# Informational renormalization group

**Takato Mori (Rikkyo University/RIKEN, Japan)**

Based on 2609.xxxxx with Teruaki Nagasawa
(Kanazawa University)

13th International Conference on the Exact Renormalization Group 2026 (ERG2026), September 2, 2026

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# Two questions

1. **What is RG for generic systems?**

Wilsonian RG:
continuum, infinite dim., momemtum=energy

Real-space RG:
spatially extended, no infinitesimal flow, mostly numerical

Generic scheme?

- analytically tractable
- computable
- discrete/continuous
- finite/infinite dims

2. **Universal RG monotone?**

c-theorem:
limited to 2d, extension to higher dim is subtle

Entropic c-function and Wilsonian cutoff?

# This talk: Quantum informational and algebraic reformulation of Wilsonian RG

Informational RG (IRG)

- Coarse graining as partial trace
- Rescaling as a product of isometries

✓ Explicit Kraus operator realization (Haar random unitary/basis scrambling)

☹ Tensor factorization assumed

Algebraic RG (ARG)

- Constructing Wilsonian low/high-energy algebra from one-particle operators
- Rescaling as one-particle isometry
- RG as conditional expectation

✓ No factorization assumed

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- Rescaling as one-particle isometry
- RG as conditional expectation

✓ No factorization assumed

☺ **Resource-theoretic formulation leads to a rigorous RG monotone!**

☺ **Explicit relation between QI metric and Zamolodchikov metric/central charge!**

☺ **Holographic calculation!**

# Recap: Wilsonian RG (WRG)

$$Z_\Lambda = \int \mathscr{D}\phi_< \int \mathscr{D}\phi_> \, e^{-S_\Lambda[\phi_<, \phi_>]} = \int \mathscr{D}\phi_< \, e^{-S'_{\Lambda(s)}[\phi_<]}$$

**Integrating out high-momentum modes**

$$\phi_> := \tilde{\phi}(p \geq e^{-s}\Lambda)$$

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**Rescale each mode**

$$\phi'(p' = e^s p \leq \Lambda)$$
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**Rescale each mode**

$$\phi'(p' = e^s p \leq \Lambda)$$
$$:= \phi_<(p \leq e^{-s}\Lambda)$$

➡ This iterative, infinitesimal procedure defines the effective action $S^{(s)}_\Lambda[\phi]$ at each renormalization scale $s$

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Intuition: Coarse graining = scramble high-energy modes

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$$\mathscr{E}(\rho) := \int dU_> \, K_U^{(\sigma)} \rho \, K_U^{(\sigma)\dagger} = (\mathrm{Tr}_> \rho) \otimes \sigma_>$$

agrees with [Costa-Brink-Nogueira-Klein ’22, ’23]

$$K_U^{(\sigma)} = \sqrt{d_>}\,(\mathbf{1}_< \otimes \sqrt{\sigma_>})(\mathbf{1}_< \otimes U_>)$$

high-E reference state

scrambling high-E modes

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high-E reference state

scrambling high-E modes

Can be extended to a more general ensemble (e.g. symmetry constrained)

By following each trajectory $U$, a pure state RG can be formulated.

# Informational RG (IRG)

$$\frac{\Lambda}{\Delta p} \sim \text{\# of modes}$$

🤔 Rescaling is a mere relabeling? Consider a discrete system

$$\phi(p_0 = 0) \qquad \phi(p_1 = \Delta p) \qquad \phi(p_2 = 2\Delta p = \Lambda)$$

integrating out $\int d\phi(p_2)$ $\downarrow$

$$\phi(p_0 = 0) \qquad \phi(p_1 = \Delta p = e^{-s}\Lambda)$$

relabel $p \to p' = e^s p$ $\downarrow$

$$\phi(p_0' = 0) \qquad \phi(p_2' = 2\Delta p = \Lambda)$$

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$$\boxed{\phi(p_0 = 0) \qquad \phi(p_1 = \Delta p) \qquad \phi(p_2 = 2\Delta p = \Lambda)}$$

$\updownarrow$

Dimensions disagree!

