Mass generation at a fixed point: A Functional Renormalization Group Study of the tricritical $O(N)$ model in $d = 3$ and $N = \infty$

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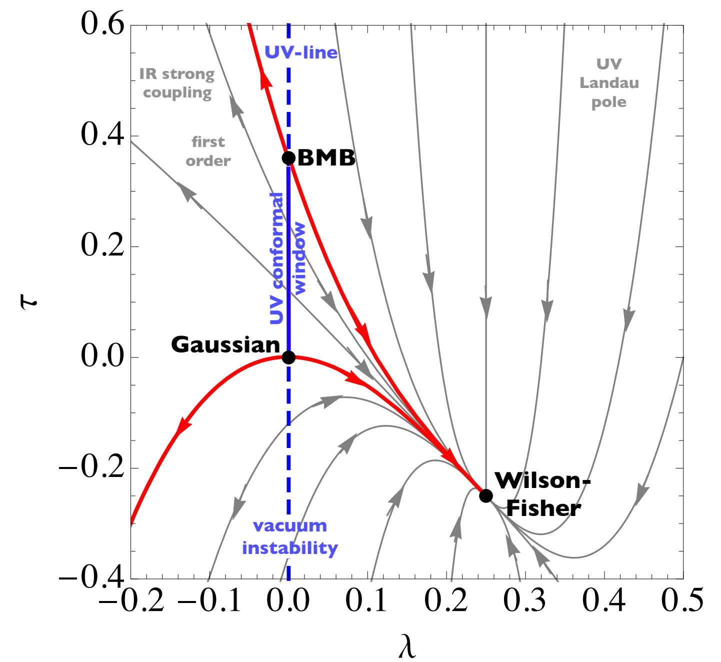
S. Yabunaka and B. Delamotte, arxiv 2606.12269

Bardeen-Moshe-Bander (BMB) Phenomena at $d = 3$ and $N = \infty$

In $O(N)$ models with $\tau(\varphi^2)^3$ interaction,

- Line of fixed points (FPs) at $d = 3$ and $N = \infty$
  $\cdots$Starts at Gaussian FP and ends at the BMB FP
- Asymptotically Safe: FPs are UV-stable on the critical plane
- Critial exponents along the line $0 \leq \tau < \tau_{BMB}$
  $\cdots$Interacting FP, but all exponents are Gaussian ($\nu = 1/2$)



D. F. Litim, M. J. Trott,
Phys. Rev. D 98, 125006 (2018)

At the BMB FP,

- Discontinuous jump to the critical exponent $\nu = 1/3$
- Singularity at small fields $\bar{U}(\bar{\phi}) \simeq \mathrm{const}\,|\bar{\phi}|$, considered to be relevant for spontaneous symmetry breaking of scale invariance

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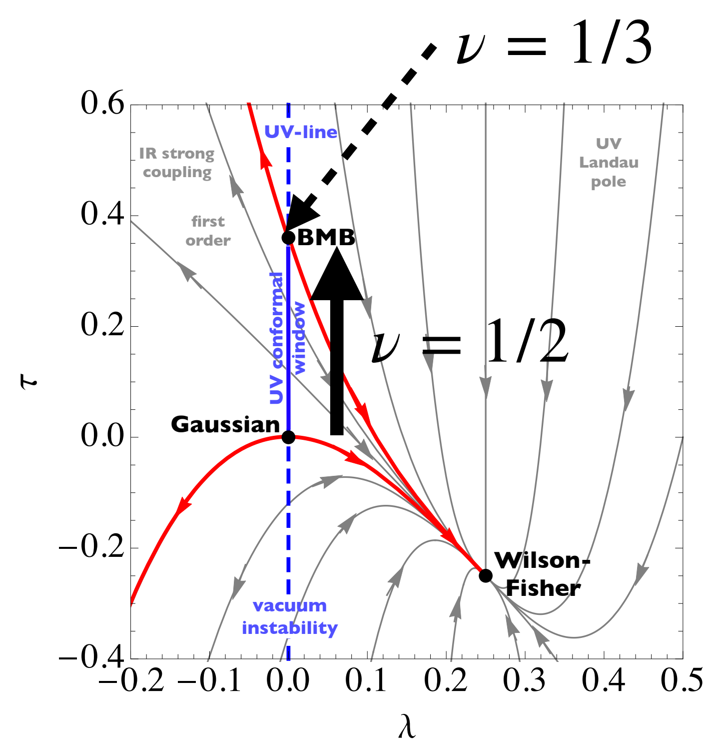

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Open questions revisited by NPRG

Mechanism of Mass Generation: Can we dynamically realize an arbitrary physical mass $m^2(k=0)$ by simply tuning a smooth bare potential at $k=\Lambda$?

Abrupt Jump of Exponent ν: How can ν jump discontinuously from 1/2 along the line to 1/3 right at the BMB endpoint?

Nondimensionalized LPA NPRG eq.

- Scaling solutions can be found as FPs solution of **nondimensionalized** NPRG eq.

$$\tilde{\phi} = \sqrt{Z_k} k^{\frac{2-d}{2}} \phi \qquad \tilde{\rho} = Z_k k^{2-d} \rho \qquad \tilde{U}_t(\tilde{\rho}) = k^{-d} U_k(\rho)$$

Litim cutoff $\quad y = \dfrac{q^2}{k^2} \qquad R_k(q^2) = Z_k k^2 y r(y) \qquad r(y) = (1/y - 1)\theta(1-y)$

Under LPA,

$$\partial_t \tilde{U}_t(\tilde{\phi}) = -d\,\tilde{U}_t(\tilde{\phi}) + \frac{1}{2}(d-2)\tilde{\phi}\,\tilde{U}_t'(\tilde{\phi}) + (N-1)\,\frac{\tilde{\phi}}{\tilde{\phi} + \tilde{U}_t'(\tilde{\phi})} + \frac{1}{1 + \tilde{U}_t''(\tilde{\phi})}.$$

Usual large N limit of the LPA flow

Rescaled finite N equation $\tilde{U}_t = N\bar{U}_t$ $\tilde{\phi} = \sqrt{N}\bar{\phi}$

$$0 = -d\,\bar{U} + \frac{1}{2}(d-2)\bar{\phi}\,\bar{U}' + \left(1 - \frac{1}{N}\right)\frac{\bar{\phi}}{\bar{\phi} + \bar{U}'} + \frac{1}{N}\frac{1}{1 + \bar{U}''}.$$

