$SO(1, d+1)$ symmetry of the Exact RG equation

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Motivation

1. Holography, a feature of quantum gravity, states the duality between the $d+1$ dimensional bulk gravitational theory and d-dimensional boundary field theory. AdS_{d+1}/CFT_d is a concrete and successful example towards this *(Witten 1998, Maldacena 1999)*.
2. **Symmetry in holography:** One of the explanation why AdS is a natural candidate for the bulk dual is the fact that its isometry group is $SO(1, d+1)$ (For Euclidean AdS), same as conformal group of the d dimensional CFT.
3. On the other hand, another important aspect of holography is **emergence of extra dimension**: the scale of the boundary can be interpreted as the extra dimension in the bulk. This stems from the **UV/IR duality or Holographic Renormalization Group** : UV divergence in the boundary is dual to IR divergence in the bulk.
4. **ERG apprach to** AdS/CFT: interpreting holographic RG purely from the boundary ERG. In this approach, one can map the evolution operator of Polchinski's equation to the AdS action, with a field redefinition and a suitable choice of cutoff function *(Sathiapalan, Sonoda 2017,2018)*.
5. **In this talk**: Following this, how to interpret the AdS isometry group from the perspective of the boundary ERG?

The Idea of Holographic RG

- Consider AdS space in Poincare patch,

$$ds^2 = \frac{1}{z^2}\{dz^2 + dx_i dx_j \eta^{ij}\}$$

- For fixed z, z looks like lattice spacing of the boundary theory. So speculation: change in z $\rightarrow$ scale change of Boundary theory.

- In bulk: $S_{bulk}[\phi] = \int \sqrt{g}\mathcal{L}d^{d+1}x + S_B[\phi, \epsilon]; \quad \epsilon \rightarrow 0.$

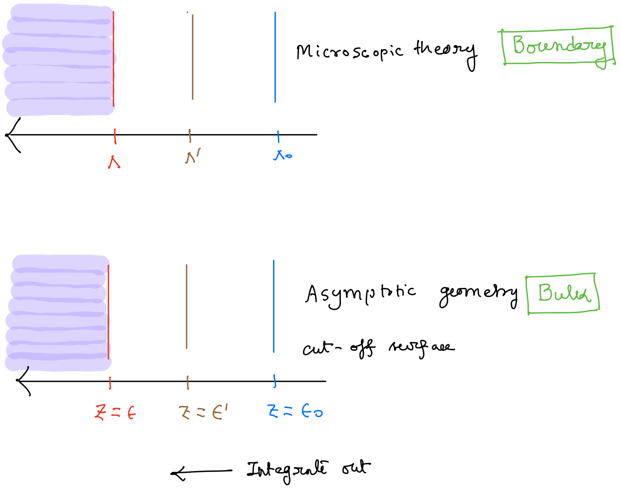
- In boundary: I_{eff} is the Wilson action.

- S_B is interpreted as the I_{eff}, higher modes $0 < z < \epsilon$ are integrated out.

- $\frac{\partial S_B}{\partial \epsilon}$ gives Hamiltonian of radial evolution **(Faulkner et al. '11)**.

- This is interpreted as RG of boundary - from the AdS/CFT correspondence.

In picture


Microscopic theory
Boundary
Λ
Λ'
Λ_0
Asymptotic geometry
Bulk
cut-off surface
$z = \epsilon$
$z = \epsilon'$
$z = \epsilon_0$
Integrate out

Literature on holographic RG

1. **E.T. Akhmedov (1998)**: classical EOM of SUGRA as RG on SYM side.
2. **Haro, Skendris, Soloudukhin (2000)** : IR regularisation of on shell bulk action, calculation of holographic stress energy tensor.
3. **Polchinski, Hemsheerk (2011)**: which radial cutoff in bulk corresponds to boundary cutoff.
4. **Faulkner, Liu, Rangamani (2011)**: Wilson action and Hamilton-Jacobi equation of the bulk.
5. **S.S. Lee (2011,2012)**: bulk dual from Wilsonian RG of matrix model.
6. **Melo, Santos (2020)**: deriving bulk dual on applying local RG on BFSS model.

ERG approach to AdS/CFT

AdS action without using AdS/CFT

• Can one obtain Holographic RG equation directly starting from RG of boundary theory (i.e. without invoking AdS/CFT conjecture)?

• We choose Exact Renormalization Group (ERG) for this purpose because concept of scale is natural in this scheme.

Simplest case

Derivation of AdS free scalar action from Boundary CFT using ERG

This is a three step procedure.

Step 1: Write Polchinski ERG Equation for d-dimensional Boundary Theory.