Cannot compare two on the same ground!

→ We need an interpolation!

$$\boxed{\phi(p_0' = 0) \qquad \phi(p_1 = \Delta p) \qquad \phi(p_2' = 2\Delta p = \Lambda)}$$

# Rescaling as isometry

For rescaling, it must

- map from a smaller dimension (#modes = $N$ ) to a larger dimension (#modes = $M$ )
- only use the coarse-grained information (unless you have a prior info)
  = preserving the coarse-grained observables
- does not mix UV and IR; sufficiently local in momentum space

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➡ a product of isometries!

$$V := \bigotimes_{i=0}^{N-1} V_i$$

$$V_i : \mathscr{H}_i \to \mathscr{H}_{C_i}, \quad V_i^\dagger V_i = \mathbf{1}_i$$

$$\{0, \ldots, M-1\} = \bigsqcup_{i=0}^{N-1} C_i, \quad n_i := |C_i|, \quad \sum_{i=0}^{N-1} n_i = M$$

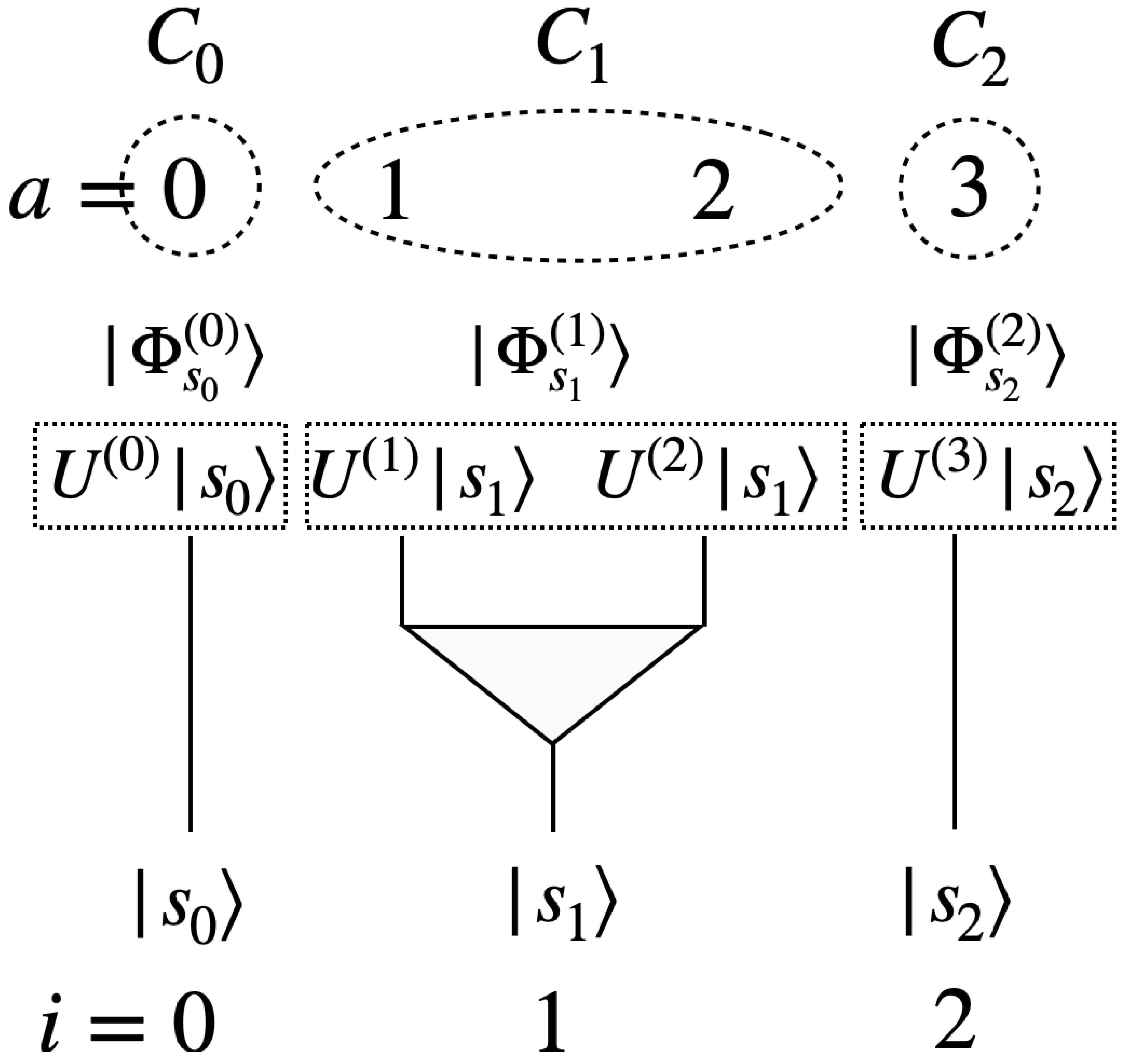
# Rescaling as isometry

$$V := \bigotimes_{i=0}^{N-1} V_i$$

$$V_i = \int \mathscr{D}\phi_i \Big( \bigotimes_{i \in C_i} U^{(i)} | \phi_i \rangle \Big) \langle \phi_i |$$

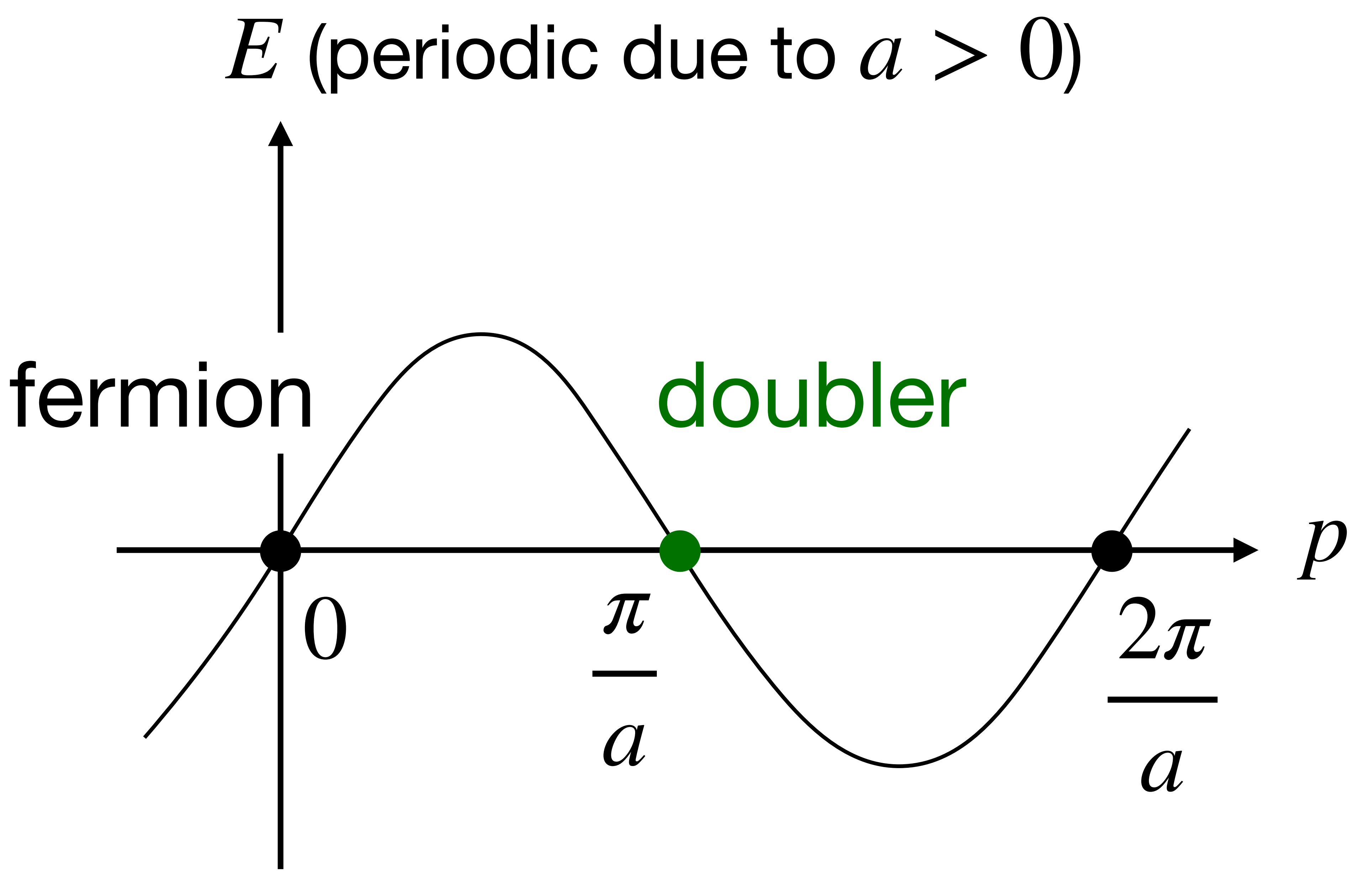
✓ does not generate a momentum-space entanglement