- The terms proportional to 1/N are assumed to be subleading.
- At N=∞, this simplifies as

$$0 = -d\,\bar{U} + \frac{1}{2}(d-2)\bar{\phi}\,\bar{U}' + \frac{\bar{\phi}}{\bar{\phi} + \bar{U}'}$$

Wilson-Polchinski version of NPRG

Transformation of the variables

$$(U,\phi) \longleftrightarrow (V,\Phi)$$

$$V(\mu) = U(\phi) + (\phi - \Phi)^2/2$$

$$\phi - \Phi = -2\Phi V'(\mu)$$

$$\mu = \Phi^2$$

Rescaling in N $\bar{\mu} = \mu/N,\ \bar{V} = V/N$

LPA FP eq.

$$0 = 1 - d\,\bar{V} + (d-2)\bar{\mu}\bar{V}' + 4\bar{\mu}\bar{V}'^2 - 2\bar{V}' - \underline{\frac{4}{N}\bar{\mu}\,\bar{V}''}.$$

$1/N$ A small parameter

$\bar{V}''$ The highest order derivative

We have to deal with singular perturbation in general.

Tricritical FP solutions in $d = 3$ and at $N = \infty$ in LPA (WP-parametrization)

$$\bar{\varrho} = \bar{\mu}/2$$

$$\bar{\varrho}_{\pm} = 1 + \frac{\bar{V}'\left(\frac{5}{2} - \bar{V}'\right)}{\left(1 - \bar{V}'\right)^2} + \frac{\frac{3}{2}\arcsin\sqrt{\bar{V}'} \pm \sqrt{2/\tau}}{\left(\bar{V}'\right)^{-1/2}\left(1 - \bar{V}'\right)^{5/2}}$$

$\bar{\varrho}_{+} \dashrightarrow \bar{\varrho} > 1$

$\bar{\varrho}_{-} \dashrightarrow \bar{\varrho} < 1$

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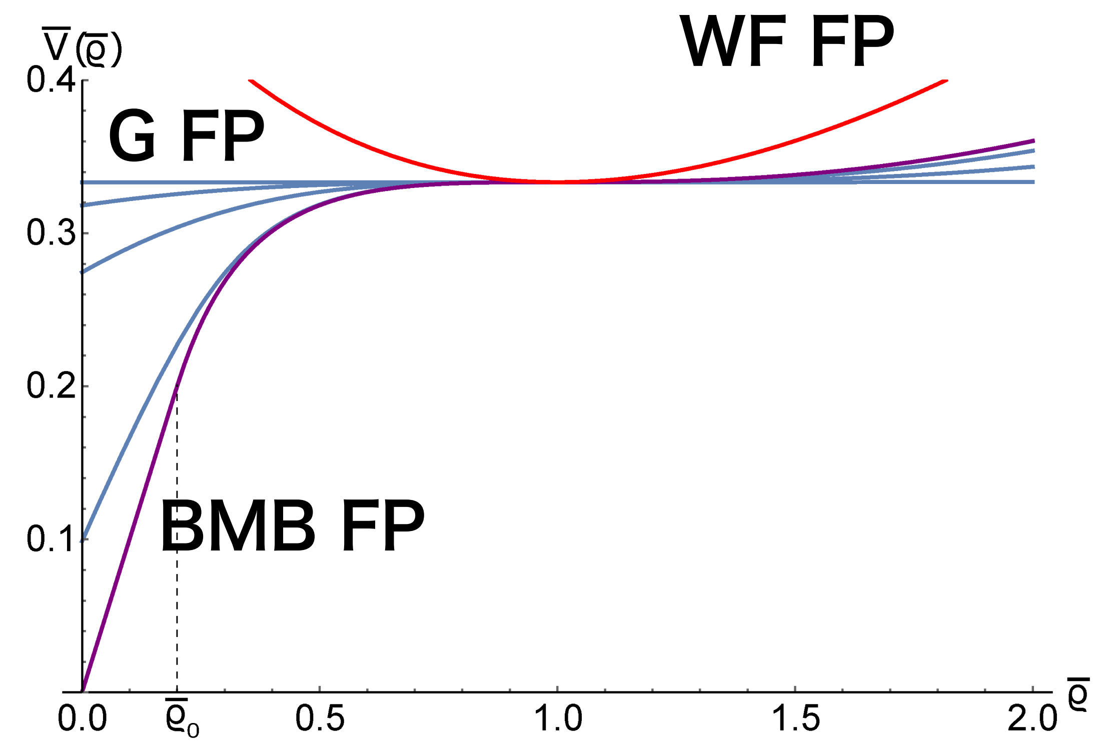
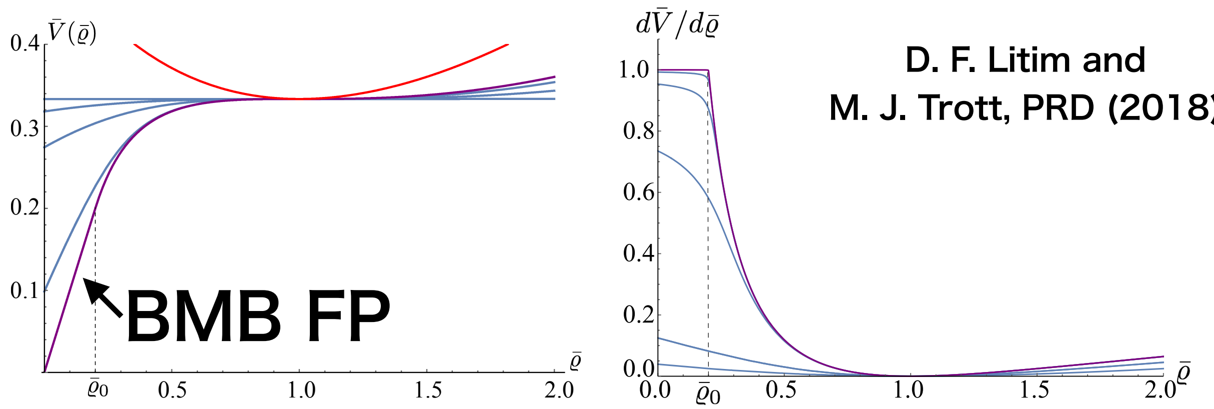