Step 2: Write Evolution Operator for ERG equation as a functional integral over a $d+1$ dimensional field theory.

Step 3: Perform field redefinition and choose a cutoff function to map the $d+1$ dimensional field theory action obtained in Step 2 to an action in $d+1$ dimensional AdS space.

Polchinski's equation and bulk action

B.Sathiapalan, H.Sonoda('17,'18)

- Consider perturbed Gaussian action :

$$S = \frac{1}{2}\int_p \phi(p)G^{-1}(p)\phi(-p) + S_I$$

- Step 1: Define $\psi = e^{-S_I}$.

The Polchinski's equation in form of diffusion equation **(Morris 1994)**:

$$\frac{\partial\psi}{\partial t} = -\frac{1}{2}\int_p \dot{G}(p)\frac{\partial^2\psi}{\partial\phi(p)\phi(-p)}$$

- Step 2: Evolution operator :

$$U(\phi_f, t_f; \phi_i, t_i) = e^{-\int_{t_i}^{t_f} H} = e^{-\frac{1}{2}\{G(t_f)-G(t_i)\}\frac{\partial^2}{\partial\phi(p)\partial\phi(-p)}}$$

- Write evolution operator in terms of a functional integral:

$$\psi(t_f, \phi_f) = \int \mathcal{D}\phi_i(p) \int_{\phi(p,t_i)=\phi_i(p),\phi(p,t_f)=\phi_f(p))} \mathcal{D}\phi(p,t)$$

$$\times e^{\underbrace{\frac{1}{2}\int_{t_i}^{t_f} dt \int_p \frac{\dot{\phi}(p)\dot{\phi}(-p)}{\dot{G}(p)}}_{d+1 \;\; dimensional \;\; field \;\; theory}} \psi(\phi_i, t_i)$$

- We have a $d+1$ dimenional field theory albeit with a non standard kinetic term.

Step 3: Map to AdS free scalar action

•Step 3:

Write action as

$$\frac{1}{2}\int dt\,\frac{1}{e^{td}}\int d^dq\,\frac{\dot{\phi}(q)\dot{\phi}(-q)}{f^2};\quad f^2=-e^{-td}\dot{G}(p,t)$$

1. Field redefinition:

$$\phi(p)=y(p)f$$

2. Change of variables:

$$z=e^t$$

3. Choose f to satisfy :

$$z^{d-1}\left(z^{-d+1}\frac{d}{dz}\right)^2 e^{-\ln f}=z^{-d+1}\left(p^2+\frac{m^2}{z^2}\right)e^{-\ln f}$$

• We get free scalar AdS action.

$$\boxed{\frac{1}{2}\int dz\int_p z^{-d+1}\left[\frac{\partial y(p)}{\partial z}\frac{\partial y(-p)}{\partial z}+\left(p^2+\frac{m^2}{z^2}\right)y(p)y(-p)\right]}$$

AdS action

- $y(p)$ and $\frac{1}{f}$ satisfy same equation.

- Form of $\frac{1}{f}$: $Az^{d/2}K_\nu(pz) + Bz^{d/2}I_\nu(pz)$ with $\nu^2 = m^2 + \frac{d^2}{4}$.

- Taking low energy limit $pz \to 0$ one can determine exponent of p, as $f^2 \propto \dot{G}$, so ν is related to conformal dimension of the boundary composite operator. Putting this value gives correct value of bulk mass m^2.

- Once we have the AdS action the correlator can be calculated in the standard way.

Literature on ERG approach to AdS/CFT

1. **Sathiapalan, Sonoda (2017, 2018)** : Evolution operator of Polchinski's equation for a scalar field theory maps to free AdS action. Extended for anomalous dimension and generating functional.
2. **Sathiapalan (2020, 2024)**: Composite operator in O(N) model. The bulk action contains cubic potential term consistent with literature; extension to flat bulk dual space.
3. **Dharanipragada, Dutta, Sathiapalan (2021)**: Derivation of EOM of vector boson and graviton from the ERG equation of vector current and stress tensor.
4. **Dharanipragada, Dutta, Sathiapalan (2022)**: attempt to understand dS/CFT from analytic continuation of ERG equation.
5. **Dharanipragada, Sathiapalan (2023)**: Uniqueness of map to AdS in terms of locality in 3-scalar and graviton-2 scalar coupling.
6. **Dharanipragada, Sathiapalan (2024)** : Construction of non-abelian spin 1 gauge field with cubic interaction from a free $U(2N)$ theory in the boundary.