✓ becomes unitary in infinite modes limit



# Information to Algebra

- IRG still assumes a tensor product structure $\mathscr{H} = \mathscr{H}_< \otimes \mathscr{H}_>$
- Beyond free theories, tensor product structure is not guaranteed
- Without translational invariance, momentum ≠ energy



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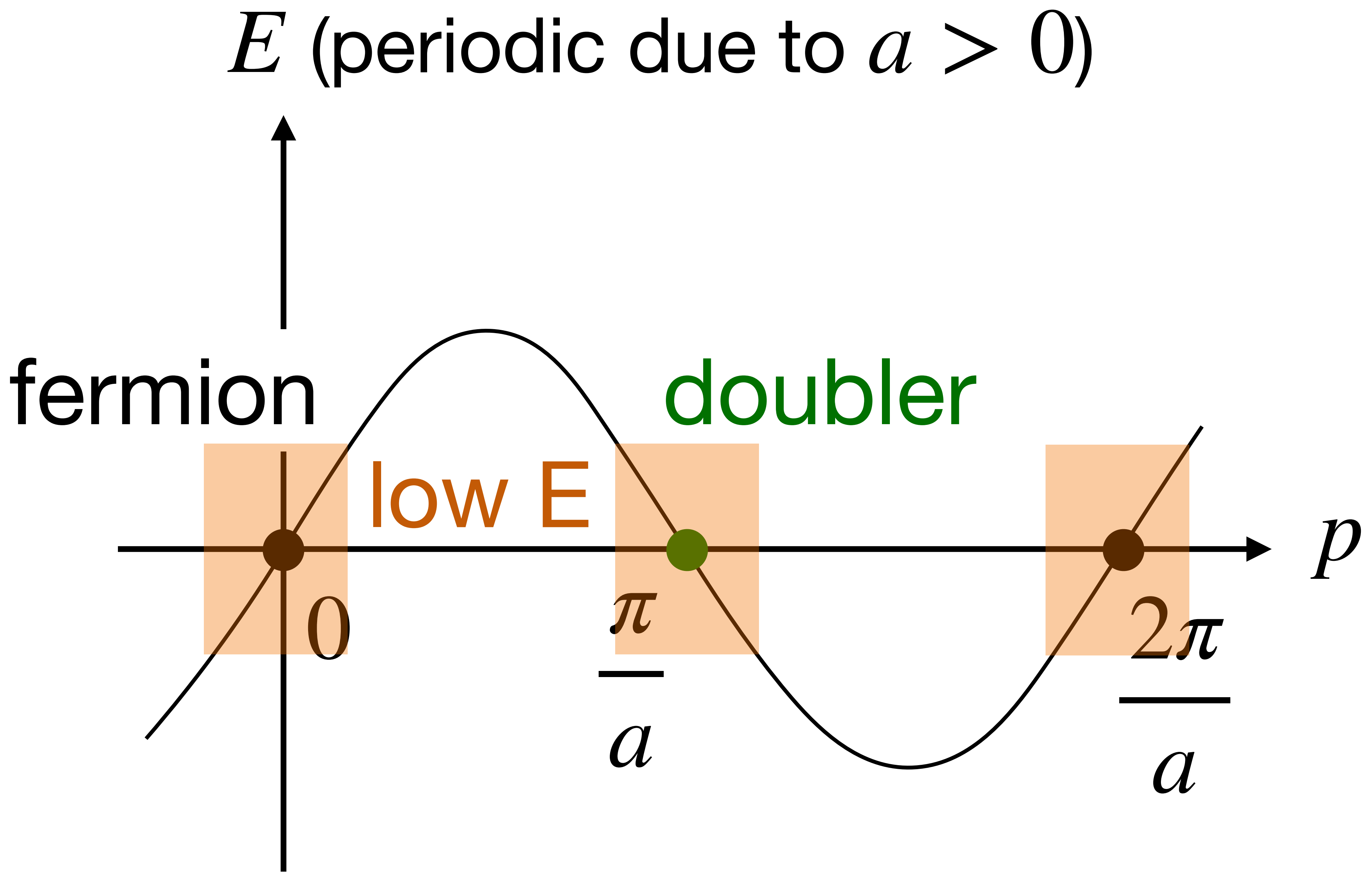

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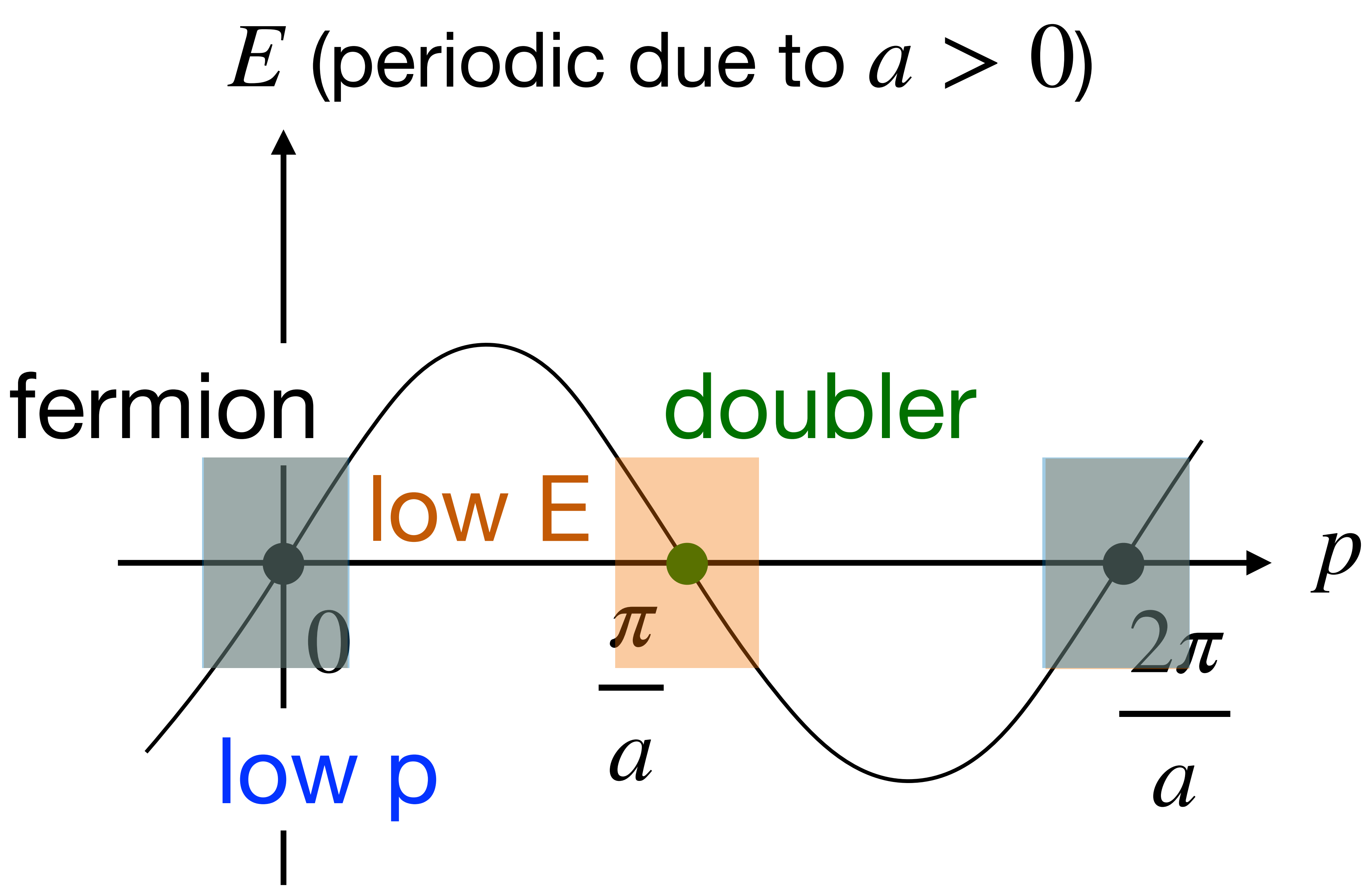

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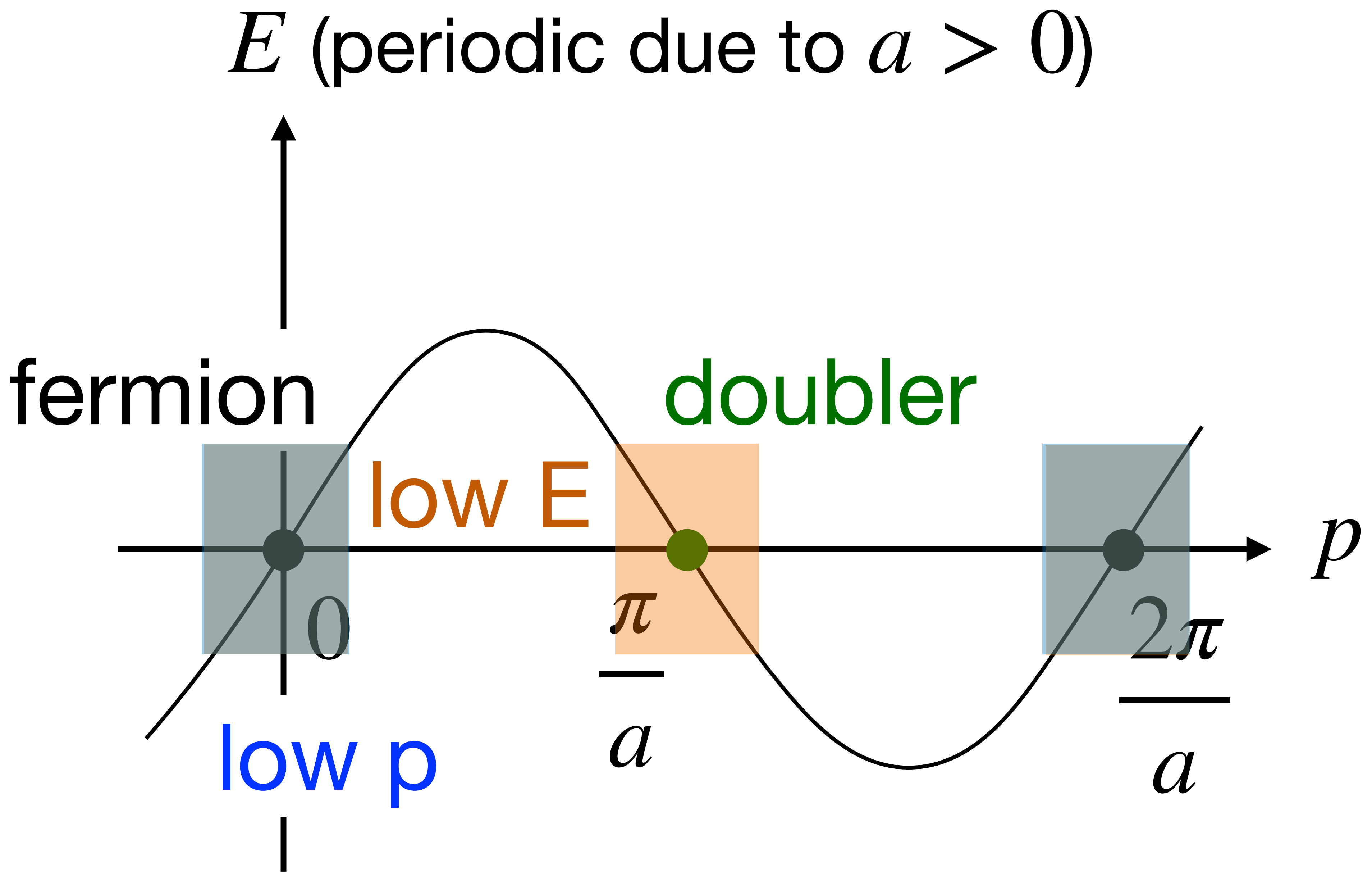


We first need to define “low/high energy”… What does it mean?

# Algebraic RG (ARG)

We should distinguish two notions of energy:

- Total energy: $E = \int_{\mathbf{p}} \omega_{\mathbf{p}}$
- $\boxed{\text{Modewise energy = one-particle energy } \omega_{\mathbf{p}}}$ ← 'energy' in Wilsonian RG

Collective excitations can carry a large amount of energy but this is not a UV phenomenon, e.g. electromagnetism

# Algebraic RG (ARG)

In free theory

1. Choose 'one-particle' (1p) excitations
   $S_{1\mathrm{p}} = \{\mathbf{1}\} \cup \{O_i\}_i \Rightarrow \mathscr{V}_1 = \mathrm{span}\{S_{1\mathrm{p}}\}$
   s.t. they are separating on $|\Omega\rangle$

$$a_p^\dagger$$

$$\{a_p^\dagger|\Omega\rangle = |p\rangle\}_p$$

→ Define 1p Hilbert space & projector

$\mathscr{H}_1 = \mathrm{span}\{O|\Omega\rangle \,|\, O \in \mathscr{V}_1\}, \quad P_1\mathscr{H} = \mathscr{H}_1$

$$P_1 = \sum_p |p\rangle\langle p|$$

→ 1p Hamiltonian $H_1 = P_1 H P_1 = \sum_i \varepsilon_i |\varepsilon_i\rangle\langle\varepsilon_i|$

$$\sum_p \frac{3\omega_p}{2} |p\rangle\langle p|$$

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$\sum_p \frac{3\omega_p}{2}|p\rangle\langle p|$

2. Low-E 1p projection & Hilbert space
   $P_{1,L} = \sum_{\varepsilon_i \leq \Lambda} |\varepsilon_i\rangle\langle\varepsilon_i| \Rightarrow \mathscr{H}_{1,L} = P_{1,L}\mathscr{H}_1$

$P_{1,L} = \sum_{p\leq\Lambda} |p\rangle\langle p| \qquad |p \leq \Lambda\rangle \in \mathscr{H}_{1,L}$

→ Low-E 1p generators $\mathscr{V}_{1,L} = \{O \in \mathscr{V}_1 \,|\, O|\Omega\rangle \in \mathscr{H}_{1,L}\}$

$a_{p\leq\Lambda}^\dagger$

# Algebraic RG (ARG)

In free theory

3. