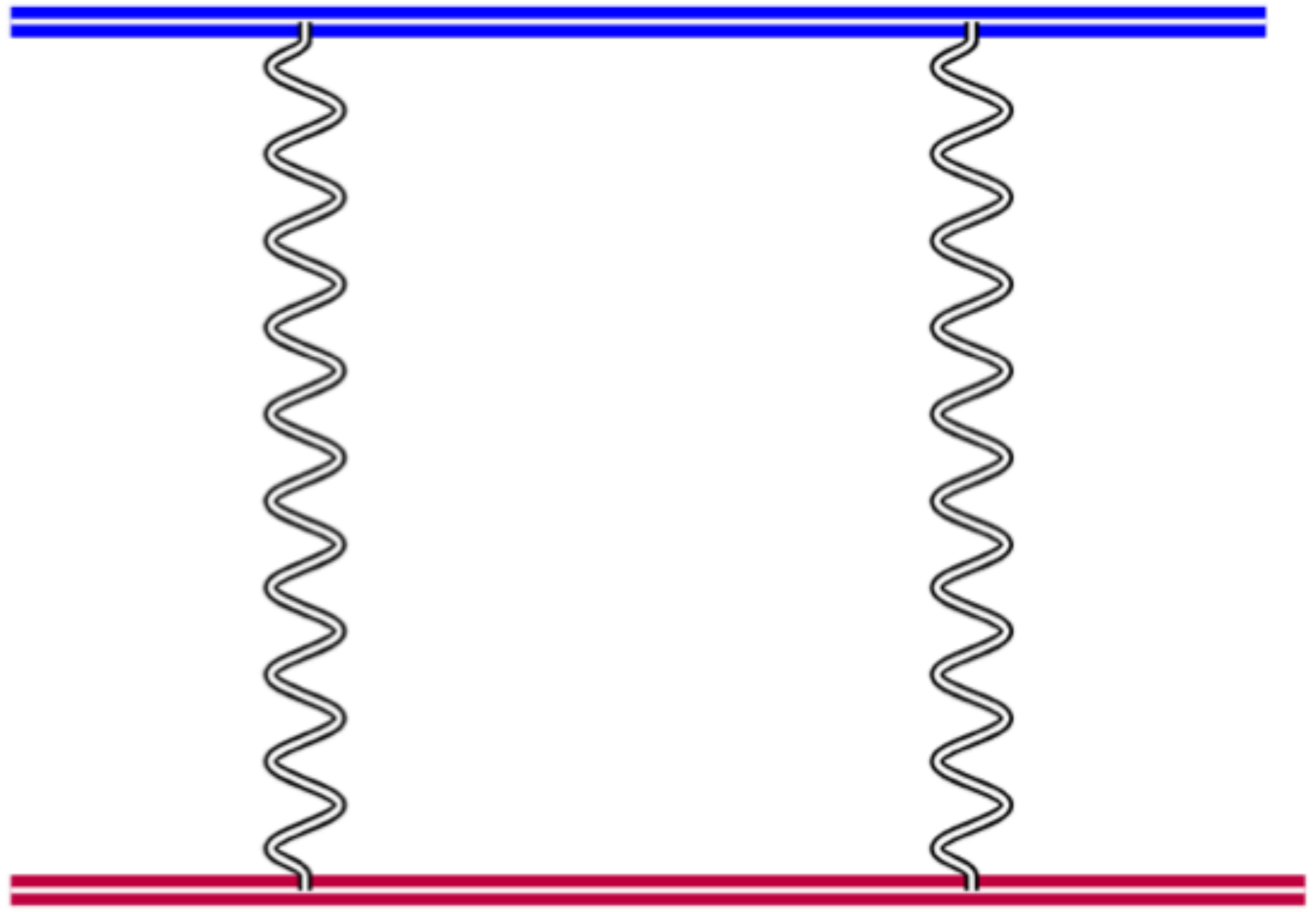


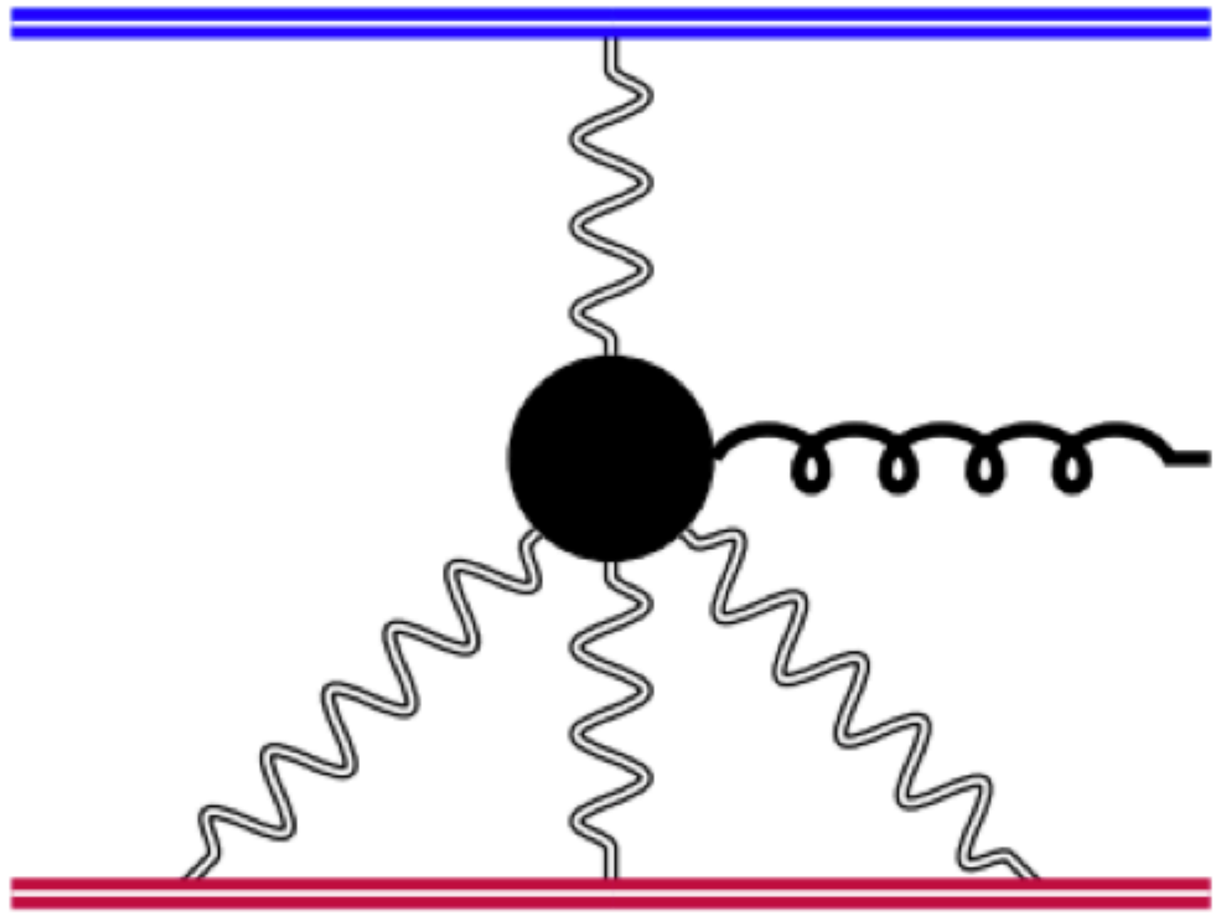
$$\mathcal{A}_{2\to 2} \propto T^a \cdot \delta^{ab}\left(\frac{s}{t}\right) e^{L_{12}\,\tau_g} \cdot T^b$$



$$\mathcal{A}_{2\to 3} \propto T^a \cdot \left(\frac{s_{34}}{s_{23}}\right) e^{L_{34}\,\tau_g} \cdot f^{abc} V_\lambda \cdot \left(\frac{s_{45}}{s_{51}}\right) e^{L_{45}\,\tau_g} \cdot T^b$$

LL AND BEYOND





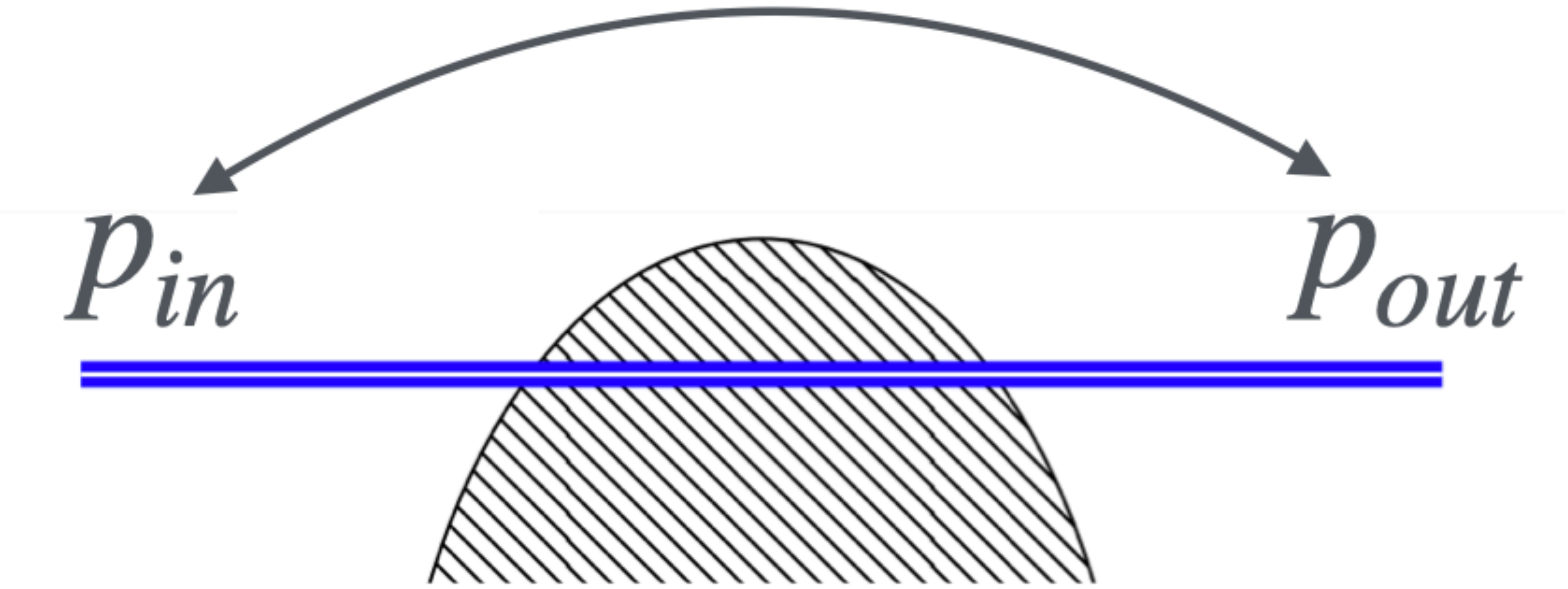
?!

Factorization breaking contributions!

Only start appearing in the "odd" amplitude at NNLL

SIGNATURE AND COLOR

Make crossing symmetry explicit



$$\mathscr{A}^{\pm} = \frac{1}{2}\left(\mathscr{A}(s,t) \pm \mathscr{A}(-s-t,t)\right)$$

$$\mathscr{A}^{(\sigma_a,\sigma_b)}(p_1,p_2,p_3,p_4,p_5) \sim \mathscr{A}(p_1,p_2,p_3,p_4,p_5) + \sigma_a \mathscr{A}(p_5,p_2,p_3,p_4,p_1)$$
$$+\sigma_b \mathscr{A}(p_1,p_3,p_2,p_4,p_5) + \sigma_a\sigma_b \mathscr{A}(p_5,p_3,p_2,p_4,p_1)$$