Symmetry in AdS/CFT

1. $SO(1, d+1)$ algebra: Euclidean AdS_{d+1} isometry group and d dimensional conformal group

$$[D, T_\mu] = T_\mu,\ [D, K_\mu] = -K_\mu,\ [K_\mu, T_\nu] = 2\delta_{\mu\nu} D - 2M_{\mu\nu}$$
$$[M_{\mu\nu}, T_\sigma] = \delta_{\nu\sigma} T_\mu - \delta_{\mu\sigma} T_\nu,\ [M_{\mu\nu}, K_\sigma] = \delta_{\nu\sigma} K_\mu - \delta_{\mu\sigma} K_\nu$$
$$[M_{\mu\nu}, M_{\rho\sigma}] = \delta_{\nu\rho} M_{\mu\sigma} - \delta_{\mu\rho} M_{\nu\sigma} - \delta_{\nu\sigma} M_{\mu\rho} + \delta_{\mu\sigma} M_{\nu\rho}$$

The conformal transformation in d dimensional boundary:

$$\begin{aligned}
&\textit{Translation}, T_\mu : p_\mu, \\
&\textit{Rotation}, M_{\mu\nu} : p_\mu \frac{\partial}{\partial p^\nu} - p_\nu \frac{\partial}{\partial p^\mu}, \\
&\textit{Dilatation}, D : p.\frac{\partial}{\partial p} + (d - \Delta_\phi), \\
&\textit{Special conformal}, K_\mu : 2(d - \Delta)\frac{\partial}{\partial p^\mu} + 2p^\rho \frac{\partial^2}{\partial p^\rho \partial p^\mu} - p_\mu \frac{\partial^2}{\partial p^\rho \partial p_\rho},
\end{aligned}$$

AdS_{d+1} generators in Poincare co-ordinates

$$\textit{Translation}, T_\mu : p_\mu,$$

$$\textit{Rotation}, M_{\mu\nu} : p_\mu \frac{\partial}{\partial p^\nu} - p_\nu \frac{\partial}{\partial p^\mu},$$

$$\textit{Dilatation}, D : p.\frac{\partial}{\partial p} + (d - \Delta_\phi) - z\frac{\partial}{\partial z},$$

$$\textit{Special conformal}, K_\mu :$$

$$2(d-\Delta)\frac{\partial}{\partial p^\mu} + 2p^\rho \frac{\partial^2}{\partial p^\rho \partial p^\mu} - p_\mu \frac{\partial^2}{\partial p^\rho \partial p_\rho} - 2z\frac{\partial^2}{\partial z \partial p^\mu} + z^2 p_\mu.$$

Symmetry in ERG approach-I

1. The evolution operator keeps the fixed point theory invariant, hence naturally it should also inherit $SO(1, d+1)$ symmetry of a scale (conformal) invariant theory.
2. We show this is indeed true, the evolution operator for any arbitrary cutoff is invariant under a non-standard SO(1,d+1) transformations, where the conformal transformation is dependent of cutoff function.
3. For special choice of a cutoff function, which takes the evolution operator to *AdS* action, the non-standard transformation reduces to *AdS* isometry

Symmetry in ERG approach-II

1. How to explain the scale and cutoff function dependent part in the AdS isometry a.k.a symmetry transformation of the evolution operator of ERG equation?
2. Naively one can take the condition of scale invariance of Wilson action as follows.

$$\Sigma\Psi = \int_p \frac{\delta}{\delta\phi(p)} \left[\left(p^\mu \frac{\delta}{\delta p^\mu} - \Delta^p_\phi \right) \phi\psi \right] = 0, \psi = e^{-S_\Lambda}.$$

3. Even for the free theory this is satisfied with only $\Lambda \to \infty$ or $K(p,\Lambda) = 1$.
4. For finite cutoff , e^{-S_Λ} evolves with Λ, hence we need to replace

   $\Sigma \to [\Sigma] = U\Sigma U^{-1}$.

   U: evolution operator of the Polchinski's equation.
5. Composite operators: $[\phi(p)] = \frac{1}{K(p)}\phi(p) + \frac{1-K}{K}\frac{\delta}{\delta\phi(-p)}$, $\left[\frac{\delta}{\phi(p)}\right] = K(p)\frac{\delta}{\delta\phi(p)}$.

6. Because of these composite operators there is scale dependent part in the transformations.
7. It can be shown that the scale evolved version of the above condition is the fixed point ERG equation.

$$[\Sigma]\Psi = 0 = (-\Lambda \frac{\partial}{\partial \Lambda} + \Sigma)\Psi$$

8. So we expect the scale transformation law to change by above scale dependent contribution.
9. Similar reasoning should hold for special conformal transformation too, though it is not straightforward to understand, instead we have derived that explicitly.

Literature on conformal symmetry in ERG

1. **Delamotte, Tissier, Wschebor (2016); Polsi, Tissier, Wschebor (2018,2019)**: How existence and renormalization of certain vector operators determines the implication of conformal invariance in a scale invariant theory for a large class of systems.
2. **Cabrera, Polsi, Wschebor (2024)**: Fixing non-physical regulator parameters from the constraint of conformal invariance, invoking the principle of maximal conformality, equivalent to the principle of minimal sensitivity.
3. **Delamotte, Polsi, Tissier, Wschebor (2024**: Balog, Polsi, Tissier, Wschebor (2024): To fix non-physical regulator parameters and fixing regulator.
4. **Poster Session: Ibanez, Tissier, Polsi (2026)**: Optimizing regulator and convergence of derivative expansion.

DETAILS OF COMPUTATIONS

We would like to find the symmetry transformation of the evolution operator in $\phi(p,t)$ and redefined variable $y(p,z)$.

- $$\mathcal{S}[\phi] = -\frac{1}{2}\int_{t_i}^{t_f} dt \int \frac{d^d p}{(2\pi)^d} \frac{\dot{\phi}(p,t)\dot{\phi}(-p,t)}{\dot{G}(p,t)}.$$

- $$\mathcal{S}[y] = \int dz \int \frac{d^d p}{(2\pi)^d} \left[z^{-d+1} \left(\frac{\partial y(p,z)}{\partial z} \right)^2 + C(p,z) z^{-d+1} y(p,z)^2 \right],$$

  where

  $$C(p,z) z^{-d+1} = -\frac{\partial}{\partial z}\{z^{-d+1}\frac{d\ln f}{dz}\} + z^{-d+1}\left(\frac{d\ln f}{dz}\right)^2.$$

  and

  $$f^2 = -e^{-td}\dot{G}(p,t)$$

Key Result

Modified transformations in $\phi(p,t)$

$$D_\phi = p^\mu \frac{\partial}{\partial p_\mu} + (d - \Delta_\phi) - \frac{\partial}{\partial t}$$

$$C_{\mu,\phi} = 2(d-\Delta_\phi)\frac{\partial}{\partial p^\mu} + 2p^\rho \frac{\partial^2}{\partial p^\rho \partial p^\mu} - p_\mu \frac{\partial^2}{\partial p^\rho \partial p_\rho} - 2\frac{\partial^2}{\partial t \partial p^\mu} - \dot{G}(p,t)\frac{\partial}{\partial p^\mu}\frac{1}{\dot{G}(p,t)}\frac{\partial}{\partial t}$$

1. Modified $C_{\mu,\phi}$ and D_ϕ we obtained along with other conformal transformations forms $SO(1,d+1)$ group.
2. This can be proved using the properties of cutoff function.
3. We confirm the same in Hamiltonian formalism.

Modified transformations in $y(p, z)$

$$\boxed{C_{\mu,y} = 2(d - \Delta_y)\frac{\partial}{\partial p^\mu} + 2p^\rho \frac{\partial^2}{\partial p^\rho \partial p^\mu} - p_\mu \frac{\partial^2}{\partial p^\rho \partial p_\rho} - 2z\frac{\partial^2}{\partial z \partial p^\mu} + \left[\int_z z\frac{\partial}{\partial p^\mu} C(p, z)\right]}$$

where $C(p, z)z^{-d+1} = -\frac{\partial}{\partial z}\{z^{-d+1}\frac{d\ln f}{dz}\} + z^{-d+1}\left(\frac{d\ln f}{dz}\right)^2$.

One can also check consistency of $C_{\mu,\phi}$ and $C_{\mu,y}$ by using the field redefinition $\delta\phi(p, z) = f(p, z)\delta y(p, z)$.

Map to AdS isometry

1. Recall map from bulk action in y to AdS map.

$$\mathcal{S}[y] = \int dz \int \frac{d^d p}{(2\pi)^d} \underbrace{\left[z^{-d+1} \left(\frac{\partial y(p,z)}{\partial z} \right)^2 + C(p,z) z^{-d+1} y(p,z)^2 \right]}_{AdS: \left[z^{-d+1} \left(\frac{\partial y}{\partial z} \right)^2 + \left(p^2 + \frac{m^2}{z^2} \right) z^{-d+1} y^2 \right]} .$$

2. Hence we need to put $C(p,z) = \left(p^2 + \frac{m^2}{z^2} \right)$ in modified special conformal transformation in y.
3. The special conformal transformation becomes AdS isometry,

$$C_\mu^{ads} = 2(d - \Delta_y) \frac{\partial}{\partial p^\mu} + 2p^\rho \frac{\partial^2}{\partial p^\rho \partial p^\mu} - p_\mu \frac{\partial^2}{\partial p^\rho \partial p_\rho} - 2z \frac{\partial^2}{\partial z \partial p^\mu} + z^2 p_\mu .$$

Input: simple properties of cutoff function

1. We assume a reasonable form of cutoff function ($\Lambda = \Lambda_0 e^{-t}$),

$$G(p,\Lambda) = \frac{K(p/\Lambda)}{p^{2\nu}}.$$

2. This gives rises to two properties of the cutoff function

$$|p| - dependence : p_\mu \frac{\partial}{\partial p^\rho} \frac{1}{\dot{G}(p,t)} = p_\rho \frac{\partial}{\partial p^\mu} \frac{1}{\dot{G}(p,t)},$$
$$scaling\ relation : \frac{\partial}{\partial t} \frac{1}{\dot{G}(p,t)} - p^\rho \frac{\partial}{\partial p^\rho} \frac{1}{\dot{G}(p,t)} = -\frac{2\nu}{\dot{G}(p,t)}.$$

3. Similar relations hold for any functions of $G(p,t)$.

4. Using these properties one can find the symmetry transformations of the evolution operator, both in $\phi(p,t)$ and redefined field $y(p,z)$.

More results

A. Hamiltonian formalism

1. We can construct special conformal transformation in Hamiltonian formalism, where we rotate to Minkowski spacetime.

$$\bar{\epsilon}^j K_j^0 = \bar{\epsilon}^j \int_k \delta(k) \left[\underbrace{\left(2\frac{\partial^2}{\partial k_i \partial k^j} - \delta_{ij} \frac{\partial^2}{\partial k^l \partial k^l} \right) \Theta^0{}_i(k, z_M)}_{K_j^0|_I} + \underbrace{2z \frac{\partial}{\partial k^j} \Theta^0{}_0(k, z_M)}_{K_j^0|_{II}, RG \;\; time \;\; supplemented \;\; part} \right]$$

2. $[K_j^0|_{II}, P_m] = -2iR_{jm} - 2iD$; $[K_j^0|_{II}, P_m] = -2\delta_{jm} z_M H(z_M)$.
3. Action of $K_j^0|_{II}$ on field ϕ gives the scale dependent part of the modified conformal transformation.

B. Extension to full ERG equation

1. The ERG equation for the full action, $\Psi[\phi, t] = e^{-S_t[\phi]}$ as

$$\partial_t \Psi[\phi, t] = -\frac{1}{2}\dot{G}(p,t)\frac{\partial}{\partial\phi(p)}\left(\frac{\partial}{\partial\phi(p)} + \frac{2\phi(p)}{G(p,t)}\right)\Psi[\phi, t]. \quad (1)$$

2. This evolution operator can be recast in the same form as diffusion equation.

$$U(t_f, t_i) = \int \mathcal{D}\phi e^{\int_{t_i}^{t_f} dt \frac{\dot{\chi}(p,t)^2}{2\dot{F}(p,t)}}. \quad (2)$$

3. The field $\chi(p, t)$ is related to $\phi(p, t)$ by

$$\chi(p, t) = F(p, t)\phi(p, t), \; F(p, t) = \frac{1}{G(p, t)}.$$

4. This evolution operator too form a $SO(1, d+1)$ group with the modified special conformal transformation.

$$
\begin{aligned}
&C_{\mu,\chi}(p,t) \\
&= 2(d-\Delta_\chi)\frac{\partial}{\partial p^\mu} + 2p^\rho \frac{\partial^2}{\partial p^\rho \partial p^\mu} - p_\mu \frac{\partial^2}{\partial p^\rho \partial p_\rho} - 2z\frac{\partial^2}{\partial z \partial p^\mu} - \dot{F}(p,t)\frac{\partial}{\partial p^\mu}\frac{1}{\dot{F}(p,t)}\frac{\partial}{\partial t}
\end{aligned}
$$

5. This implies we can apply ERG apprach to holography in this more general case too.

Future directions

1. We have not considered boundary terms in our work. Any change of boundary term will change the boundary Wilson action. One need to carefully choose the boundary conditions to understand exact symmetry.
2. Analysis of symmetry in non-trivial boundary composite operators like vector current and stress tensor. This corresponds to vector boson and graviton in the bulk.
3. Polschinski's equation can be mapped to the Wetterich-Morris equation. Hence, one can explicitly verify the group structure in the functional RG.

Thank You!