- $\tau = 0$ $\cdots$Gaussian (G) FP
- $\tau \in [0, \tau_{\mathrm{BMB}} = 32/(3\pi)^2]$ $\cdots$ FPs on the BMB line
- $\sqrt{2/\tau} = 0$ $\cdots$Wilson-Fisher (WF) FP

Non-Analyticity of the BMB FP: Dual representation

Wilson-Polchinski



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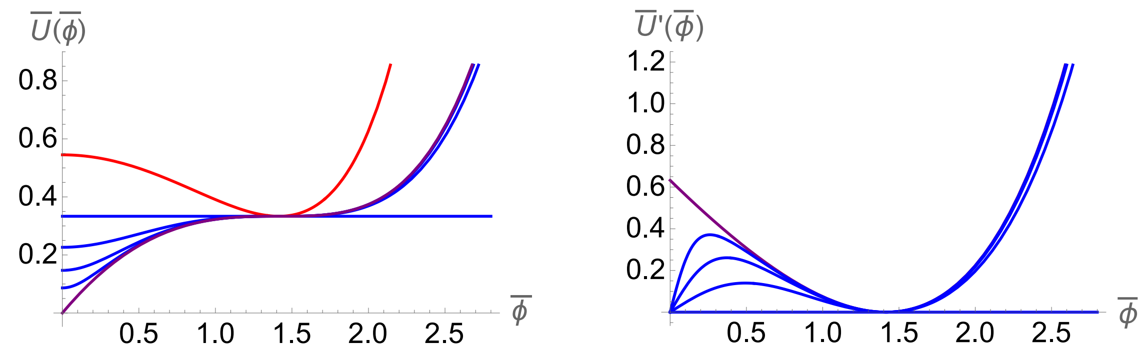
- $d^2\bar{V}/d\bar{\rho}^2$ becomes discontinuous at $\bar{\varrho} = \bar{\varrho}_0$.

Wetterich



The linear part $0 < \bar{\varrho} < \bar{\varrho}_0$ in WP is **mapped directly onto the origin** $\bar{\phi} = 0$!

Cusp-like potential: $\bar{U}(\bar{\phi}) \sim$ const $\cdot\, |\,\bar{\phi}\,|$.

Non-zero derivative at origin: $\bar{U}'(\bar{\phi} = 0_+) \neq 0$.

Dimensionful mass: $m^2(k) = k^2\bar{U}''(\bar{\phi} = 0)$ in the Wetterich parametrization.

Since $\bar{U}'(0_+) \neq 0$, evaluating strictly at the fixed point yields an **undefined / singular limit as** $k \to 0$.

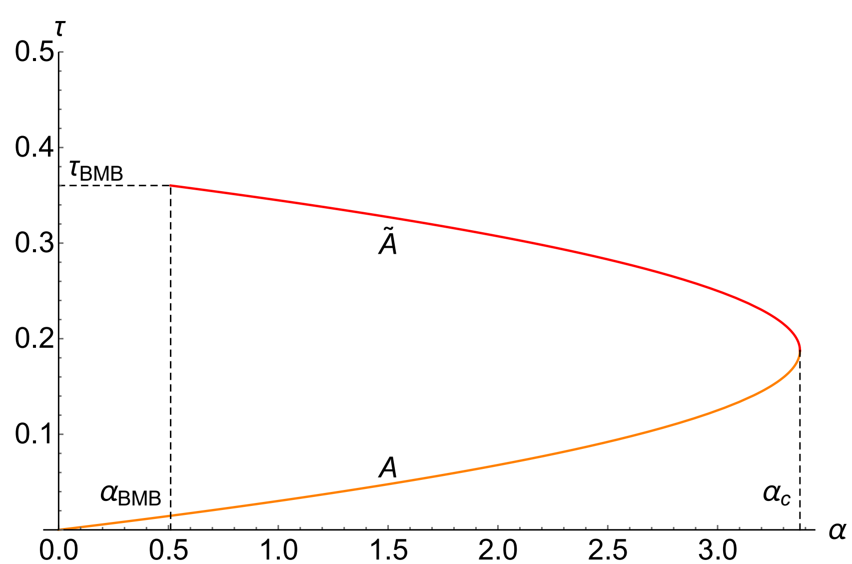
Question…"How can a finite, well-defined physical mass $m^2(k = 0)$ actually be generated along RG flow starting from a smooth bare potential at $k = \Lambda$?"

Finite-N realization of the regular BMB line (PRD 2020)

- For each $\mathscr{A}(\tau)$ ($d=3, N=\infty$), there is one finite-N FP that converges to $\mathscr{A}(\tau)$ under the double limit:

$$d = 3 - \alpha/N\ , N \to \infty$$

- The relation between α and τ is $\boxed{\alpha - 36\tau + 96\tau^2 = 0}$



$$\mathscr{A} = A \cup \tilde{A}$$

3 relevant directions at finite N

2 relevant directions at finite N

Singular FPs constructed from the FPs on the BMB line

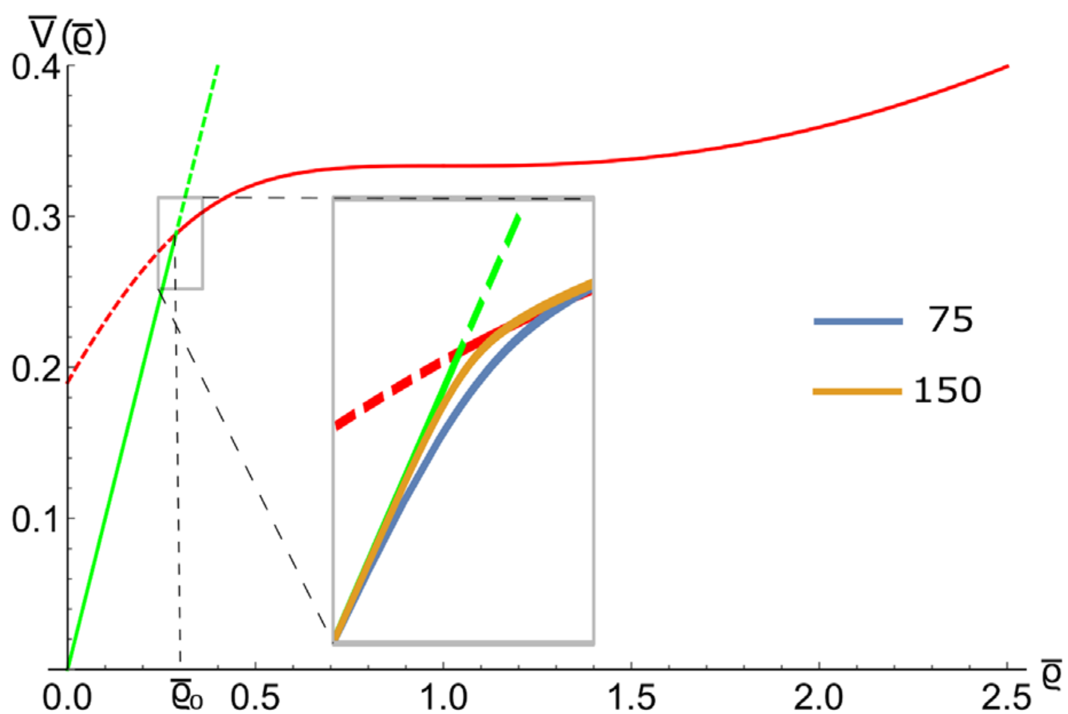
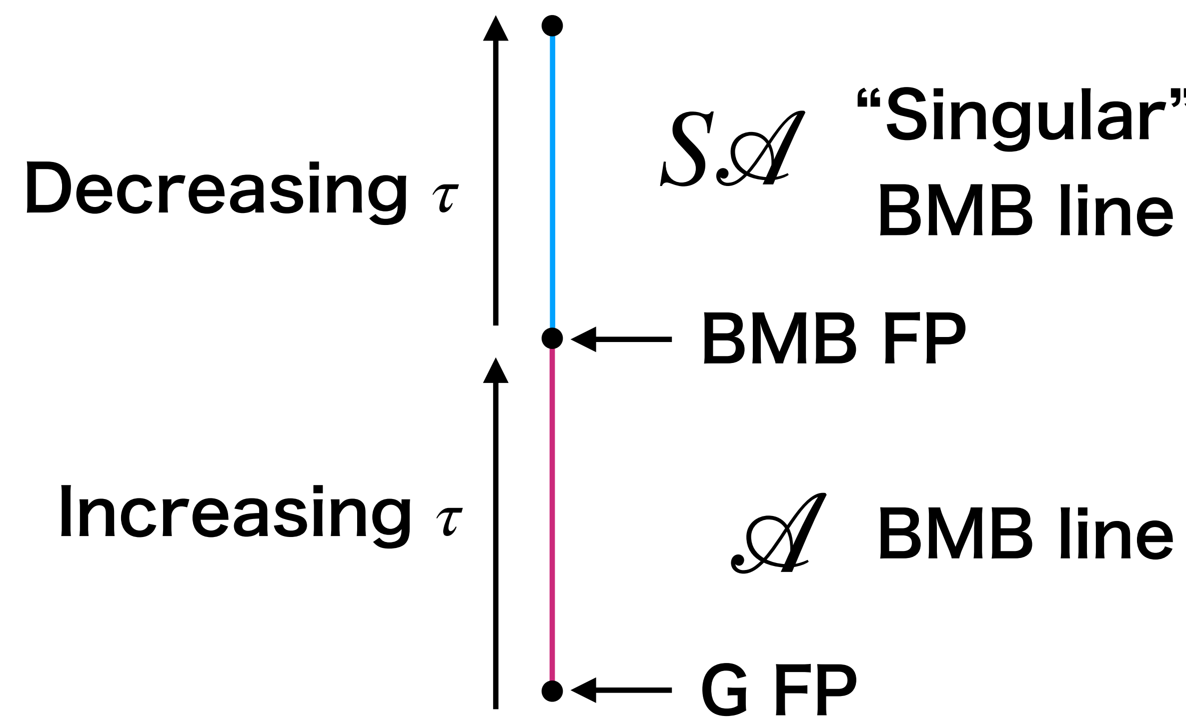


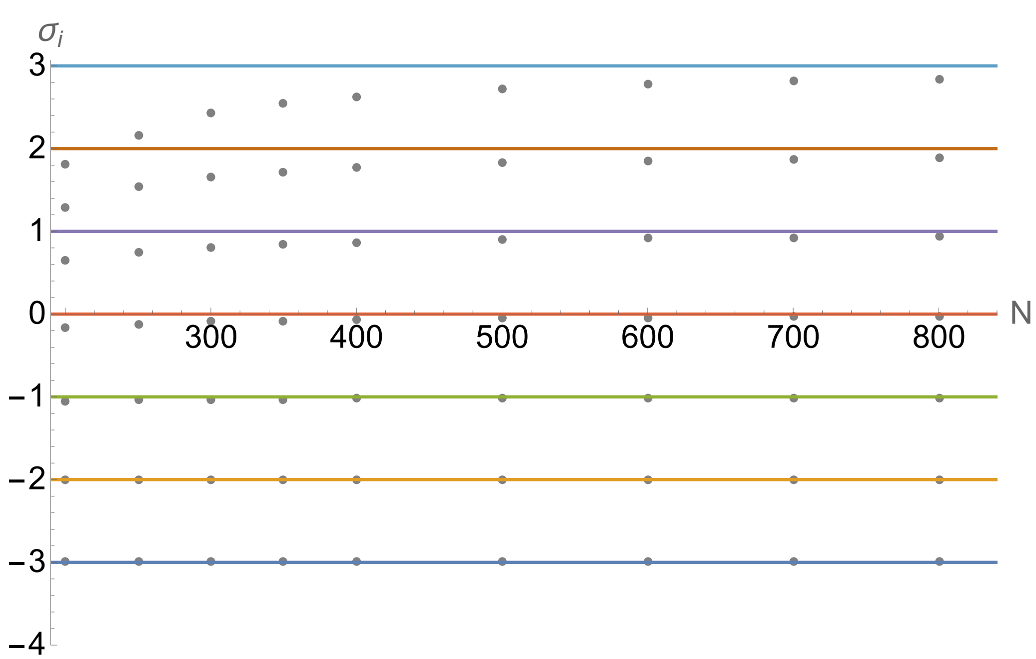
FIG. 2. $N = \infty$ and $d = 3$: Singular potential of $S\mathcal{A}(\tau = 0.33)$ from the potential of $\mathcal{A}(\tau = 0.33)$ given by the red and dashed red curves, Eq. (7). The green and dashed green curves show $\bar{V}(\bar{\varrho}) = \bar{\varrho}$. The potential of $S\mathcal{A}(\tau = 0.33)$ is made of the plain green and red curves that meet at $\bar{\varrho}_0(\tau = 0.33)$. Inset: zoom of the region around the cusp and its rounding at finite N within the boundary layer.