$2 \to 2$

$\mathscr{A}^{(-)}$ odd

$\mathscr{A}^{(+)}$ even

$2 \to 3$

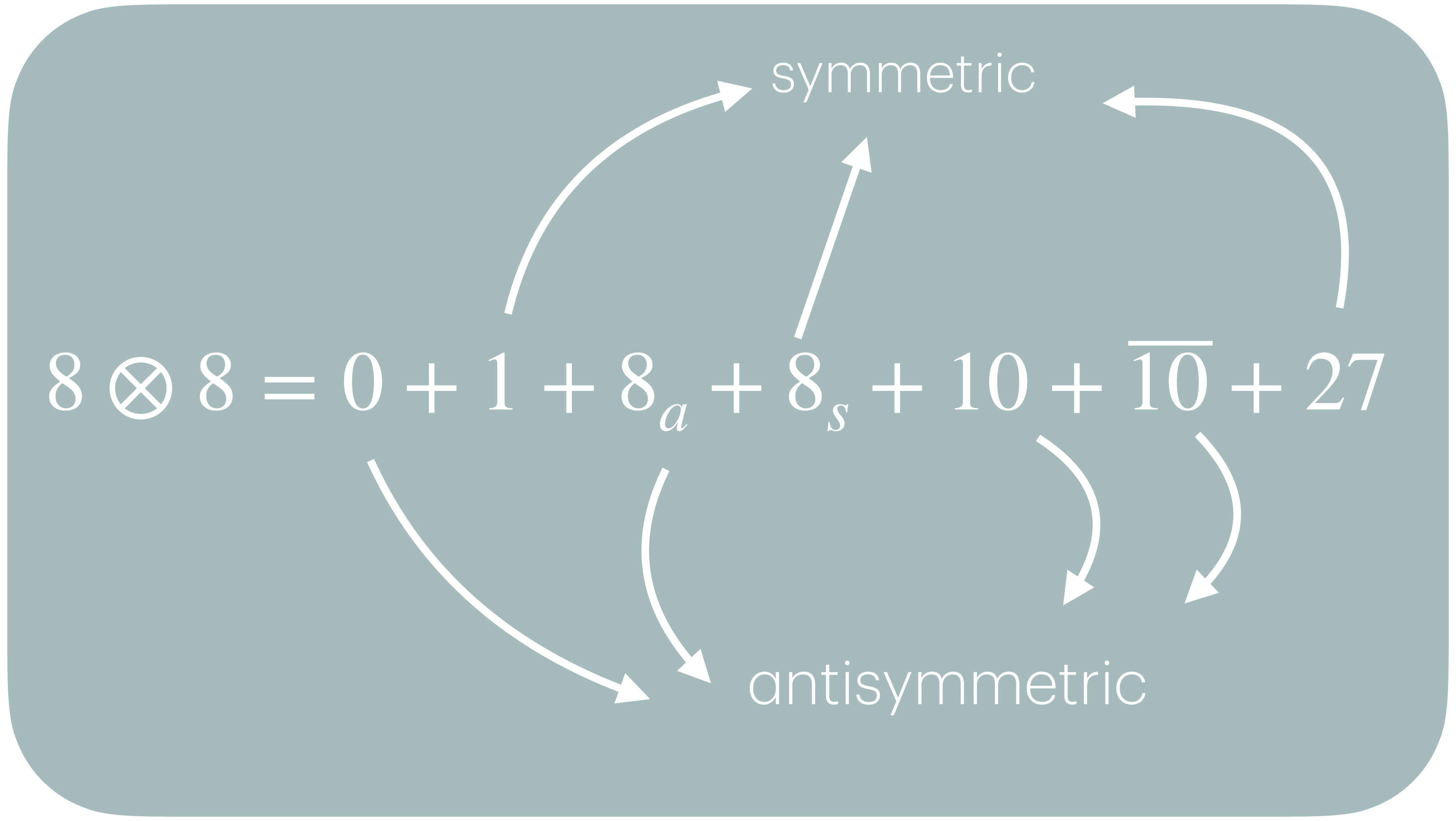
$\mathscr{A}^{(-,-)}$ odd, odd

$\mathscr{A}^{(+,+)}$ even, even

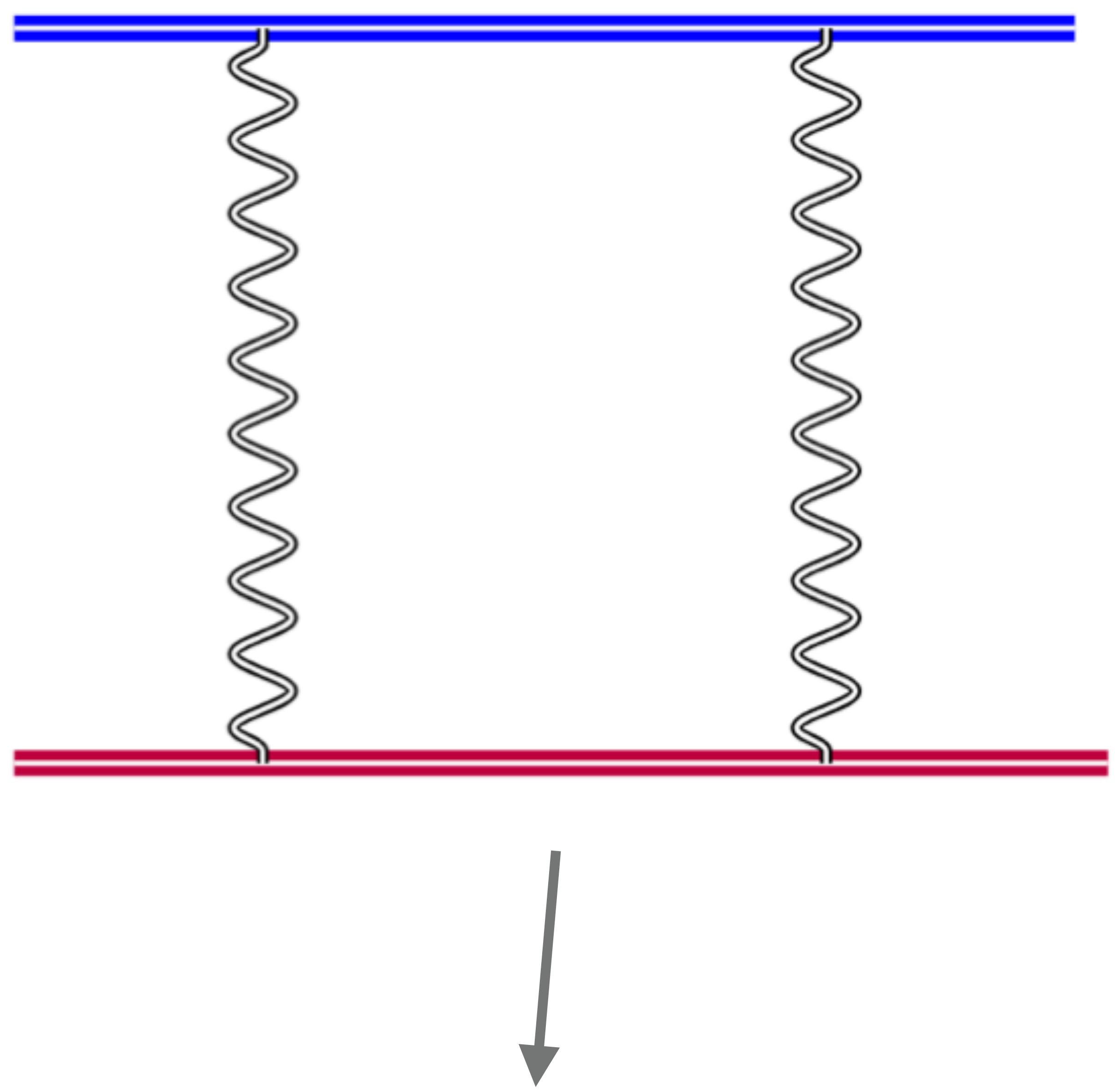
$\mathscr{A}^{(+,-)}$ even, odd

$\mathscr{A}^{(-,+)}$ odd, even

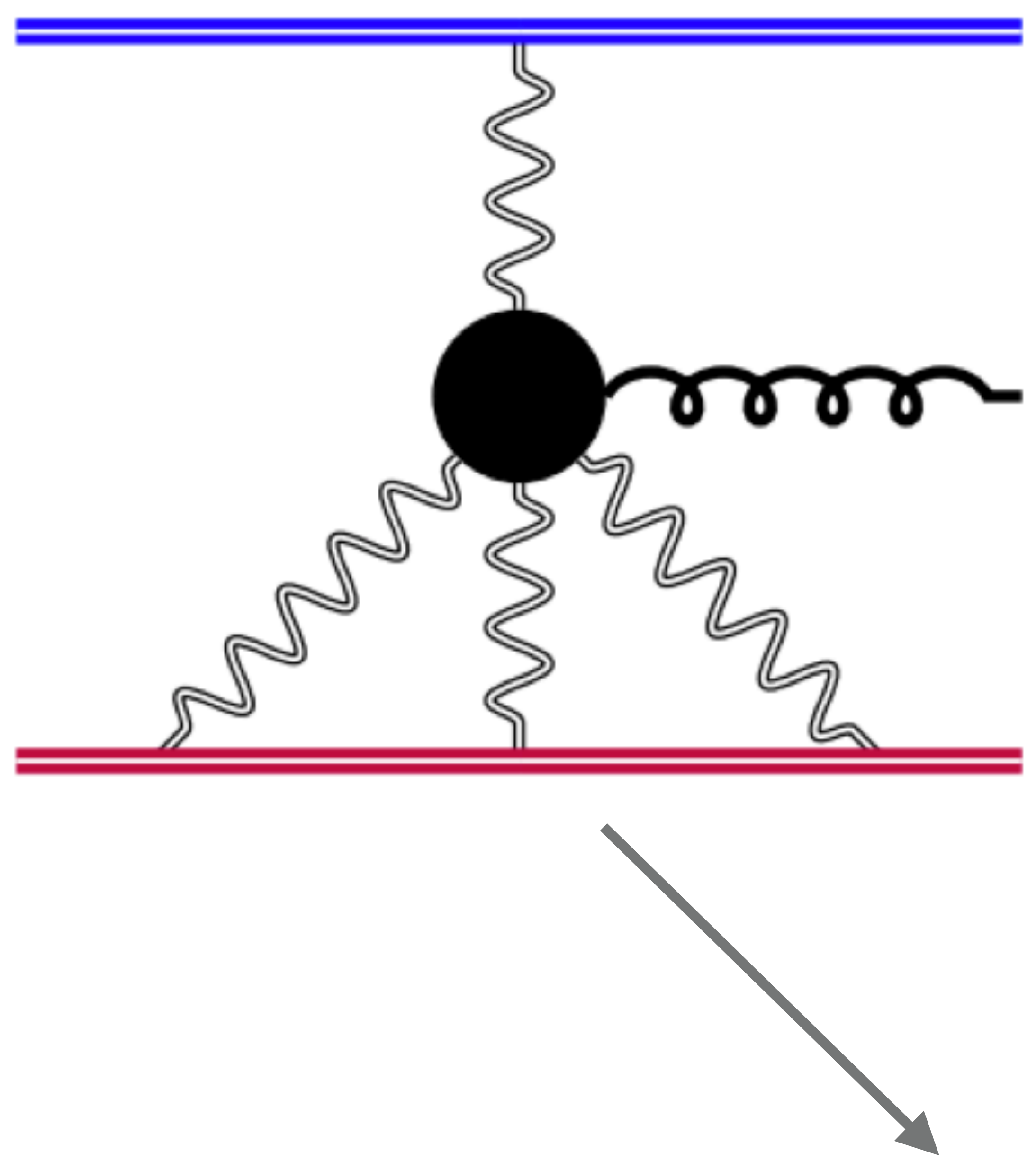
t-channel in color irrep



“Regge pole” (factorized structure) sits in $\mathscr{A}^{(-)}_{(8_a)}$ and $\mathscr{A}^{(-,-)}_{(8_a,8_a)}$!

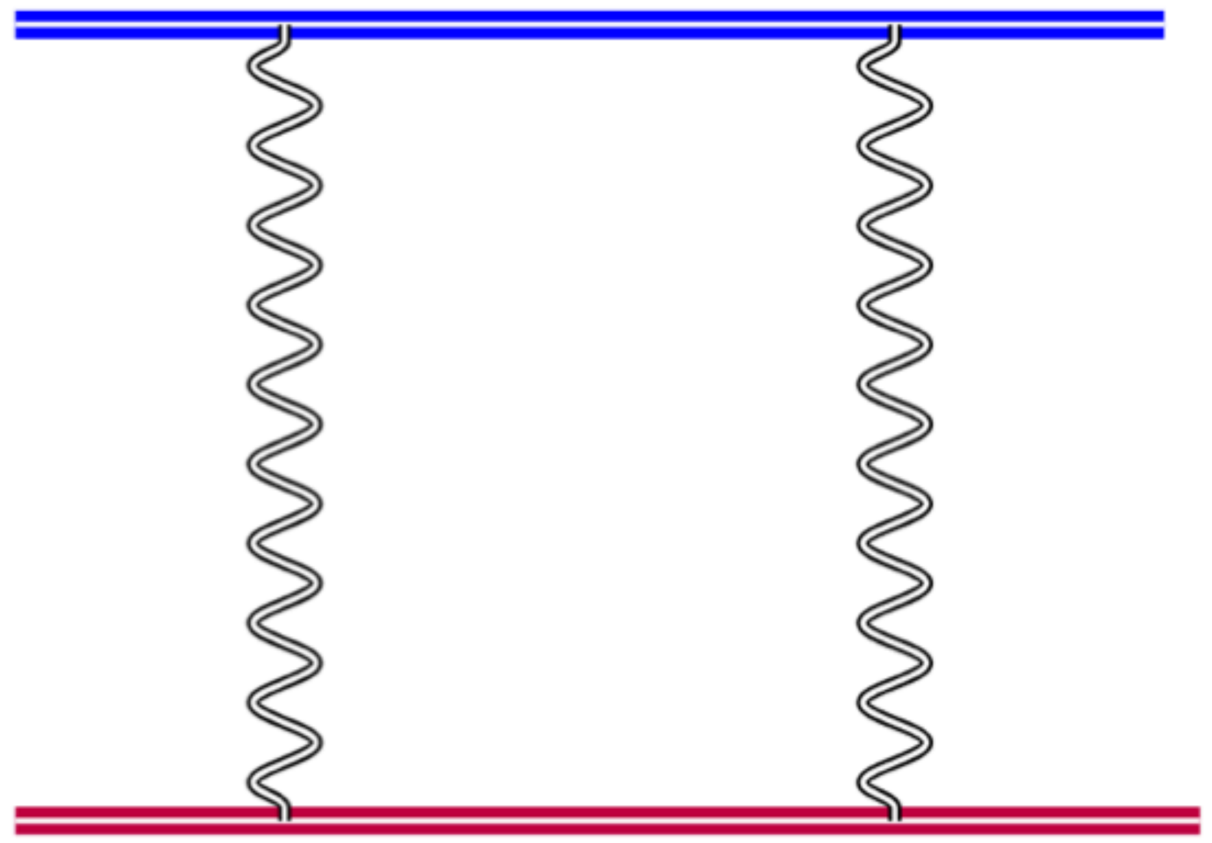


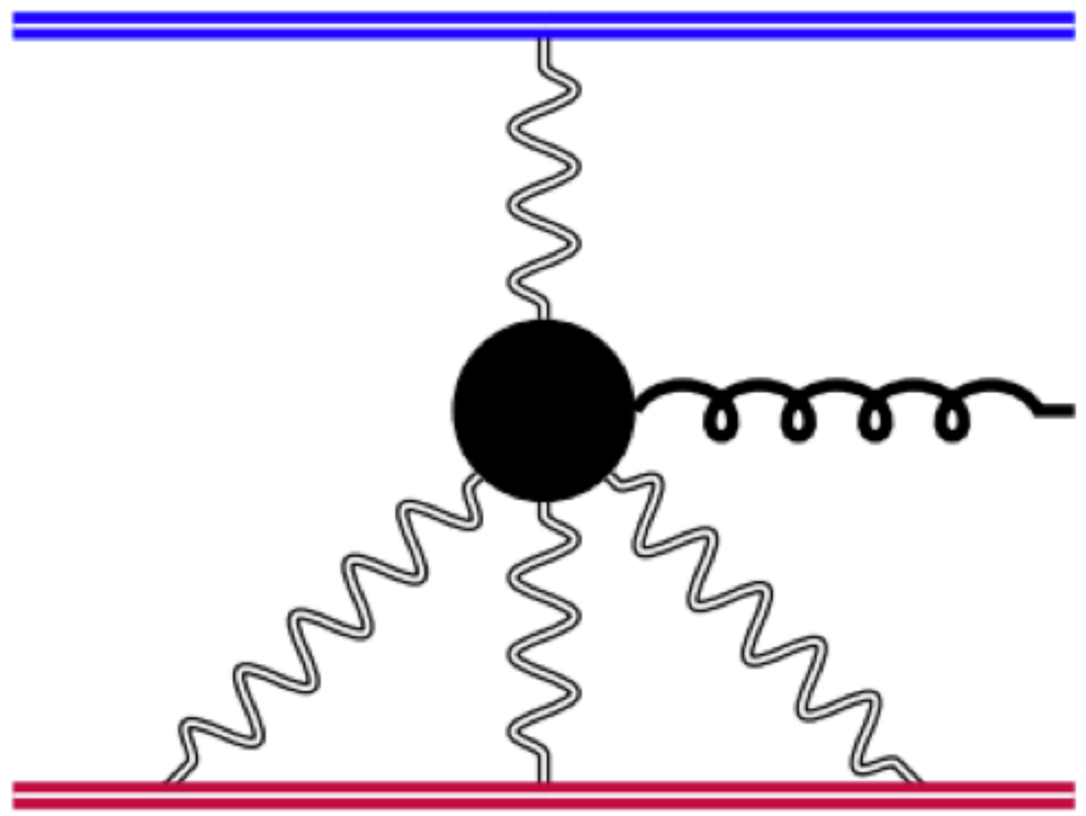
Contributes to even amplitude!
Regge factorization holds at NLL
[Fadin, Lipatov hep-ph/9802290]



Enter $\mathscr{A}^{(-,-)}_{(8_a,8_a)}$! Multi-reggeon exchanges
break factorization starting at NNLL

“Regge pole” (factorized structure) sits in $\mathscr{A}^{(-)}_{(8_a)}$ and $\mathscr{A}^{(-,-)}_{(8_a,8_a)}$!





Can they be isolated/ computed?

Contributes to even amplitude!
Regge factorization holds at NLL
[Fadin, Lipatov hep-ph/9802290]

Enter $\mathscr{A}^{(-,-)}_{(8_a,8_a)}$! Multi-reggeon exchanges
break factorization starting at NNLL

DEALING WITH REGGE CUTS

Luckily not the end of the story; we have tools

This talk

When Does The Gluon Reggeize?

Simon Caron-Huot[a,b]

[1309.6521]

G. Falcioni et al [2112.11098,2006.01267,2012.00613,2111.10664]

Balitsky/JIMWLK shockwave formalism

+

Diagrammatic approach

V. Fadin, L.N. Lipatov [1712.09805], V. Fadin [Phys. Atom. Nucl. 84 (2021) 100],[2409.01698]

+

Glauber SCET

I.Z. Rothstein, I.W. Stewart [1601.04695]; I.Z. Rothstein, M. Saavedra [2410.06283]; I. Moult et al [2207.02859]; A. Gao et al [2401.00931]

Amplitudes from shockwave formalism

Mueller, Balitsky, Kovchegov, Jalilian-Marian,
Iancu, McLerran, Weigert, Leonidov, Kovner

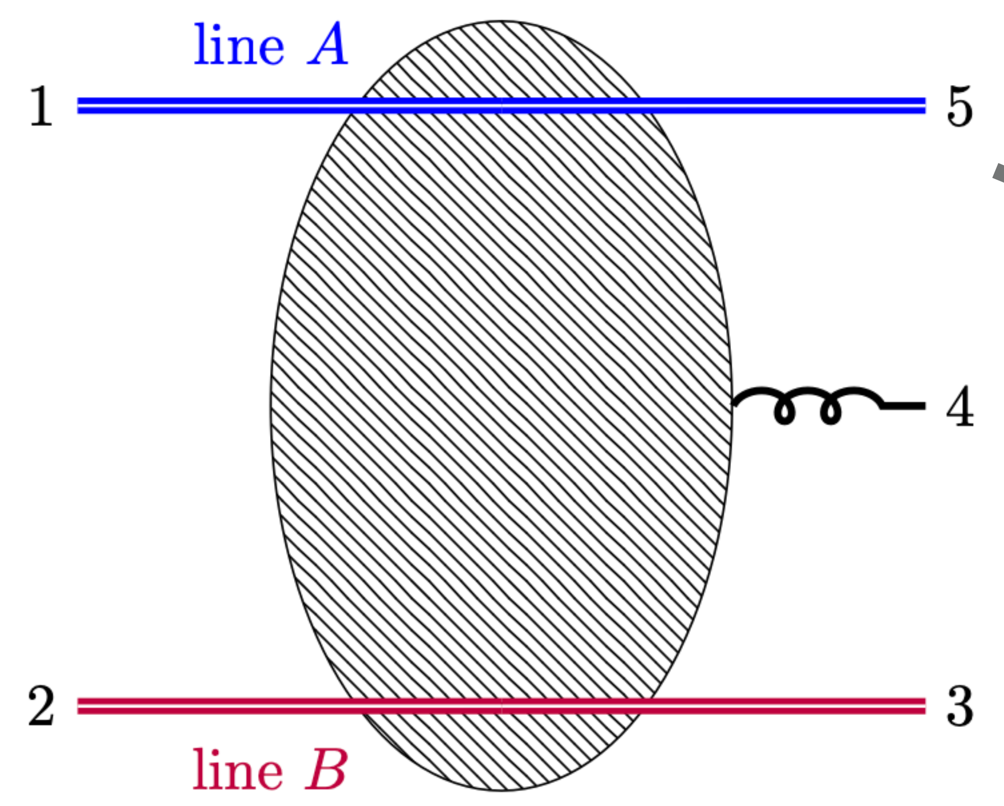
When Does The Gluon Reggeize?

Simon Caron-Huot[a,b]

MRK AMPLITUDE FROM BALITSKY/JIMWLK

Single, infinite Wilsone line localized in $x^+ = 0$ lightcone

➤ Framework to analyse systems with large rapidity gaps



Path ordering in color space

Divergent, can be regulated working at finite rapidity

$$U^{\{r\}}(z) = \mathcal{P} e^{i g_s \int_{-\infty}^{\infty} dx^- A^a_-(x^+=0, x^-, \mathbf{z}) T^a_r}$$

$$-\frac{d}{d\eta}[U(z_1)U(z_2)\ldots U(z_n)] = H\cdot[U(z_1)U(z_2)\ldots U(z_n)]$$

Balitsky/JIMWLK eqn

To get the MRK amplitude:

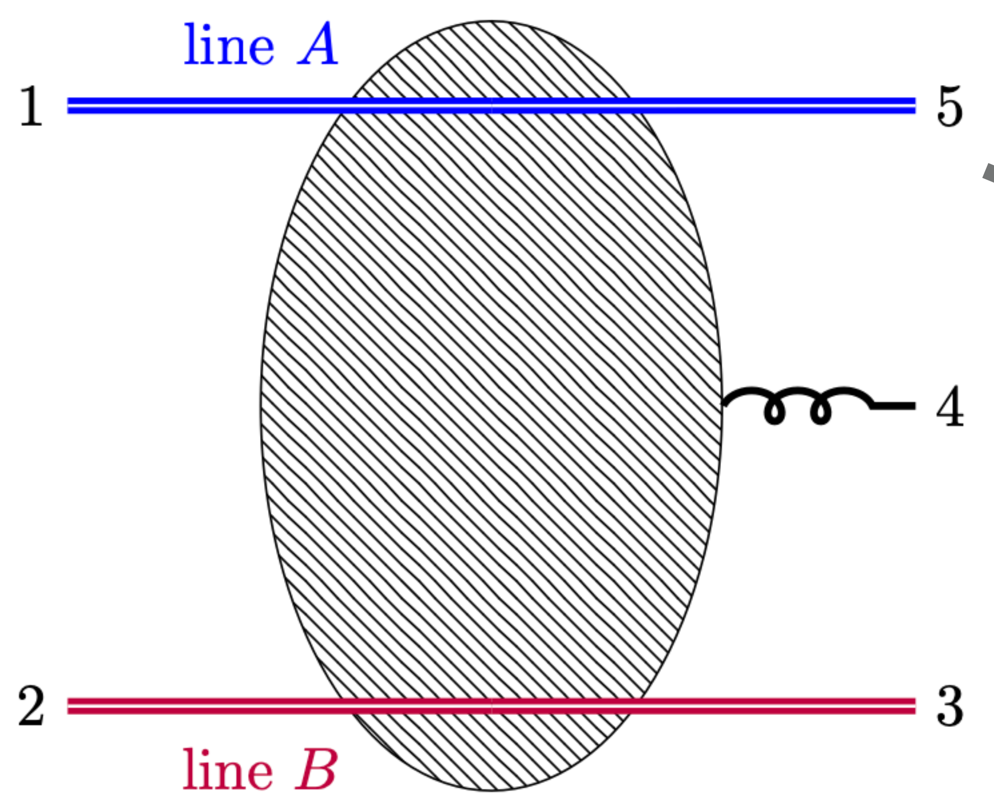
1) Evolve upper Wilson line to rapidity of the centrally emitted gluon

2) Consider interaction of the gluon with Wilson line

3) Evolve down to rapidity of B and compute correlator

MRK AMPLITUDE FROM BALITSKY/JIMWLK

➤ Framework to analyse systems with large rapidity gaps



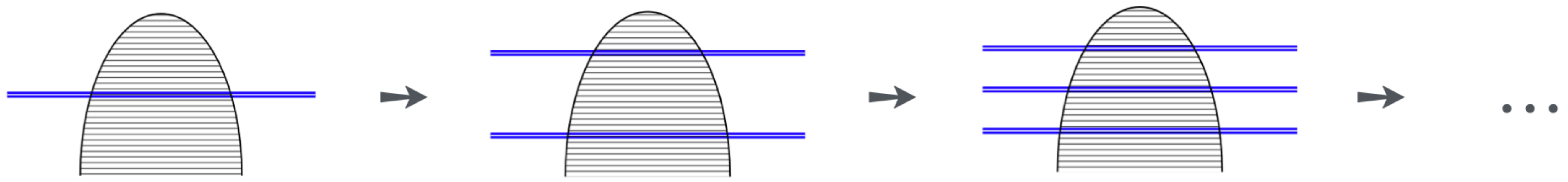
Path ordering in color space

$$U^{\{r\}}(z) = \mathcal{P} e^{ig_s \int_{-\infty}^{\infty} dx^- A_-^a(x^+=0, x^-, \mathbf{z}) T_r^a}$$

$$-\frac{d}{d\eta}[U(z_1)U(z_2)\dots U(z_n)] = H\cdot[U(z_1)U(z_2)\dots U(z_n)]$$

Balitsky/JIMWLK eqn

Rapidity evolution is non trivial (non-linear equation)

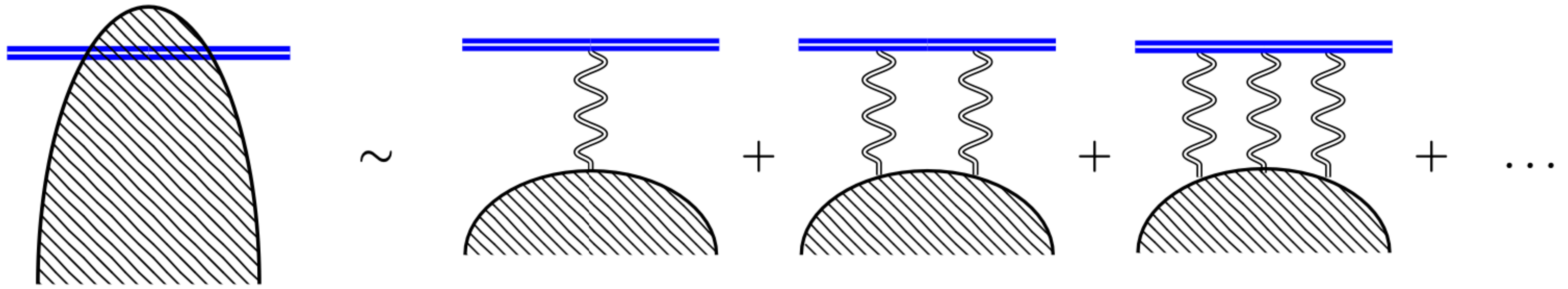


Dilute field approximation

DILUTE FIELD AND LINEARIZED EVOLUTION: PROJECTILE

$$U_\eta(\mathbf{z}) \equiv \exp\left\{ig_s T^a W^a_\eta(\mathbf{z})\right\} = 1 + (ig_s)W^a_\eta T^a + \frac{1}{2}(ig_s)^2 W^a_\eta W^b_\eta T^a T^b + \dots$$

$g_s W_\eta \ll 1$



Dashed blob: background generated by the other projectile (shockwave formalism)

Eikonal vertex

Radiatively-generated impact factors

$$[\![O_1 O_2 \dots O_n]\!]^{ab}_r \equiv (T^{c_1}_r)_{aa_1}(T^{c_2}_r)_{a_1 a_2}\dots(T^{c_n}_r)_{a_{n-1}b}\, O^{c_1}_1 O^{c_2}_2 \dots O^{c_n}_n$$

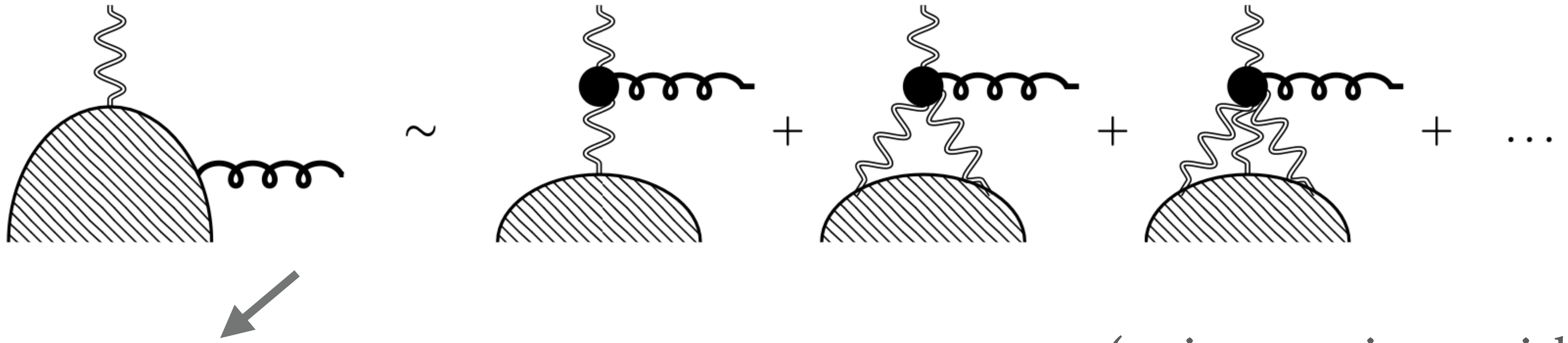
$$[\![O_1 O_2 \dots O_n]\!]^{ab} \equiv [\![O_1 O_2 \dots O_n]\!]^{ab}_{\text{adj}}\,,$$

$$a^{a_1,\dagger}_{\lambda_1}(p_1)a^{a_5}_{\lambda_5}(p_5) \sim 2\pi\delta(p_1^+ - p_5^+)\delta_{\lambda_1\lambda_5} \times 2p_1^+ \times \Big\{(ig_s)\mathcal{J}(\mathbf{q}_A)[\![W(\mathbf{q}_A)]\!]_A +$$
$$+ \frac{(ig_s)^2}{2!}\int\{\mathrm{d}\mathbf{q}\}\left[1 + \mathcal{J}'(\mathbf{q}_A,\mathbf{q})\right]\, [\![W(\mathbf{q}_A - \mathbf{q})W(\mathbf{q})]\!]_A +$$
$$+ \frac{(ig_s)^3}{3!}\int\{\mathrm{d}\mathbf{q}_1\}\{\mathrm{d}\mathbf{q}_2\}\,[\![W(\mathbf{q}_A - \mathbf{q}_1)W(\mathbf{q}_1 - \mathbf{q}_2)W(\mathbf{q}_2)]\!]_A + \dots\Big\}_{\eta=\eta_A}$$

DILUTE FIELD AND LINEARIZED EVOLUTION: EXTERNAL GLUON OPE

$$U_\eta(\mathbf{z}) \equiv \exp\left\{ig_s T^a W^a_\eta(\mathbf{z})\right\} = 1 + (ig_s)W^a_\eta T^a + \frac{1}{2}(ig_s)^2 W^a_\eta W^b_\eta T^a T^b + \dots$$

Interaction of "linearized" Wilson line with external gluon (in a background)



$$W(\mathbf{p})^b\, a^a_\lambda(q) \sim$$

$$2g_s [\![W]\!]^{ab}(\mathbf{q}+\mathbf{p})\left[\frac{\boldsymbol{\varepsilon}^*_\lambda\cdot\mathbf{p}}{\mathbf{p}^2} + \frac{\boldsymbol{\varepsilon}^*_\lambda\cdot\mathbf{q}}{\mathbf{q}^2}\right] +$$

$$+\, ig_s^2 \int \{\mathrm{d}\mathbf{k}_1\} [\![W(\mathbf{q}+\mathbf{p}-\mathbf{k}_1)W(\mathbf{k}_1)]\!]^{ab}\left[\frac{\boldsymbol{\varepsilon}^*_\lambda\cdot\mathbf{p}}{\mathbf{p}^2} + \frac{\boldsymbol{\varepsilon}^*_\lambda\cdot(\mathbf{k}_1-\mathbf{p})}{(\mathbf{k}_1-\mathbf{p})^2}\right] +$$

$$+\, g_s^3 \int \{\mathrm{d}\mathbf{k}_1\}\{\mathrm{d}\mathbf{k}_2\} [\![W(\mathbf{q}+\mathbf{p}-\mathbf{k}_1)W(\mathbf{k}_1-\mathbf{k}_2)W(\mathbf{k}_2)]\!]^{ab} \times$$

$$\times \left[\frac{1}{6}\left(\frac{\boldsymbol{\varepsilon}^*_\lambda\cdot(\mathbf{k}_1-\mathbf{p})}{(\mathbf{k}_1-\mathbf{p})^2}\right) - \frac{1}{2}\left(\frac{\boldsymbol{\varepsilon}^*_\lambda\cdot(\mathbf{k}_2-\mathbf{p})}{(\mathbf{k}_2-\mathbf{p})^2}\right) - \frac{1}{3}\left(\frac{\boldsymbol{\varepsilon}^*_\lambda\cdot\mathbf{p}}{\mathbf{p}^2}\right)\right] + \dots,$$

(+ interactions with multiple Ws…)

These contributions receive perturbative corrections beyond LO

SOME REMARKS ON THE W FIELD

- W != Reggeized gluon
- However, up to two loops it is an eigenstate of the rapidity evolution equation, with eigenvalue given by the Regge trajectory:

$$-\frac{\mathrm{d}}{\mathrm{d}\eta} W_\eta(\mathbf{p}) = \tau_g(\mathbf{p}) W_\eta(\mathbf{p})$$

- Single W exchange therefore clearly contributes to 'Regge pole' factorization
- In principle B/JIMWLK Hamiltonian induces transitions $n \to n+2, n \to n+4, \ldots$ but these are suppressed by powers of strong coupling and do not enter our analysis
- At LO Ws are free: same rapidity correlators computed with Wick's theorem

$$\left\langle \mathbb{T}\left[W(\mathbf{p}_1)\cdots W(\mathbf{p}_n)\right]_\eta \left[\widetilde{W}(\mathbf{q}_1)\cdots \widetilde{W}(\mathbf{q}_m)\right]_\eta \right\rangle =$$

$$\delta_{nm} \sum_{\sigma \in S_n} G(\mathbf{p}_1, \mathbf{q}_{\sigma(1)}) \ldots G(\mathbf{p}_n, \mathbf{q}_{\sigma(n)}) + \mathcal{O}(\alpha_s)$$

$$G(\mathbf{p},\mathbf{q}) = \left\langle \mathbb{T}\, W^a_\eta(\mathbf{p}) \widetilde{W}^b_\eta(\mathbf{q}) \right\rangle = (2\pi)^{2-2\epsilon} \delta^{2-2\epsilon}(\mathbf{p}-\mathbf{q}) \frac{i\delta^{ab}}{\mathbf{p}^2} + \mathcal{O}(\alpha_s)$$

EXAMPLE: LL (-,-) AMPLITUDE FROM WILSON LINE EFT

At LL, only single W contributions (multiple Ws suppressed by g_s)

$$\mathcal{S}_{\rm LL} = \phi^{[AB]} \left[2\pi\delta(p_1^+ - p_5^+)\delta_{\lambda_1\lambda_5} \times 2p_1^+\right] \left[2\pi\delta(p_2^- - p_3^-)\delta_{\lambda_2\lambda_3} \times 2p_2^-\right]$$
$$\times \left\langle \mathbb{T}\left(ig_s[\![W_{\eta_A}(\mathbf{q}_A)]\!]_{r_A}^{c_5c_1}\right) a_{\lambda_4}^{a_4}(p_4) \left(ig_s[\![\widetilde{W}_{\eta_B}(\mathbf{q}_B)]\!]_{r_B}^{c_3c_2}\right)\right\rangle .$$

NB: rapidity evoloution generates large log(x)!

1) Evolve projectile A to central rapidity (remember single W is eigenstate of H)

$$W_{\eta_A}^a(\mathbf{q}_A) = e^{\Delta\eta_{A4}\tau_g(\mathbf{q}_A)} W_{\eta_4}^a(\mathbf{q}_A)$$

LO

$$\left(\frac{\alpha_s}{4\pi}\right)\tau_g^{(1)}(\mathbf{q}) = \left(\frac{\alpha_s}{4\pi}\right) 2N_c \frac{e^{\epsilon\gamma_E}\Gamma(1-\epsilon)^2\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)\epsilon}\left(\frac{\mu^2}{\mathbf{q}^2}\right)^\epsilon$$

2) Interaction with central gluon

$$W_{\eta_4}^a(\mathbf{q}_A) a_{\lambda_4}^{a_4}(p_4) \to 2g_s[\![W]\!]_{\eta_4}^{a_4a}(\mathbf{q}_A+\mathbf{p}_4)\left[\frac{\boldsymbol{\varepsilon}_{\lambda_4}^*\cdot\mathbf{p}_4}{\mathbf{p}_4^2} + \frac{\boldsymbol{\varepsilon}_{\lambda_4}^*\cdot\mathbf{q}_A}{\mathbf{q}_A^2}\right] + \dots$$

3) Evolve to rapidity of B and compute correlator

EXAMPLE: LL (-,-) AMPLITUDE FROM WILSON LINE EFT

3) Evolve to rapidity of B and compute correlator

$$\mathcal{A}_{\rm LL} = \mathcal{A}_{\rm LL}^{(--)} = \phi^{[AB]}\, 4 g_s^3 s_{12} \left[(T_{r_B})^b_{c_3 c_2} (T_{r_A})^a_{c_5 c_1}\, i f^{a b a_4} \right] \times$$

$$e^{\tau_g(\mathbf{q}_A)\Delta\eta_{A4}} \frac{1}{\mathbf{q}_B^2} \left[\frac{\varepsilon^*_{\lambda_4}\cdot \mathbf{p}_4}{\mathbf{p}_4^2} + \frac{\varepsilon^*_{\lambda_4}\cdot \mathbf{q}_A}{\mathbf{q}_A^2} \right] e^{\tau_g(\mathbf{q}_B)\Delta\eta_{4B}}$$

$$\varepsilon^*_+ = \frac{1}{\sqrt{2}}\left(\frac{\bar{p}_{4,\perp}}{p_4^+}, 1, -i, -\frac{\bar{p}_{4,\perp}}{p_4^+}\right), \qquad \varepsilon^*_- = \frac{1}{\sqrt{2}}\left(\frac{p_{4,\perp}}{p_4^+}, 1, i, -\frac{p_{4,\perp}}{p_4^+}\right)$$

$$\frac{1}{\sqrt{2}} \times V_{\lambda_4}(\mathbf{q}_A, \mathbf{p}_4) \times \frac{1}{\mathbf{q}_A^2}$$

$$V_+(\mathbf{q}_A, \mathbf{p}_4) = \frac{\bar{q}_{A,\perp}\, q_{B,\perp}}{p_{4,\perp}}, \qquad V_-(\mathbf{q}_A, \mathbf{p}_4) = \frac{q_{A,\perp}\, \bar{q}_{B,\perp}}{\bar{p}_{4,\perp}}$$

t-channel structure apparent, iterates to all multiplicities!

LO central emission vertex

EXAMPLE: NLL (-,-) AMPLITUDE FROM WILSON LINE EFT

Straightforward generalization to NLL

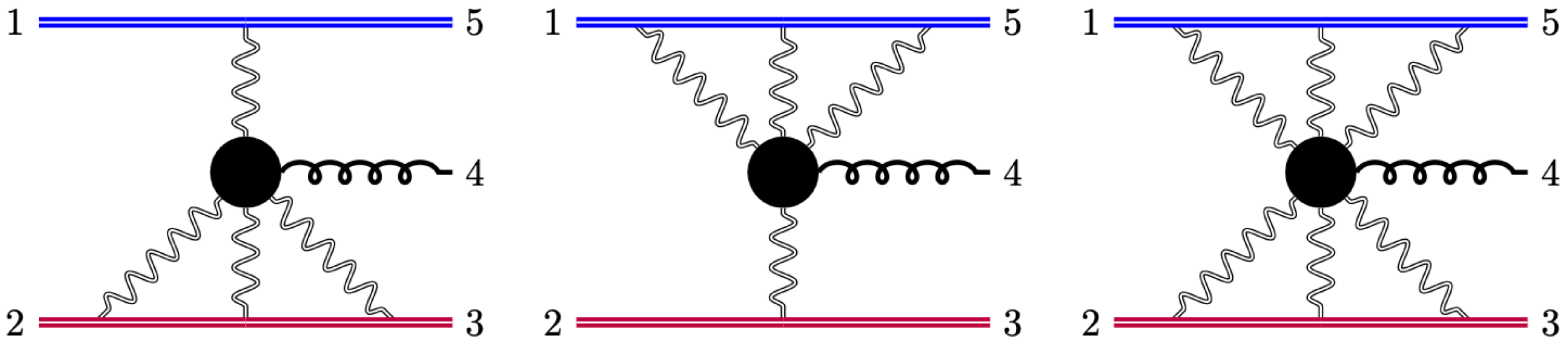
Only single W exchange in the (-,-) signature

$$\mathcal{A}^{(--)}_{\mathrm{NLL}} = \phi^{[AB]}\, 2\sqrt{2}\, g_s^3 s_{12} \times (T_{r_A})^a_{c_5 c_1} \mathcal{J}(\mathbf{q}_A) e^{\tau_g(\mathbf{q}_A)\Delta\eta_{A4}} \\ \times \frac{1}{\mathbf{q}_A^2}\left[i f^{aba_4} V_{\lambda_4}(\mathbf{q}_A, \mathbf{p}_4) \mathcal{W}_{\lambda_4}(\mathbf{q}_A, \mathbf{p}_4)\right] \frac{1}{\mathbf{q}_B^2} e^{\tau_g(\mathbf{q}_B)\Delta\eta_{4B}} \mathcal{J}(\mathbf{q}_B)(T_{r_B})^b_{c_3 c_2}$$

Same structure, however radiative corrections to impact factors, vertex and Regge trajectory. These are NOT predicted by effective theory, need to be matched to e.g. full kinematics $2 \to 2$ and $2 \to 3$ scattering amplitudes

GLANCE AT NNLL COMPLEXITY

At NNLL situation is more complex: ‘straightforward’ corrections to single W exchange mix with multi-W exchanges



Rapidity evolution of 3-W state does not contribute at this order (i.e. up to 2 loops)

$$\mathcal{A}^{(--)}_{\mathrm{NNLL}} = \mathcal{A}^{(--)}_{\mathrm{NNLL},\{1,1\}} + \mathcal{A}^{(--)}_{\mathrm{NNLL},\{1,3\}} + \mathcal{A}^{(--)}_{\mathrm{NNLL},\{3,1\}} + \mathcal{A}^{(--)}_{\mathrm{NNLL},\{3,3\}}$$

Crucial: all Ws and gluon interactions are LO in the EFT, multi-W contributions ‘straightforward’ to calculate

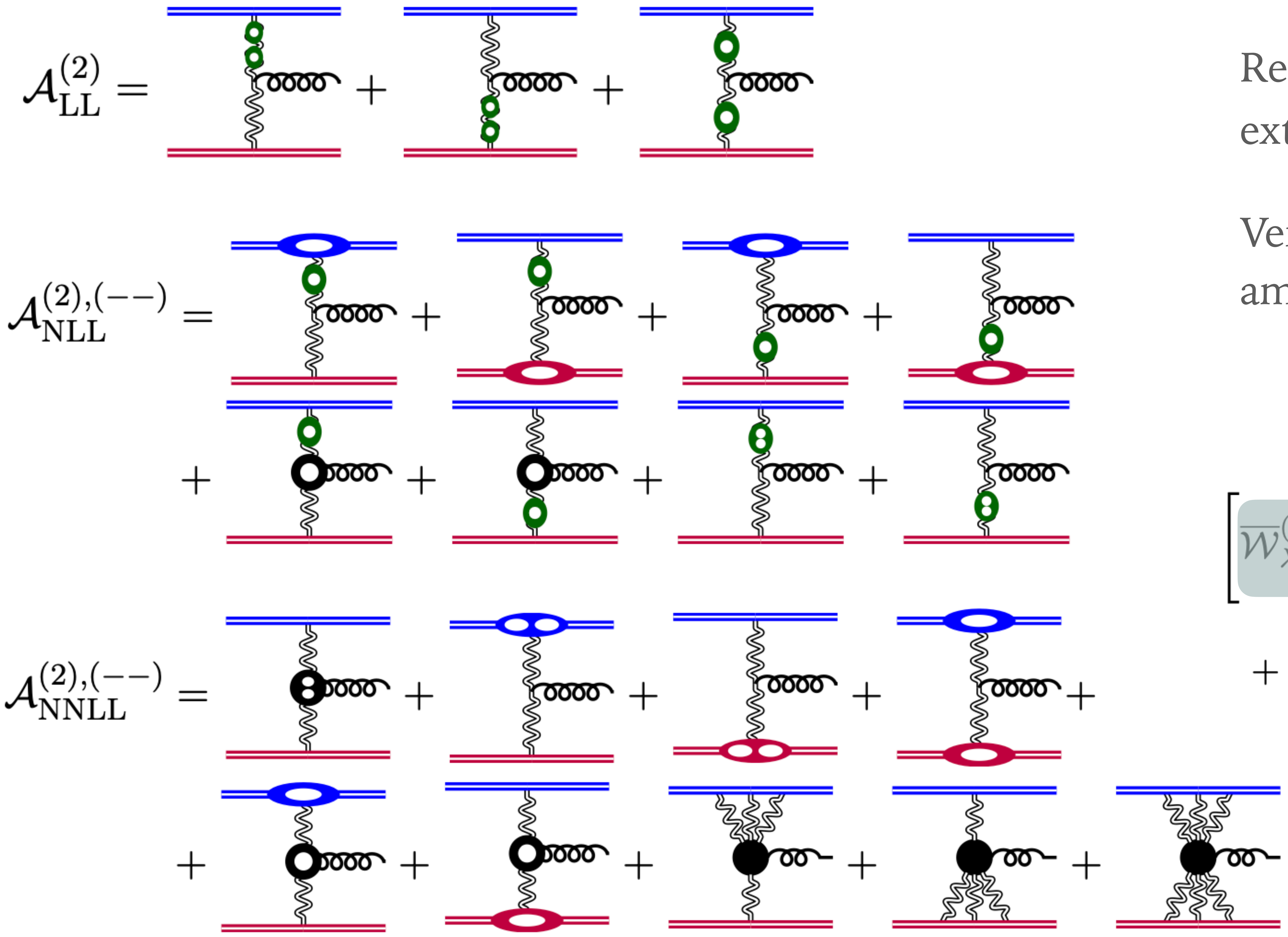
Non-diagonal color operator generates LC contributions in the octet, but also (the only) non-universal SLC contributions in other color structures

$$\mathcal{A}^{(2),(--)}_{\mathrm{NNLL},\{1,3\}} = \pi^2 \mathcal{K}^{(2)}_{\{1,3\}} \left[2\mathcal{T}^2_{+-} + \frac{2}{3}\mathcal{T}^2_{++} + \frac{2N_c}{3}\mathcal{T}_{++} \right] \mathcal{A}^{(0)}$$

$$\mathcal{K}^{(2)}_{\{1,3\}} = \left[\mathcal{K}^{(0)}_{\{1,1\}}\right]^{-1} \int \frac{\mathfrak{D}\mathbf{k}_1 \mathfrak{D}\mathbf{k}_2}{\mathbf{k}_2^2 (\mathbf{k}_1 - \mathbf{k}_2)^2 (\mathbf{q}_B - \mathbf{k}_1)^2} \times \left[\frac{1}{6} \left(\frac{\boldsymbol{\varepsilon}^*_{\lambda_4} \cdot (\mathbf{k}_1 - \mathbf{q}_A)}{(\mathbf{k}_1 - \mathbf{q}_A)^2} \right) - \frac{1}{2} \left(\frac{\boldsymbol{\varepsilon}^*_{\lambda_4} \cdot (\mathbf{k}_2 - \mathbf{q}_A)}{(\mathbf{k}_2 - \mathbf{q}_A)^2} \right) - \frac{1}{3} \left(\frac{\boldsymbol{\varepsilon}^*_{\lambda_4} \cdot \mathbf{q}_A}{\mathbf{q}_A^2} \right) \right]$$

Pure, single-valued and UT

2-LOOP (-,-): SUMMARY

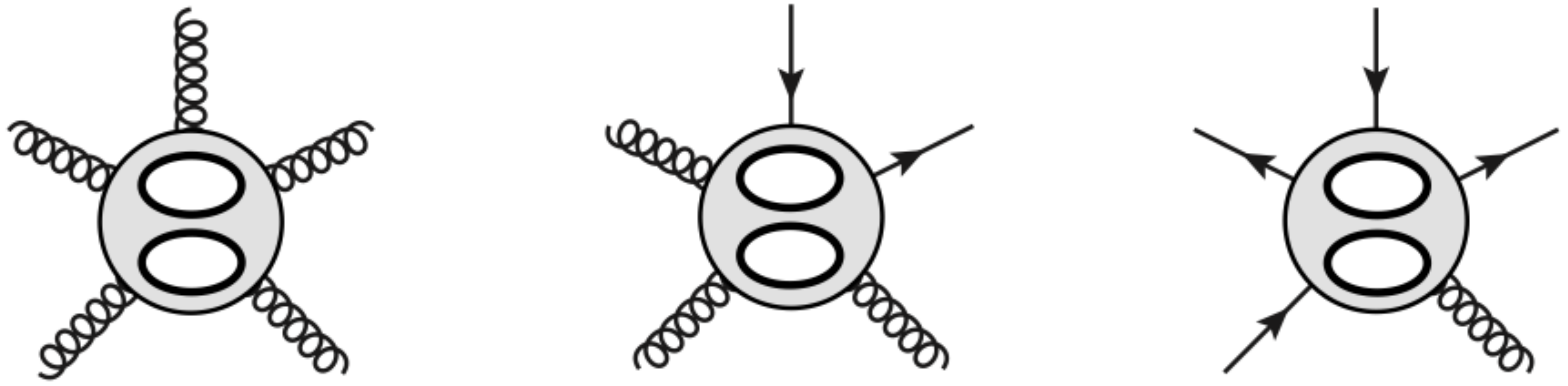


Regge trajectory and impact factors: can be extracted from 2-loop $2 \to 2$ amplitudes ✓

Vertex at 2 loops: match to 2-loop $2 \to 3$ amplitudes NEW!!

$$\Big[\overline{\mathcal{W}}^{(2)}_{\lambda_4} + \bar{\mathcal{J}}^{(2)}_A + \bar{\mathcal{J}}^{(2)}_B + \bar{\mathcal{J}}^{(1)}_A \bar{\mathcal{J}}^{(1)}_B + \overline{\mathcal{W}}^{(1)}_{\lambda_4}(\bar{\mathcal{J}}^{(1)}_A + \bar{\mathcal{J}}^{(1)}_B) + (i\pi)^2\Big(B^{(2)}_{+-}\boldsymbol{\mathcal{T}}^2_{+-} + B^{(2)}_{--}\boldsymbol{\mathcal{T}}^2_{--} + B^{(2)}_{-+}\boldsymbol{\mathcal{T}}^2_{-+} - B^{(2)}_d \frac{N_c^2}{4}\Big)\Big]\mathcal{A}^{(0)}$$

Matching amplitude to “EFT”



De Laurentis, Ita, Klinkert, Sotnikov: 2311.10086(PRD)

De Laurentis, Ita, Sotnikov: 2311.18752(PRD)

Agarwal, Buccioni, **FD**, Gambuti, von Manteuffel, Tancredi: 2311.16907(PRD)

$$\mathcal{H} = \sum_{i,c} R_{ic}\, F_i\, \mathcal{C}_c$$

Rational (R_{ic}), Transcendental (F_i), Colour ($\mathcal{C}_c$)

MRK scaling parameter x

Strategy: expand the amplitude and match it to the "EFT" prediction

$$\{s_{12}, s_{23}, s_{34}, s_{45}, s_{51}\} \to \left\{\frac{s}{x^2}, s_{23}, \frac{s_1}{x}, \frac{s_2}{x}, s_{51}\right\}$$

Expansion of the amplitude in MRK

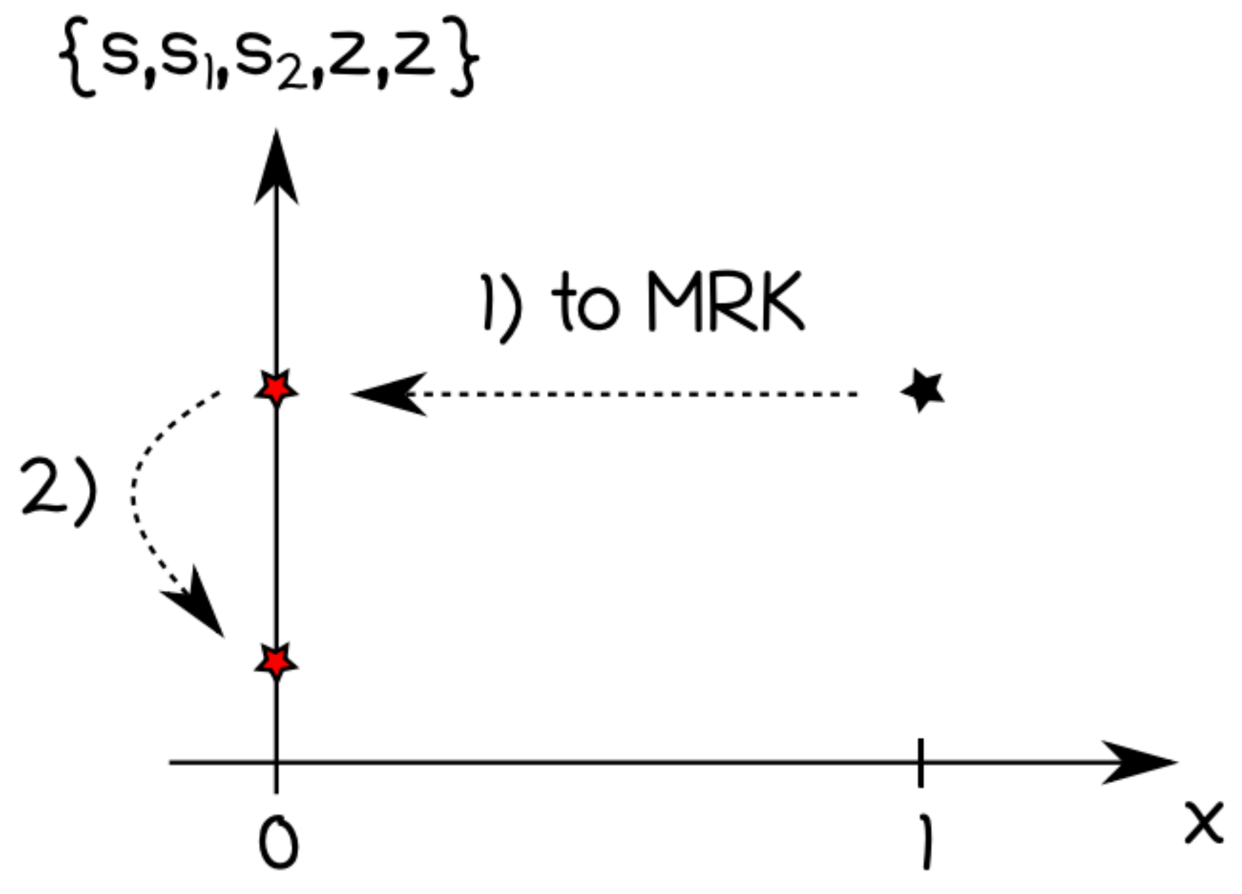
Rational

Transcendental

Scaling parameter
$x = 1$: “physical point”,
$x = 0$:MRK

Set of PF and BC from [Chicherin, Sotnikov: 2009.07803]

$$\{s_{12}, s_{23}, s_{34}, s_{45}, s_{51}\} \to \{s, s_1, s_2, z, \bar{z}\} + x$$



$$dI_i(\mathbf{s}) = \epsilon\, dA_{ij}(\mathbf{s})\, I_j(\mathbf{s}) \qquad dA_{ij}(\mathbf{s}) = \sum_{n=1} a_{ij}^n\, d\log(W_n)$$

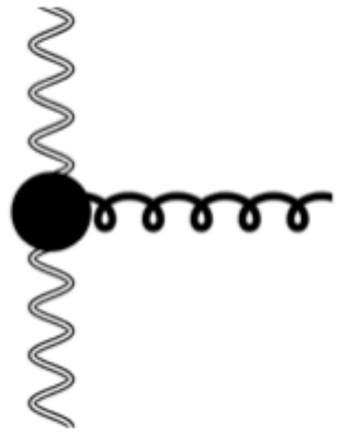
$$W_n \to W_n(x)$$

$$\begin{cases} \dfrac{\partial}{\partial x}\mathbf{f}(x,y,\epsilon) = \epsilon\, A_x(x,y)\, \mathbf{f}(x,y,\epsilon) \\ \dfrac{\partial}{\partial y}\mathbf{f}(x,y,\epsilon) = \epsilon\, A_y(x,y)\, \mathbf{f}(x,y,\epsilon) \end{cases}$$

[Caron-Huot, Chicherin, Henn, Zhang, Zoia 2003.03120]

$$f^{(w)}(\mathbf{s};x) = \sum_{n=0} \sum_{m=0}^{w} f_{mn}^{(w)}(\mathbf{s})\, x^n \log^m x$$

$$\{x\},\ \{\frac{s_1 s_2}{s}\},\ \{s_1, s_2, s_1 - s_2, s_1 + s_2\},$$
$$\{z, \bar{z}, 1 - z, 1 - \bar{z}, z - \bar{z}, 1 - z - \bar{z}\}$$

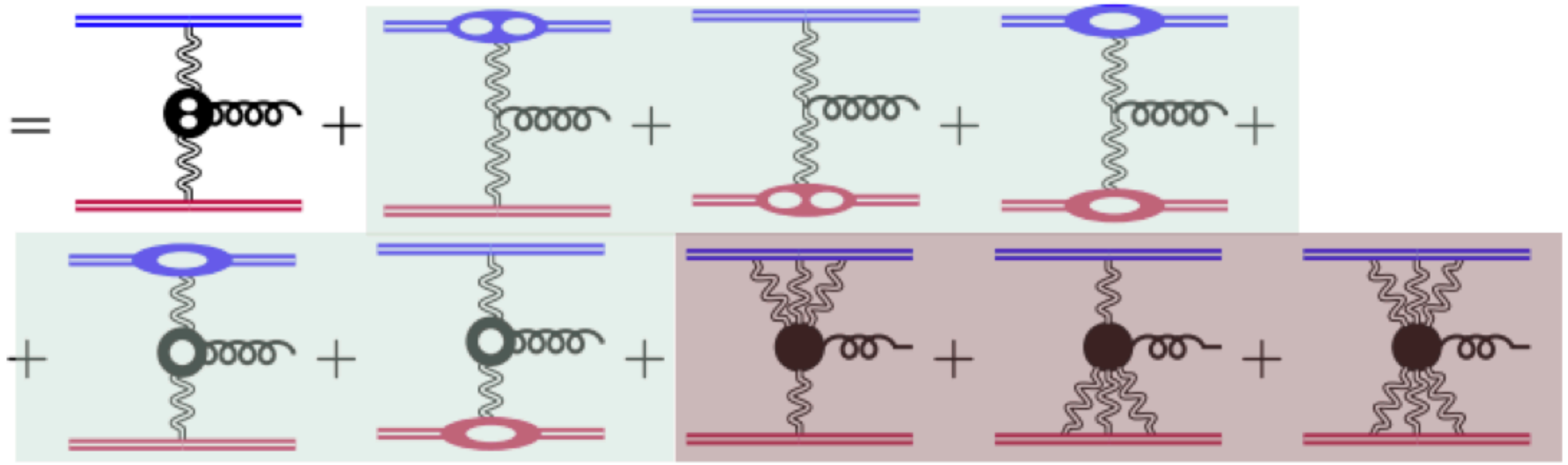


$$V_\lambda = V_\lambda^{LL} \cdot \left(1 + \alpha_s \mathcal{W}^{(1)} + \alpha_s^2 \mathcal{W}^{(2)} + \dots\right)$$

Disentangling the Regge cut and Regge pole in perturbative QCD

Giulio Falcioni,[1,*] Einan Gardi,[1,†] Niamh Maher,[1,‡] Calum Milloy,[2,§] and Leonardo Vernazza[2,3,¶]

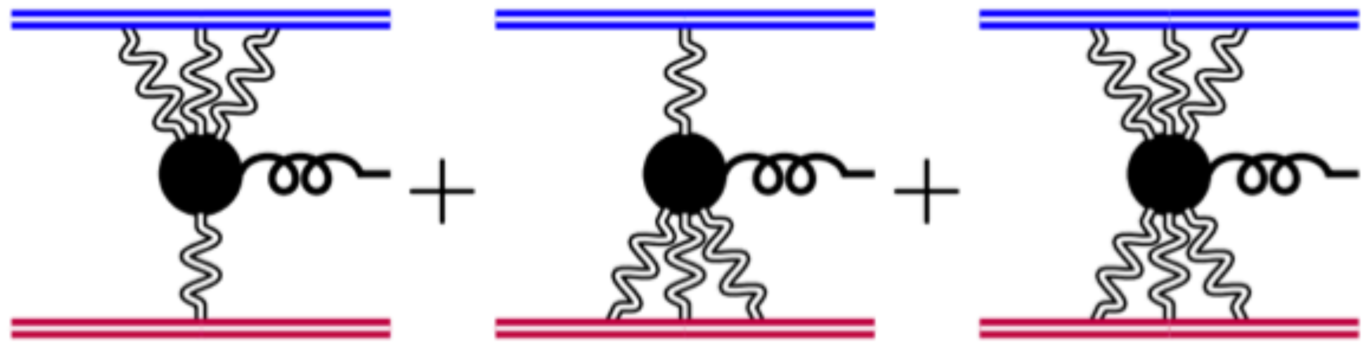
$\mathcal{A}^{(2),(--)}_{\rm NNLL} =$



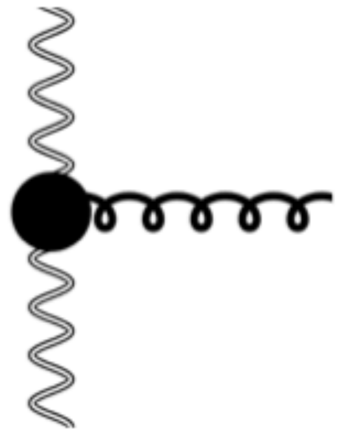
$$= \Big[\overline{\mathcal{W}}^{(2)}_{\lambda_4} + \bar{\mathcal{J}}^{(2)}_A + \bar{\mathcal{J}}^{(2)}_B + \bar{\mathcal{J}}^{(1)}_A \bar{\mathcal{J}}^{(1)}_B + \overline{\mathcal{W}}^{(1)}_{\lambda_4}(\bar{\mathcal{J}}^{(1)}_A + \bar{\mathcal{J}}^{(1)}_B)$$

$$+ (i\pi)^2 \left(B^{(2)}_{+-} \boldsymbol{T}^2_{+-} + B^{(2)}_{--} \boldsymbol{T}^2_{--} + B^{(2)}_{-+} \boldsymbol{T}^2_{-+} - B^{(2)}_d \frac{N_c^2}{4}\right)\Big] \mathcal{A}^{(0)}$$

Additional universal (factorized) contributions



$$\approx (i\pi)^2 \frac{N_c^2}{4} \left(B^{(2)}_{+-} + B^{(2)}_{--} + B^{(2)}_{-+} - B^{(2)}_d\right) \mathcal{A}^{(0)}$$



$$V_\lambda = V_\lambda^{LL} \cdot \left(1 + \alpha_s \mathcal{W}^{(1)} + \alpha_s^2 \mathcal{W}^{(2)} + \dots\right)$$

Disentangling the Regge cut and Regge pole in perturbative QCD

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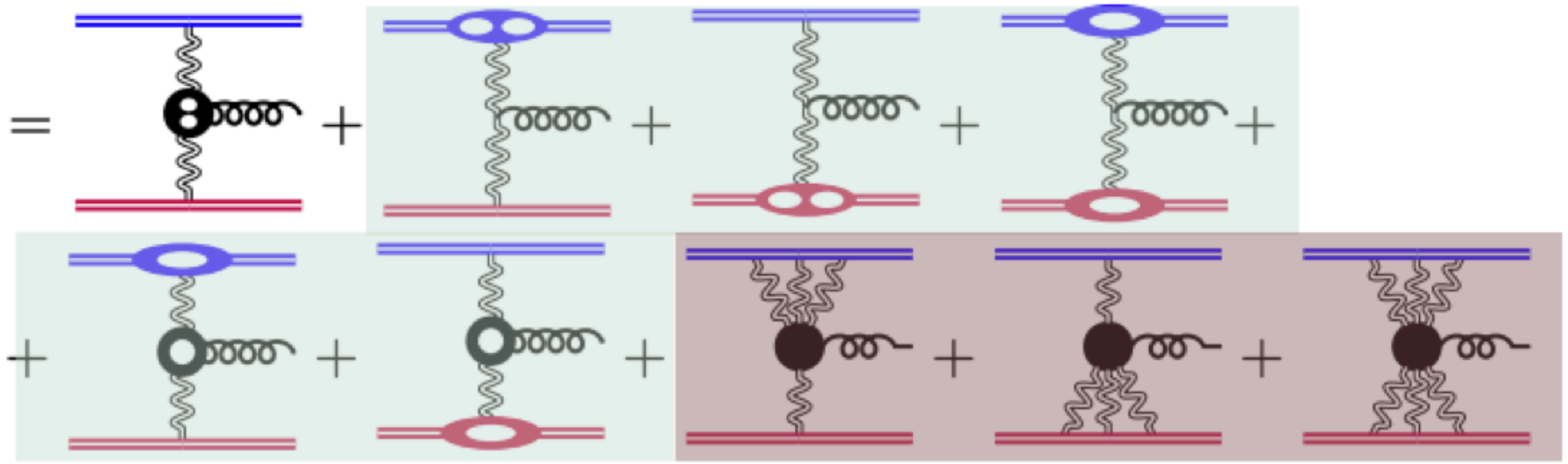
$\mathcal{A}^{(2),(--)}_{\mathrm{NNLL}} =$



$$= \Bigg[\overline{\mathcal{W}}^{(2)}_{\lambda_4} + \bar{\mathcal{J}}_A^{(2)} + \bar{\mathcal{J}}_B^{(2)} + \bar{\mathcal{J}}_A^{(1)}\bar{\mathcal{J}}_B^{(1)} + \overline{\mathcal{W}}^{(1)}_{\lambda_4}(\bar{\mathcal{J}}_A^{(1)} + \bar{\mathcal{J}}_B^{(1)})$$

$$+ (i\pi)^2\left(B^{(2)}_{+-}\mathcal{T}^2_{+-} + B^{(2)}_{--}\mathcal{T}^2_{--} + B^{(2)}_{-+}\mathcal{T}^2_{-+} - B^{(2)}_{d}\frac{N_c^2}{4}\right)\Bigg]\mathcal{A}^{(0)}$$

Additional universal (factorized) contributions

Mismatch between W and reggeized gluon at NNLL

THE QCD CENTRAL EMISSION VERTEX

NEW

Investigating the universality of five-point QCD scattering amplitudes at high energy

Federico Buccioni,[a] Fabrizio Caola,[b,c] Federica Devoto,[b,d] Giulio Gambuti[b]

The Two-Loop Lipatov Vertex in QCD

Samuel Abreu,[a,b] Giuseppe De Laurentis,[a] Giulio Falcioni,[c,d] Einan Gardi,[a] Calum Milloy,[e] Leonardo Vernazza[e]

$h_{w,i}$

Transcendental (weight w)

r_j

Rational

$$\hat{\mathcal{U}}^{(1)}_{+,QCD} = \frac{N_c}{2}\left(5\zeta_2 - h_{1,2}\left(h_{1,2}+3r_3\right) - i\pi h_{1,1}\right) - \frac{N_c-N_f}{3}\left(r_1 h_{1,2}+r_2\right),$$

$$\begin{aligned}
\hat{\mathcal{U}}^{(2)}_{+,QCD} = N_c^2\Bigg[&\frac{1}{144}i\pi\Big(-72\zeta_3 + h_{1,1}\left(-36\zeta_2+9h_{1,2}\left(3r_3+4h_{1,2}\right)-456\right)+464\\
&-27r_3\left(h_{1,3}h_{1,4}+8h_{2,2}-8h_{2,3}\right)\Big)+\frac{1}{432}\Big(216r_2-1809\zeta_4+216r_1h_{1,2}-2872\\
&+36\zeta_2\left(-18h_{1,1}^2+3\left(9r_3-7h_{1,2}\right)h_{1,2}+209\right)-9\big(-6h_{1,2}^4+98h_{1,2}^2+9r_3\big(2h_{1,2}^3\\
&+3\left(\left(h_{1,1}-4\right)h_{1,1}+24\right)h_{1,2}+h_{1,1}\left(h_{1,3}h_{1,4}+8h_{2,2}-8h_{2,3}\right)+64h_{3,6}\big)\big)\Big)\Bigg]\\
+N_c\left(N_c-N_f\right)\Bigg[&\frac{1}{216}i\pi\Big(36r_4+36r_2\left(h_{1,1}-1\right)+108r_3h_{1,2}+3h_{1,1}\left(3r_1h_{1,2}-40\right)\\
&-9r_1\left(12h_{1,2}+h_{1,3}h_{1,4}+8h_{2,2}-8h_{2,3}\right)+112\Big)+\frac{1}{648}\Big(36\zeta_2\left(9r_1h_{1,2}+55\right)\\
&+36\left(3\left(5r_3+r_6\right)-113r_1\right)h_{1,2}+36r_2\left(3h_{1,2}^2-15\zeta_2+6h_{1,1}-137\right)-9\big(9r_1h_{1,2}h_{1,1}^2\\
&-3\big(4r_4-12r_3h_{1,2}+r_1\left(36h_{1,2}-h_{1,3}h_{1,4}-8h_{2,2}+8h_{2,3}\right)-4\big)h_{1,1}+2\big(3r_1h_{1,2}^3\\
&+\left(6r_5+2\right)h_{1,2}^2-18\left(r_1-r_3\right)\left(h_{1,3}h_{1,4}+8h_{2,2}-8h_{2,3}\right)+96r_1h_{3,6}\big)\big)-260\Big)\Bigg]\\
+N_c\beta^{(0)}\Bigg[&\frac{1}{8}i\pi\Big(h_{1,1}^2+2h_{1,2}^2+4\zeta_2-8h_{2,1}\Big)+\frac{1}{48}\Big(-h_{1,4}^3-3h_{1,1}^2h_{1,4}+3h_{1,2}^2h_{1,4}\\
&-9h_{1,2}\left(h_{1,3}h_{1,4}+8h_{2,2}-8h_{2,3}\right)-48\left(2\zeta_2h_{1,4}-2h_{3,4}+2h_{3,5}+h_{3,7}\right)\\
&+3h_{1,3}^2h_{1,4}+232\zeta_3+3h_{1,1}\left(5h_{1,2}^2+2h_{1,3}h_{1,2}-16h_{2,1}\right)\Big)\Bigg]\\
+\frac{\left(N_c-N_f\right)^2}{54}&\Big[\left(r_2+r_1h_{1,2}\right)\left(6h_{1,1}-20\right)+3h_{1,2}h_{1,2}\Big]+\frac{N_f}{2N_c}\Big[r_2+\left(r_1-2r_3\right)h_{1,2}\Big]
\end{aligned}$$

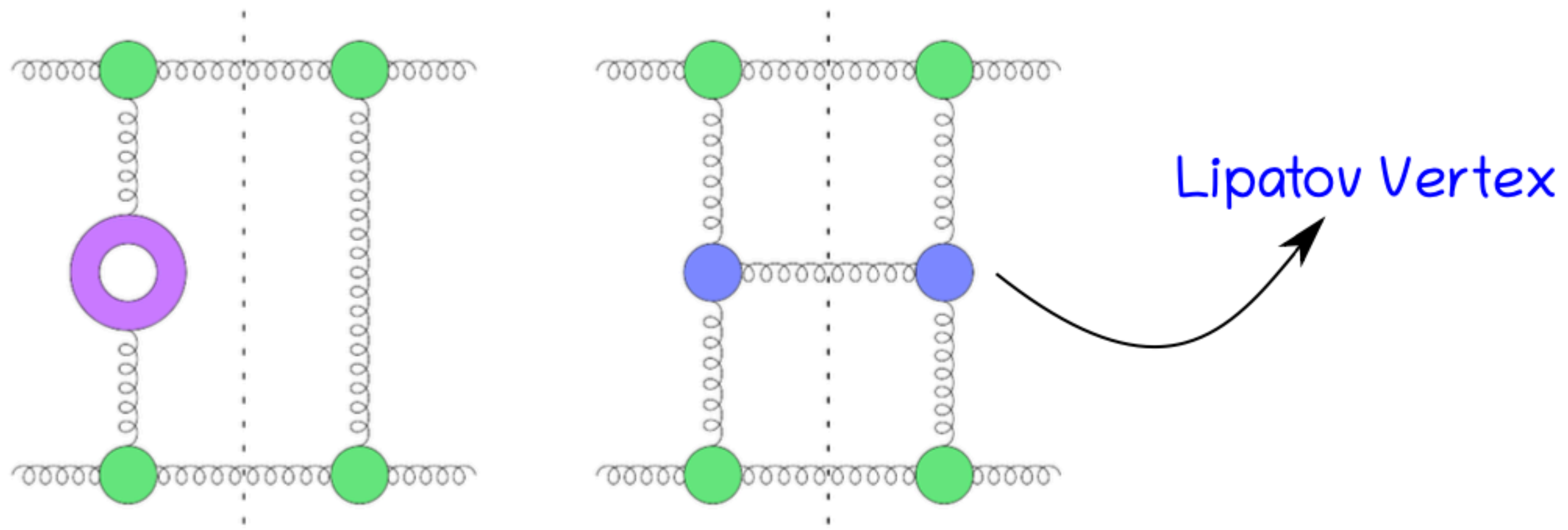
- Completely leading color (except for Nf/Nc)
- Leading color contributions of multi-W absorbed into definition of the vertex
- Finite remainder free of spurious $1 - z - \bar{z}$ and $z - \bar{z}$ singularities
- Leading transcendental part agrees with N=4
- Result in terms of single valued MPLs

$$\hat{\mathcal{U}}^{(1)}_{+,QCD} = \frac{N_c}{2}\left(5\zeta_2 - h_{1,2}\left(h_{1,2} + 3r_3\right) - i\pi h_{1,1}\right) - \frac{N_c - N_f}{3}\left(r_1 h_{1,2} + r_2\right),$$

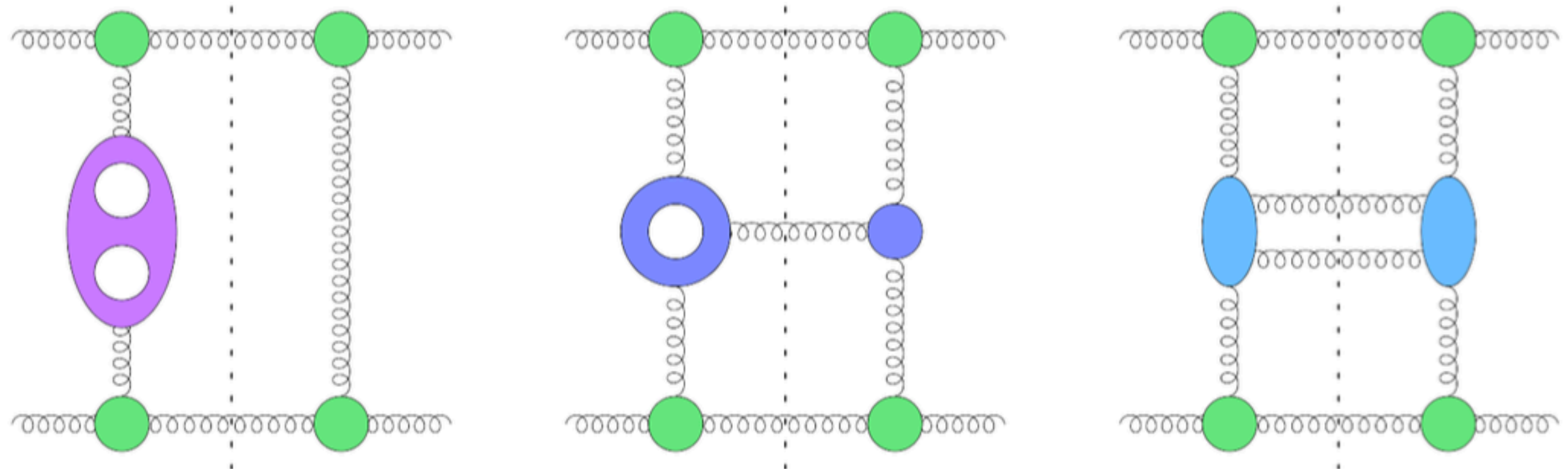
$$\begin{aligned}
\hat{\mathcal{U}}^{(2)}_{+,QCD} = N_c^2 \Bigg[& \frac{1}{144} i\pi \Big(- 72\zeta_3 + h_{1,1}\left(-36\zeta_2 + 9h_{1,2}\left(3r_3 + 4h_{1,2}\right) - 456\right) + 464 \\
& - 27r_3\left(h_{1,3}h_{1,4} + 8h_{2,2} - 8h_{2,3}\right)\Big) + \frac{1}{432}\Big(216r_2 - 1809\zeta_4 + 216r_1h_{1,2} - 2872 \\
& + 36\zeta_2\left(-18h_{1,1}^2 + 3\left(9r_3 - 7h_{1,2}\right)h_{1,2} + 209\right) - 9\big(- 6h_{1,2}^4 + 98h_{1,2}^2 + 9r_3\big(2h_{1,2}^3 \\
& + 3\left(\left(h_{1,1} - 4\right)h_{1,1} + 24\right)h_{1,2} + h_{1,1}\left(h_{1,3}h_{1,4} + 8h_{2,2} - 8h_{2,3}\right) + 64h_{3,6}\big)\big)\Big)\Bigg] \\
+ N_c\left(N_c - N_f\right) \Bigg[& \frac{1}{216} i\pi \Big(36r_4 + 36r_2\left(h_{1,1} - 1\right) + 108r_3h_{1,2} + 3h_{1,1}\left(3r_1h_{1,2} - 40\right) \\
& - 9r_1\left(12h_{1,2} + h_{1,3}h_{1,4} + 8h_{2,2} - 8h_{2,3}\right) + 112\Big) + \frac{1}{648}\Big(36\zeta_2\left(9r_1h_{1,2} + 55\right) \\
& + 36\left(3\left(5r_3 + r_6\right) - 113r_1\right)h_{1,2} + 36r_2\left(3h_{1,2}^2 - 15\zeta_2 + 6h_{1,1} - 137\right) - 9\big(9r_1h_{1,2}h_{1,1}^2 \\
& - 3\big(4r_4 - 12r_3h_{1,2} + r_1\left(36h_{1,2} - h_{1,3}h_{1,4} - 8h_{2,2} + 8h_{2,3}\right) - 4\big)h_{1,1} + 2\big(3r_1h_{1,2}^3 \\
& + \left(6r_5 + 2\right)h_{1,2}^2 - 18\left(r_1 - r_3\right)\left(h_{1,3}h_{1,4} + 8h_{2,2} - 8h_{2,3}\right) + 96r_1h_{3,6}\big)\big) - 260\Big)\Bigg] \\
+ N_c\beta^{(0)} \Bigg[& \frac{1}{8} i\pi \Big(h_{1,1}^2 + 2h_{1,2}^2 + 4\zeta_2 - 8h_{2,1}\Big) + \frac{1}{48}\Big(- h_{1,4}^3 - 3h_{1,1}^2h_{1,4} + 3h_{1,2}^2h_{1,4} \\
& - 9h_{1,2}\left(h_{1,3}h_{1,4} + 8h_{2,2} - 8h_{2,3}\right) - 48\left(2\zeta_2h_{1,4} - 2h_{3,4} + 2h_{3,5} + h_{3,7}\right) \\
& + 3h_{1,3}^2h_{1,4} + 232\zeta_3 + 3h_{1,1}\left(5h_{1,2}^2 + 2h_{1,3}h_{1,2} - 16h_{2,1}\right)\Big)\Bigg] \\
+ \frac{\left(N_c - N_f\right)^2}{54} \Bigg[& \left(r_2 + r_1h_{1,2}\right)\left(6h_{1,1} - 20\right) + 3h_{1,2}h_{1,2}\Bigg] + \frac{N_f}{2N_c}\Bigg[r_2 + \left(r_1 - 2r_3\right)h_{1,2}\Bigg]
\end{aligned}$$

SUMMARY: BFKL PERSPECTIVE

LL:



NLL:

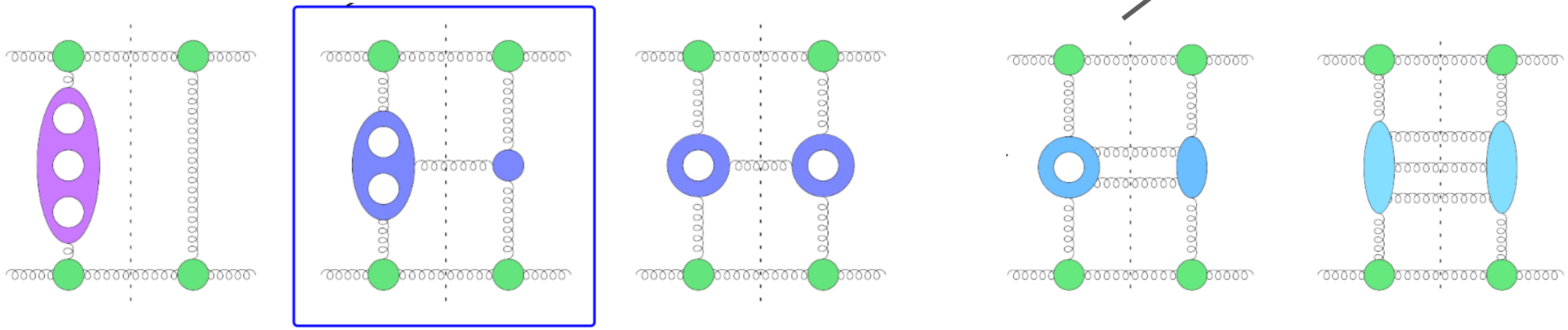


[Byrne, Del Duca, Dixon, Gardi, Smilie 2204.12459]

[Byrne 2312.15051]

Computed in N=4

NNLL:



This talk

Artwork courtesy of:
[Byrne, Del Duca, Dixon, Gardi, Smilie 2204.12459]

CONCLUSIONS AND OUTLOOK

- Amplitudes at high energy are very interesting lab to explore structures
- Review of factorization at LL and NLL, breaking of factorization at NNLL due to multi-W exchanges
- Latter can be calculated in the Wilson-line formalism
- Matching to full amplitude allows to extract 2-loop central emission vertex in QCD
- Other signatures?!
- Precise relation between W field and the reggeon?!
- NMRK limits
- Comparison with Glauber SCET

Still a lot to do to get BFKL kernel@NNLL but we're one step closer!

THANK YOU!

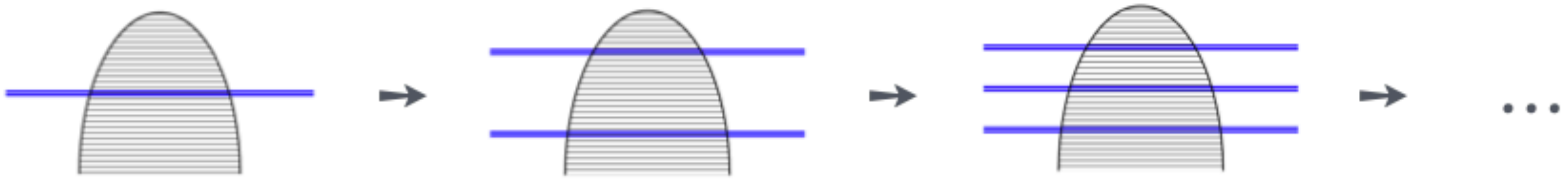
BACKUP

rapidity evolution $$-\frac{d}{d\eta}[U(z_1)U(z_2)\dots U(z_n)] = H\cdot[U(z_1)U(z_2)\dots U(z_n)]$$

Balitsky-JIMWLK eq.

dipoles at leading order $$H = \sum_{i,j} H_{ij}$$

$$H_{ij} = \frac{\alpha_s}{4\pi}\int \mathrm{d}^{2-2\epsilon} z_0 K_{ij}(z_0)\left[T^a_{i,L}T^a_{j,L} - U^{ab}_{adj.}(z_0)T^a_{i,L}T^a_{j,R} + (i\leftrightarrow j)\right]$$



unbounded system of coupled equations!

The (hard) N=4 vertex

$$\hat{\mathcal{U}}^{(1)}_{\mathcal{N}=4} = \frac{N_c}{2}\left(-h_{1,2}^2 - i\pi h_{1,1} + 5\zeta_2\right),$$

$$\hat{\mathcal{U}}^{(2)}_{\mathcal{N}=4} = -\frac{N_c^2}{4}\left(h_{1,2}^2\,(7\zeta_2 - i\pi h_{1,1}) + \zeta_2 h_{1,1}\,(6h_{1,1} + i\pi) - \frac{h_{1,2}^4}{2} + 2i\pi\zeta_3 + \frac{67}{4}\zeta_4\right)$$

$h_{w,i}$
Transcendental
(weight w)

r_j
Rational

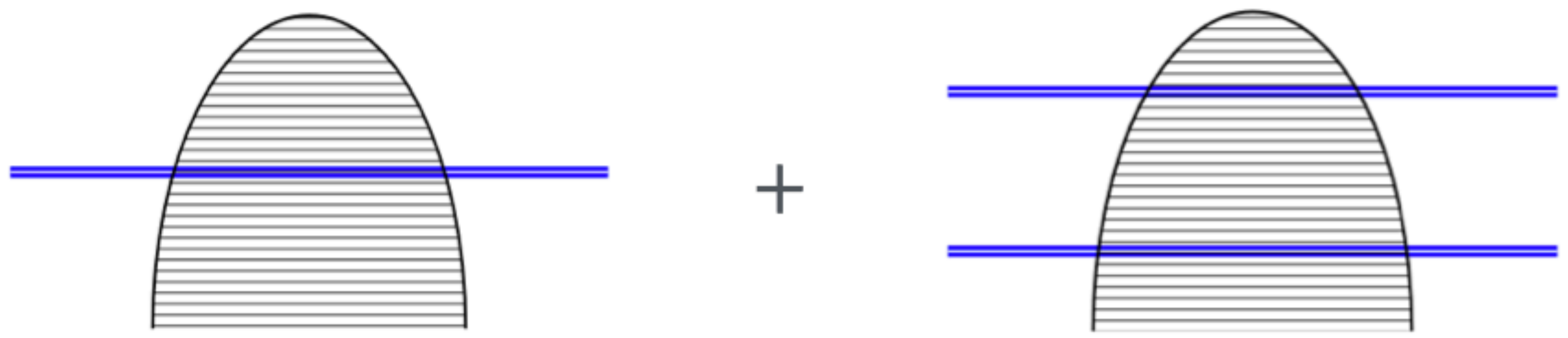
Some observations

QCD	N=4
(almost) leading colour	leading colour
logarithms @ 1loop	logarithms @ 1loop
only w3 polylogs @ 2loop	logarithms @ 2loop
simple rational functions	no rational functions

MAX transcendentality principle

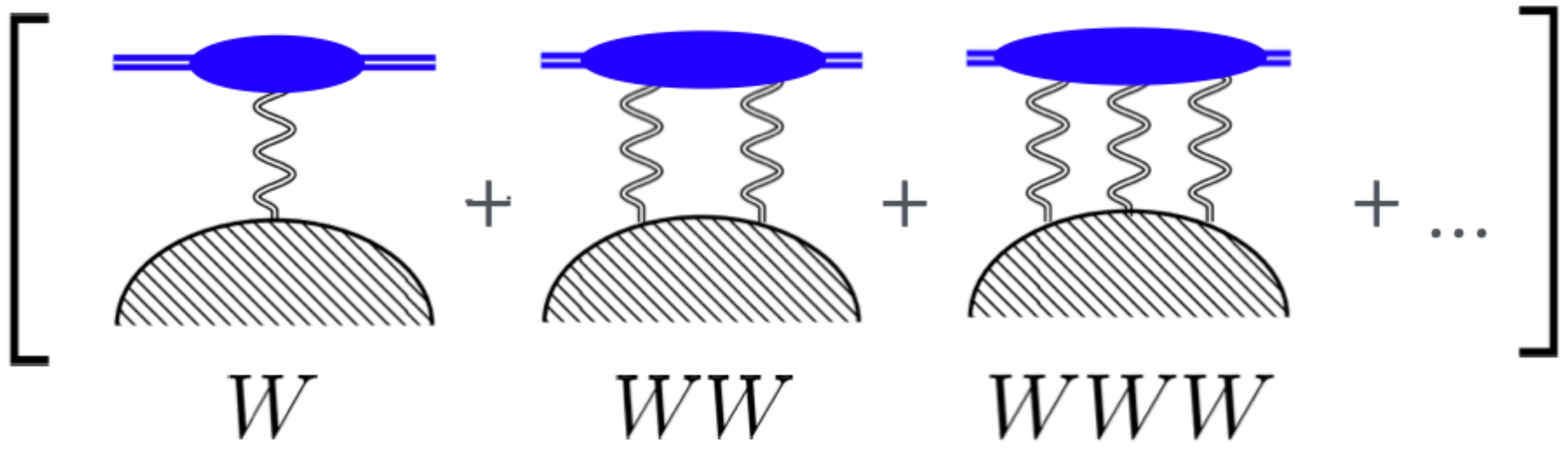
Operator Product Expansions

$$a^\dagger(p_1)a(p_n) \sim 2\pi\delta(p_1^+ - p_n^+)\times 2p_1^+ \times \left[\mathcal{J}(q)U(q) + \int\{dk\}\mathcal{J}'(q,k)U(q-k)U(k) + \ldots\right]$$

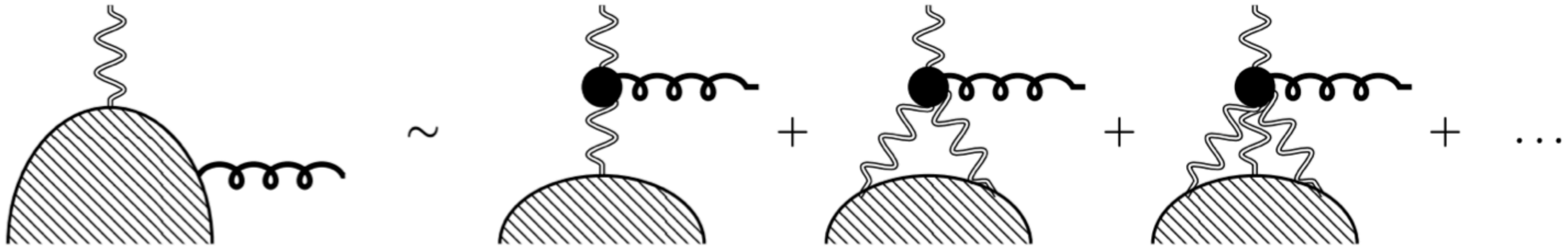


impact factors $\mathcal{J} = 1 + \mathcal{O}(\alpha_s)$ $\mathcal{J}' = \mathcal{O}(\alpha_s)$

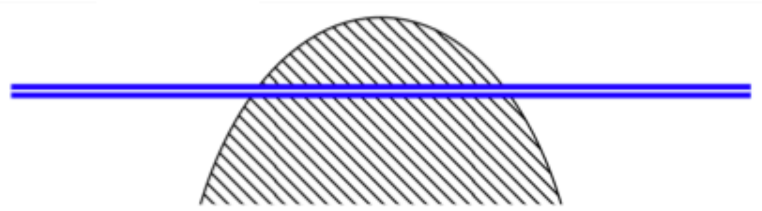
$$a^\dagger(p_1)a(p_n) \sim 2\pi\delta(p_1^+ - p_n^+)\times 2p_1^+ \times$$



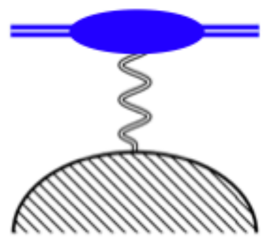
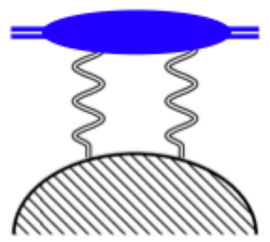
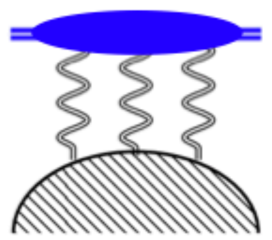
$$W(q)a(p) \sim$$



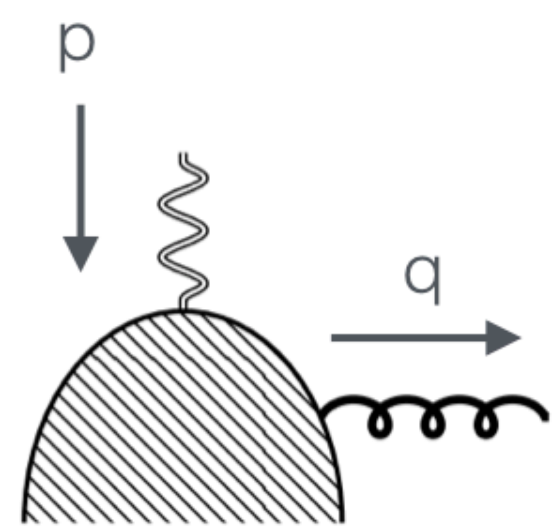
Details on projectile OPE



$$a^{a_1,\dagger}_{\lambda_1}(p_1)\,a^{a_5}_{\lambda_5}(p_5) \sim 2\pi\delta(p_1^+ - p_5^+)\delta_{\lambda_1\lambda_5} \times 2p_1^+ \times \Big\{ (ig_s)\mathcal{J}(\mathbf{q}_A)[\![W(\mathbf{q}_A)]\!]_A + \frac{(ig_s)^2}{2!}\int\{\mathrm{d}\mathbf{q}\}\,[1+\mathcal{J}'(\mathbf{q}_A,\mathbf{q})]\;[\![W(\mathbf{q}_A-\mathbf{q})W(\mathbf{q})]\!]_A + \frac{(ig_s)^3}{3!}\int\{\mathrm{d}\mathbf{q}_1\}\{\mathrm{d}\mathbf{q}_2\}\,[\![W(\mathbf{q}_A-\mathbf{q}_1)W(\mathbf{q}_1-\mathbf{q}_2)W(\mathbf{q}_2)]\!]_A + \dots \Big\}$$

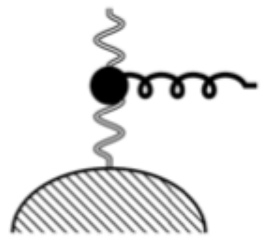
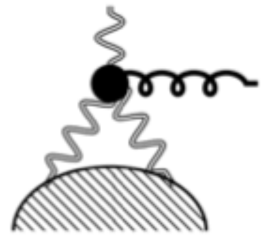
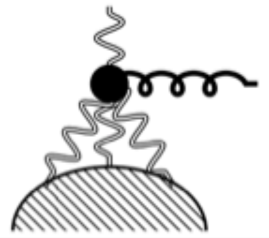
Details on emission OPE



$$[\![O_1 O_2 \dots O_n]\!]^{ab}_r \equiv (T_r^{c_1})_{aa_1}(T_r^{c_2})_{a_1a_2}\dots(T_r^{c_n})_{a_{n-1}b}\,O_1^{c_1}O_2^{c_2}\dots O_n^{c_n}$$

$$[\![O_1 O_2 \dots O_n]\!]^{ab} \equiv [\![O_1 O_2 \dots O_n]\!]^{ab}_{\mathrm{adj}}\,,$$

$$W(\mathbf{p})^b\,a^a_\lambda(q) \sim 2g_s[\![W]\!]^{ab}(\mathbf{q}+\mathbf{p})\left[\frac{\varepsilon^*_\lambda\cdot\mathbf{p}}{\mathbf{p}^2}+\frac{\varepsilon^*_\lambda\cdot\mathbf{q}}{\mathbf{q}^2}\right]\mathcal{W}_\lambda(\mathbf{p},\mathbf{q}) + ig_s^2\int\{\mathrm{d}\mathbf{k}_1\}[\![W(\mathbf{q}+\mathbf{p}-\mathbf{k}_1)W(\mathbf{k}_1)]\!]^{ab}\left[\frac{\varepsilon^*_\lambda\cdot\mathbf{p}}{\mathbf{p}^2}+\frac{\varepsilon^*_\lambda\cdot(\mathbf{k}_1-\mathbf{p})}{(\mathbf{k}_1-\mathbf{p})^2}\right] + g_s^3\int\{\mathrm{d}\mathbf{k}_1\}\{\mathrm{d}\mathbf{k}_2\}[\![W(\mathbf{q}+\mathbf{p}-\mathbf{k}_1)W(\mathbf{k}_1-\mathbf{k}_2)W(\mathbf{k}_2)]\!]^{ab}\times\left[\frac{1}{6}\left(\frac{\varepsilon^*_\lambda\cdot(\mathbf{k}_1-\mathbf{p})}{(\mathbf{k}_1-\mathbf{p})^2}\right)-\frac{1}{2}\left(\frac{\varepsilon^*_\lambda\cdot(\mathbf{k}_2-\mathbf{p})}{(\mathbf{k}_2-\mathbf{p})^2}\right)-\frac{1}{3}\left(\frac{\varepsilon^*_\lambda\cdot\mathbf{p}}{\mathbf{p}^2}\right)\right]$$

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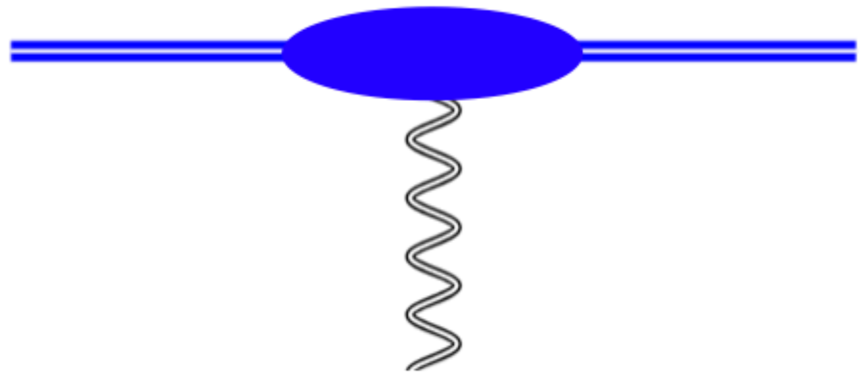
Impact Factors



$$\mathcal{J} = 1 + \alpha_s J_1 + \alpha_s^2 J_2 + \ldots$$

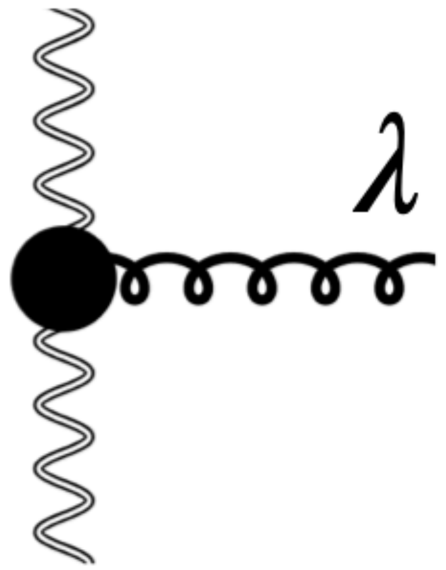
Regge Trajectory

$$e^{L\,\tau_g(\alpha)}$$

$$\tau_g(\alpha) = \alpha_s \tau_1 + \alpha_s^2 \tau_2 + \alpha_s^3 \tau_3 + \ldots$$

and similarly for all

Lipatov Vertex



$$V^\lambda = V_0^\lambda + \alpha_s V_1^\lambda + \alpha_s^2 V_2^\lambda$$

and similarly for

Rational Functions

$$r_1 = \frac{z^3 + (1-\bar{z})^3}{(1-z-\bar{z})^3}, \quad r_2 = \frac{z(1-\bar{z})}{(1-z-\bar{z})^2}\left(\frac{1}{1-z} + \frac{1}{\bar{z}}\right), \quad r_3 = \frac{1+z-\bar{z}}{1-z-\bar{z}},$$

$$r_4 = \frac{z(1-\bar{z})}{(1-z)\bar{z}}, \quad r_5 = \frac{z(1-\bar{z})}{(1-z-\bar{z})^2}, \quad r_6 = \frac{z(1-\bar{z})(z-\bar{z})}{\bar{z}(1-z)(1-z-\bar{z})},$$

(anti-)symmetric under $z \to 1-\bar{z}$

Transcendental Functions

$$g_{1,4} = \ln(z\bar{z}), \quad g_{1,5} = \ln\left((1-z)(1-\bar{z})\right),$$
$$g_{1,6} = \ln(z) - \ln(\bar{z}), \quad g_{1,7} = \ln(1-z) - \ln(1-\bar{z}),$$
$$g_{2,1} = D_2(z,\bar{z}),$$
$$g_{2,2} = \mathrm{Li}_2(z) + \mathrm{Li}_2(\bar{z}),$$
$$g_{2,3} = \mathrm{Li}_2\left(\frac{z}{1-\bar{z}}\right) + \mathrm{Li}_2\left(\frac{\bar{z}}{1-z}\right) + (g_{1,4} - g_{1,5})\ln(|1-z-\bar{z}|)$$
$$+ i\pi\,(g_{1,6} + g_{1,7})\,\mathrm{sgn}[\mathrm{Im}(z)]\Theta\left(\mathrm{Re}(z) - \frac{1}{2}\right),$$
$$g_{3,1} = D_3(z,\bar{z}),$$
$$g_{3,2} = D_3(1-z, 1-\bar{z}),$$
$$g_{3,3} = \mathrm{Li}_3(z) - \mathrm{Li}_3(\bar{z}),$$
$$g_{3,4} = \mathrm{Li}_3(1-z) - \mathrm{Li}_3(1-\bar{z})$$
$$g_{3,5} = \mathrm{Li}_3\left(\frac{z\bar{z}}{(1-z)(1-\bar{z})}\right) + \frac{1}{2}\ln(1-z-\bar{z})\ln^2\left(\frac{z\bar{z}}{(1-z)(1-\bar{z})}\right),$$
$$g_{3,6} = 2\,\mathrm{Li}_3\left(\frac{z}{1-\bar{z}}\right) - 2\mathrm{Li}_3\left(\frac{\bar{z}}{1-z}\right) - \ln\left(\frac{z\bar{z}}{(1-z)(1-\bar{z})}\right) D_2\left(\frac{z}{1-\bar{z}}, \frac{\bar{z}}{1-z}\right)$$
$$+ \frac{i\pi}{2}\left[(g_{1,4} - g_{1,5})^2 + (g_{1,6} + g_{1,7})^2\right]\mathrm{sgn}[\mathrm{Im}(z)]\,\Theta\left(\mathrm{Re}(z) - \frac{1}{2}\right),$$
$$g_{3,9} = \mathrm{Li}_3\left(\frac{1-z-\bar{z}}{(1-z)(1-\bar{z})}\right),$$

single valued!

$$D_2(z,\bar{z}) = \mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z}) + \frac{\ln(z\bar{z})}{2}\left(\ln(1-z) - \ln(1-\bar{z})\right),$$
$$D_3(z,\bar{z}) = \mathrm{Li}_3(z) + \mathrm{Li}_3(\bar{z}) - \frac{1}{2}\ln(z\bar{z})\left(\mathrm{Li}_2(z) + \mathrm{Li}_2(\bar{z})\right) - \frac{1}{4}\ln^2(z\bar{z})\ln\left((1-z)(1-\bar{z})\right)$$

Infrared Structure

$$\boldsymbol{\mathcal{H}}^{[AB]} = \lim_{\epsilon\to 0} \mathbf{Z}_{IR}^{-1}\, \boldsymbol{\mathcal{B}}^{[AB]}$$

$$\mathbf{Z}_{IR}(\epsilon, \{p\}, \mu) = \exp\left[\int_{\mu}^{\infty} \frac{\mathrm{d}\mu'}{\mu'} \boldsymbol{\Gamma}_{IR}(\{p\}, \mu')\right]$$

$$\begin{aligned}
\boldsymbol{\Gamma}_{IR} = {} & \gamma_K \mathcal{C}_A \ln\frac{-s_{51}}{\mu^2} - \frac{\gamma_K}{2}\ln\frac{-s_{51}}{\rho^2}(\mathbf{T}_+^{15})^2 + 2\gamma_A \\
& + \gamma_K \mathcal{C}_B \ln\frac{-s_{23}}{\mu^2} - \frac{\gamma_K}{2}\ln\frac{-s_{23}}{\rho^2}(\mathbf{T}_+^{23})^2 + 2\gamma_B \\
& + \gamma_K L_A (\mathbf{T}_+^{15})^2 + \gamma_K L_B (\mathbf{T}_+^{23})^2 \\
& + \frac{\gamma_K}{2}\left(-\mathcal{C}_4 \ln\frac{\mu^2}{\mathbf{p}_4^2} + \ln\frac{\rho^2}{\mathbf{p}_4^2}(\mathbf{T}_+^{15})^2 + \ln\frac{\rho^2}{\mathbf{p}_4^2}(\mathbf{T}_+^{23})^2 - i\pi\, \boldsymbol{\mathcal{T}}_{++}\right) + \gamma_4 \\
& + \frac{\gamma_K}{2} \times i\pi\, (\boldsymbol{\mathcal{T}}_{+-} + \boldsymbol{\mathcal{T}}_{--} + \boldsymbol{\mathcal{T}}_{-+})\,.
\end{aligned}$$