Eigenvalues of the linearized RG flow around the regular FP $\mathscr{A}$ and the singular FP $S\mathscr{A}$

- It is not trivial to formulate the eigenvalue problem around the BMB FP at $N = \infty$. We study it at finite but Large N.

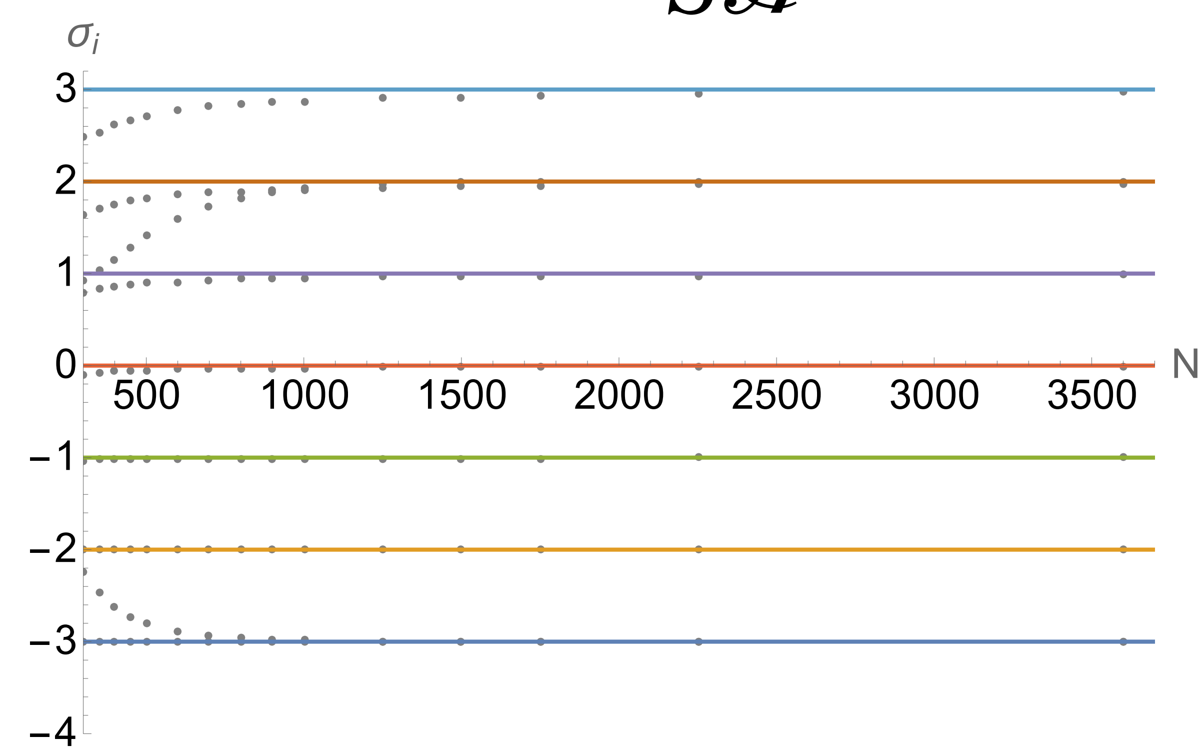
$\mathscr{A}$



$\alpha = 0.58$

$\xrightarrow{N\to\infty}$ $\{\sigma_{i=1,2,\ldots}\} = \{-3,-2,-1,0,1,2,3,\ldots\}$

$S\mathscr{A}$

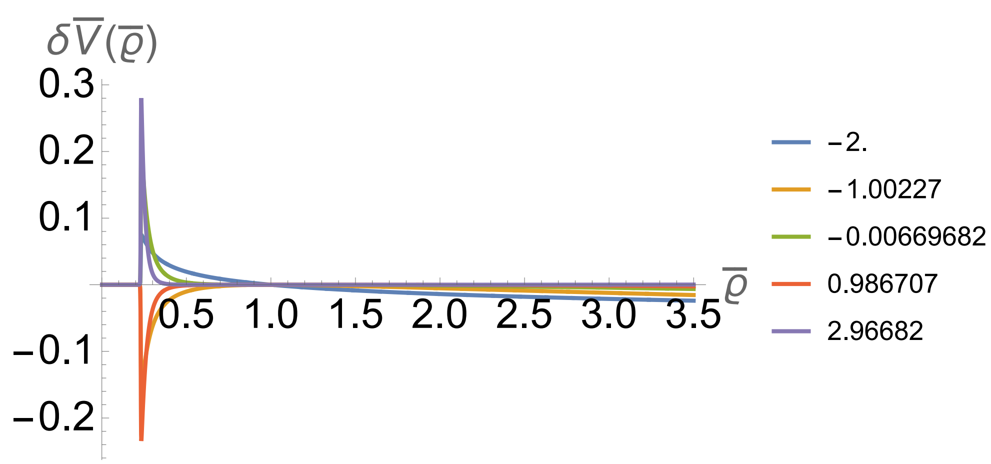


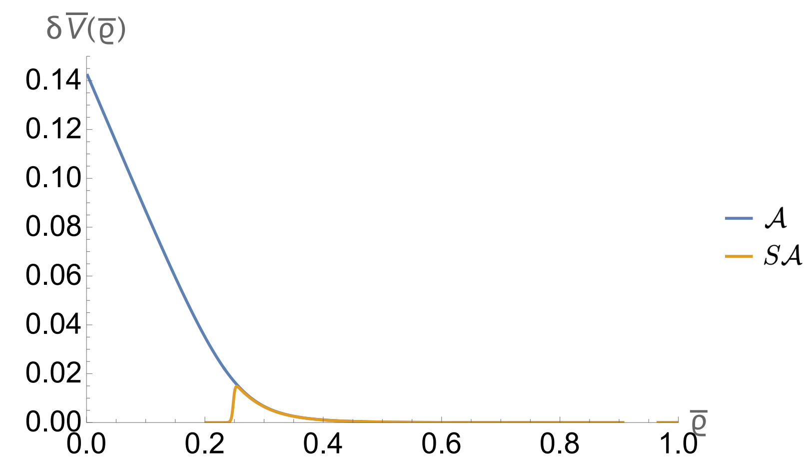
$\alpha = 0.518$

$\xrightarrow{N\to\infty}$ $\{\sigma_{i=1,2,\ldots}\} = \{-3,-3,-2,-1,0,1,2,2,3,\ldots\}$.

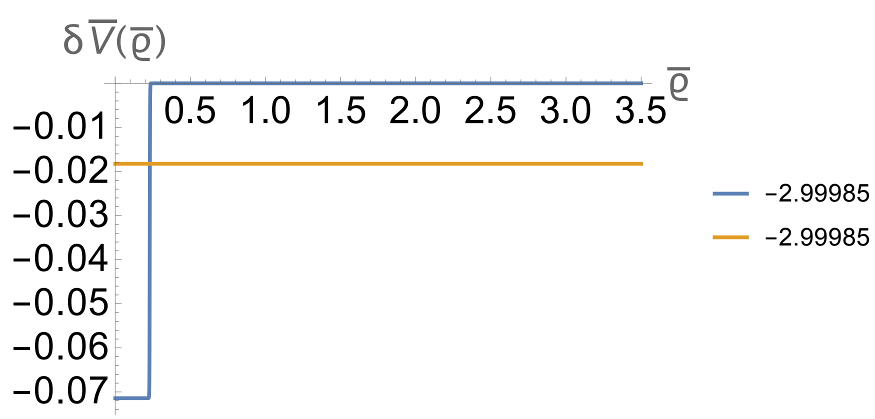
Eigenvectors of the linearized RG flow for $S\mathscr{A}$ with $\alpha = 0.518$ and $N = 3600$

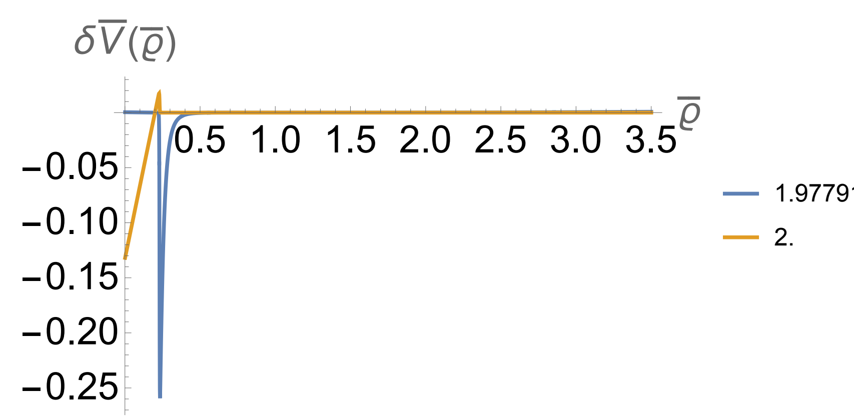
Nondegenerate ones…nonvanishing and identical to that of $\mathscr{A}$ for $\bar{\varrho} > \varrho_0$ with the same eigenvalue





Degenerate ones





Nontrivial eigenvalue $\simeq -3$ **->** $\nu = 1/3$

Subtlety: The critical exponent ν exactly at the BMB may depend on whether we approach it on the side of $\mathscr{A}$ or $S\mathscr{A}$ when taking the $N \to \infty$ limit

Eigendirection corresponding to eigenvalue 2 in $N \to \infty$

- $\bar{V}(\bar{\varrho})$ and behaves as $\tilde{a}^P \bar{\varrho}$ when $\bar{\varrho} \ll 1$

- With the eigenperturbation $\delta \bar{V}_2(\bar{\varrho}) = -C(\bar{\varrho} - 1/5)$ on the BMB FP potential, the slope at $\bar{\varrho} = 0$ should flow as $\tilde{a}_t^P = 1 - \beta e^{2t} + \cdots$ in the Polchinski parametrization.

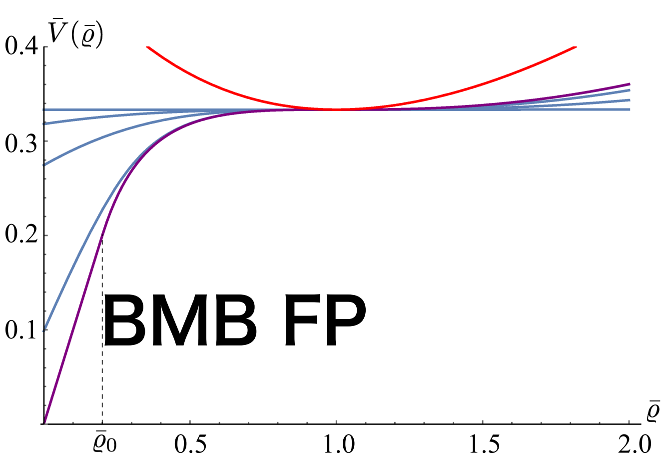
- The Wetterich potential behaves $\bar{U}_{k=\Lambda}(\bar{\rho}) = \tilde{a}_t^W \bar{\rho} + \cdots$.

  The slope at $\bar{\rho} = 0$: $\tilde{a}_t^W = \dfrac{\tilde{a}_t^P}{1 - \tilde{a}_t^P}$ flows as $\tilde{a}_t^W \simeq e^{-2t}/\beta$

  Dimensionful mass: $\boxed{\sqrt{k^2 \tilde{a}_t^W} \propto \sqrt{\cancel{e^{2t} e^{-2t}}/\beta} = 1/\sqrt{\beta} \text{ (const)}}$

Initial condition with $C \ll 1$

W-P parametrization

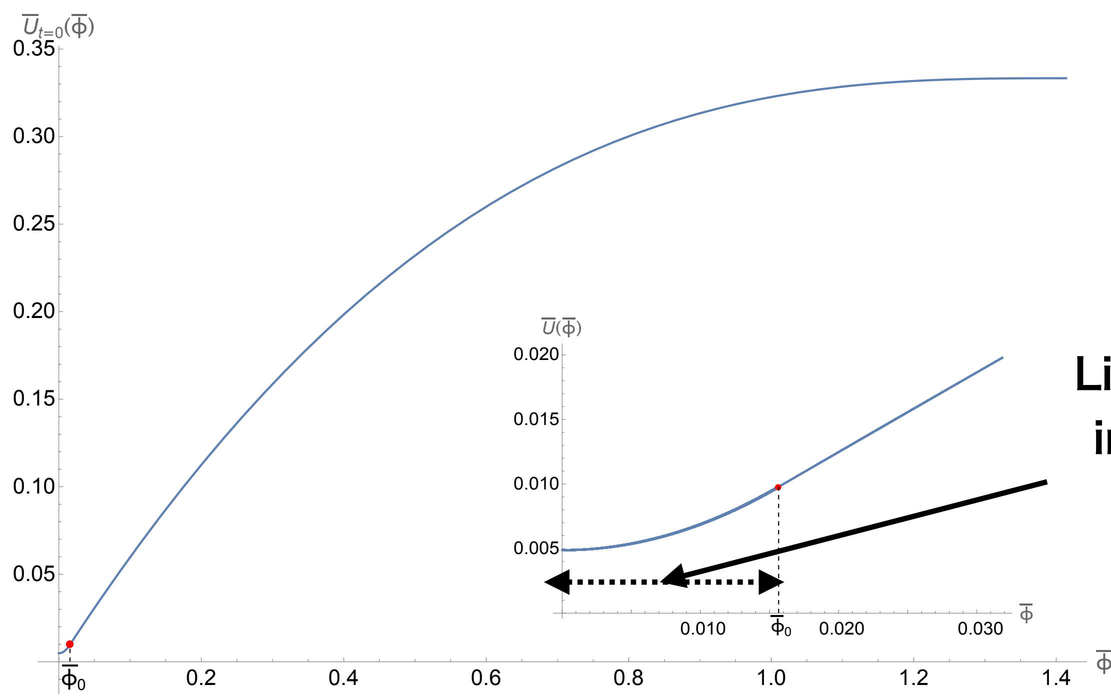


Plot of the initial condition $\bar{U}_{t=0}(\bar{\phi})$

in the Wetterich parametrization

$$\bar{U}_{t=0}\left(\bar{\phi}\right) = \bar{U}_{BMB}\left(\bar{\phi}\right) + \underline{\delta\bar{U}\left(\bar{\phi}\right)} \text{ with } C = 1/41.$$

Eigenperturbation with eigenvalue 2



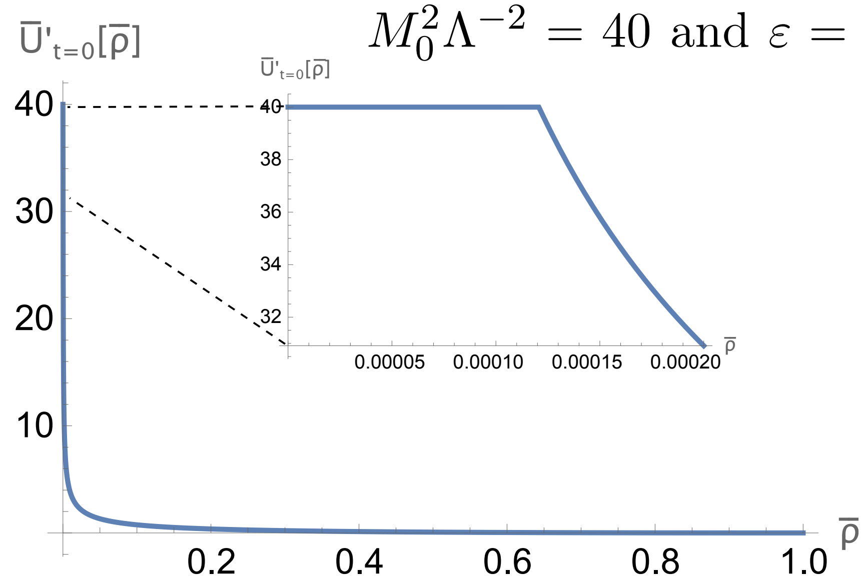
Linear region with slope 1-C in W-P is mapped onto the narrow region.

Cuspy singularity is smoothened here.

Initial condition to realize arbitrary mass

- A candidate of the initial condition $\bar{U}_{k=\Lambda}(\bar{\rho})$ at $N=\infty$ to realize finite dimensionful mass value

$M_0^2\Lambda^{-2} = 40$ and $\varepsilon = 10^{-3}$.



The divergence of $\bar{U}'_{BMB}(\bar{\rho})$ larger than $M_0^2\Lambda^{-2}$ is cutoff smoothly

$$\mathcal{T}(x) = \begin{cases} \tanh(x) - 1 & (x \le x_1) \\ (\tanh(x)-1)\frac{\psi(x_2-x)}{\psi(x-x_1)+\psi(x_2-x)} & (x_1 < x < x_2) \\ 0 & (x \ge x_2) \end{cases}$$

$$\bar{\rho}_\Lambda = 1 + \sqrt{\bar{U}'_{t=0}(\bar{\rho}_\Lambda)}F(\bar{U}'_{t=0}(\bar{\rho}_\Lambda)) - \frac{1}{4}(3\pi)\sqrt{\bar{U}'_{t=0}(\bar{\rho}_\Lambda)} + \mathcal{T}((-\bar{U}'_{t=0}(\bar{\rho}_\Lambda) + M_0^2\Lambda^{-2})/\varepsilon)$$

; $x_1 = 2$ and $x_2 = 3$.

$\psi(x) = \exp(-1/x)$ for $x > 0$ and $\psi(x) = 0$ for $x \le 0$.

Evolution of the potential and dimensionful mass

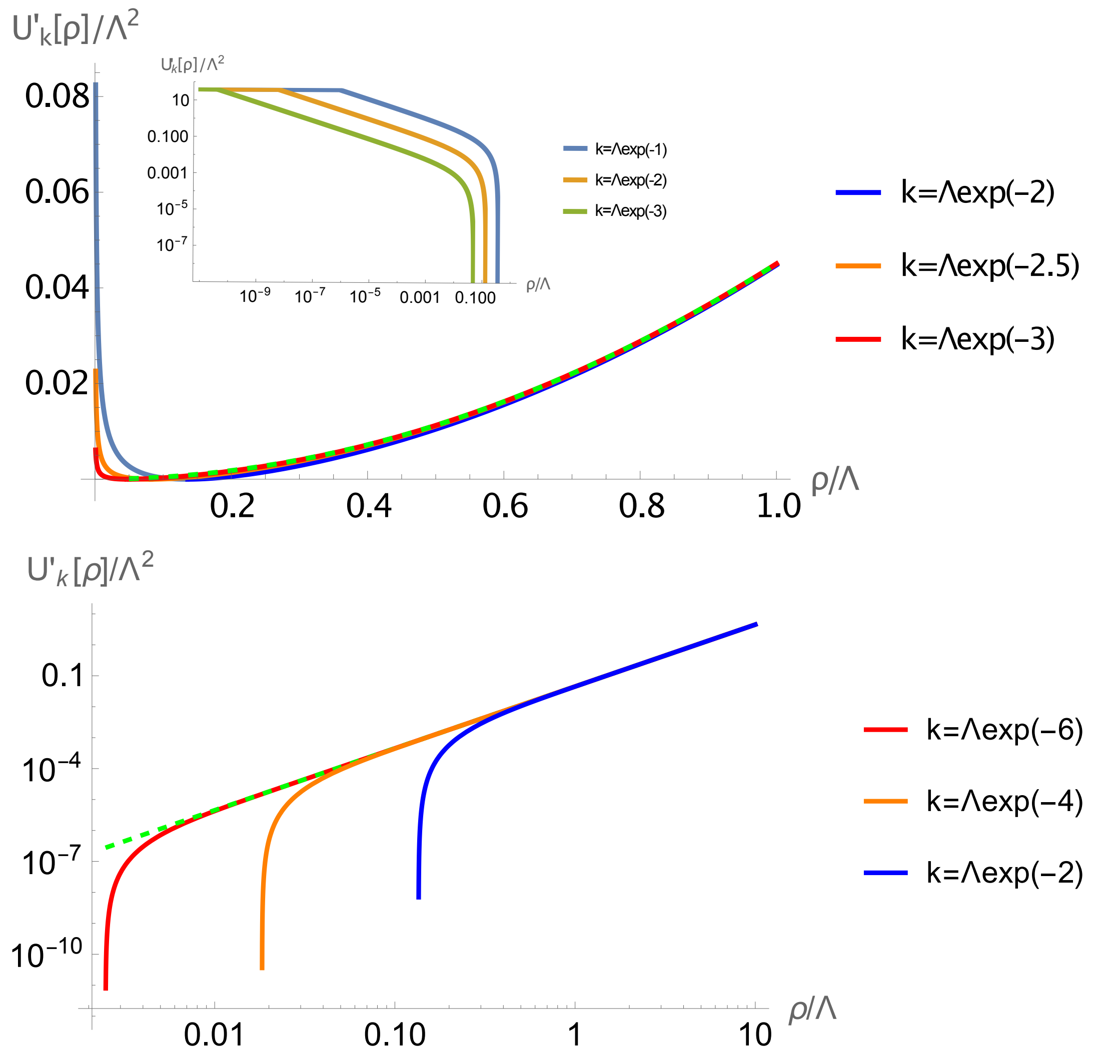
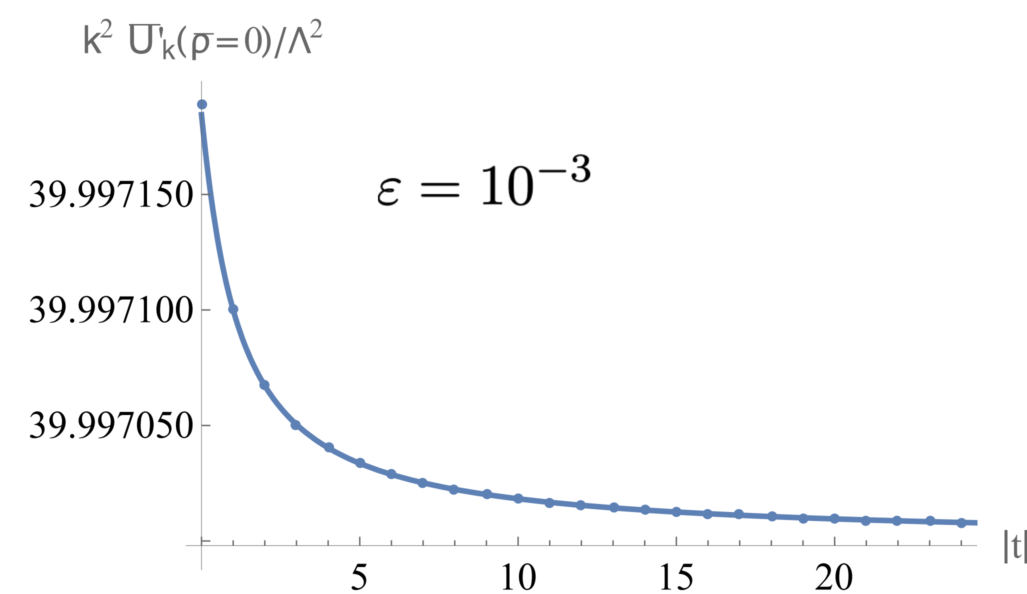


Figure 13: Evolution of the derivative of the dimensionful running potential $U_k'(\rho)$ with $M_0^2\Lambda^{-2} = 40$ and $\varepsilon = 10^{-3}$. (Top) Behavior around $\rho = 0$. (Bottom) Approach of $U_k'(\rho)$ towards the BMB potential $U'(\rho) = \rho^2/(4c_{\rm BMB}^2)$, valid for ρ strictly positive and shown as the green dashed curve. This limiting behavior is realized in the limit $k \to 0$ for each $\rho > 0$, while $U_k'(\rho = 0)$ remains almost M_0^2 all along the flow.

Evolution of dimensionful mass



$$U_k'(\rho=0)\Lambda^{-2} \approx M_0^2\Lambda^{-2} - x_2\varepsilon + \varepsilon\left[\frac{1}{5|t|+C_1} - \frac{K}{(5|t|+C_1)^3}\right] + O(|t|^{-3})$$

We use the analytical global flow $\bar{U}_t(\bar{\phi})$ at $N = \infty$

(D. F. Litim, E. Marchais and P. Mati, Phys. Rev. D 95, 125006 (2017))

Summary

- An arbitrary physical mass $m(k=0)$ can be generated by tuning smooth UV initial conditions that regularize the BMB small-field singularity.

- The singular perturbation mode with **eigenvalue +2** plays an important role in yielding a constant mass at $k=0$.

- $\nu = 1/2$ along the regular BMB line jumps to $\nu = 1/3$ on the singular BMB line with the BMB FP being the pivotal point between these two regimes.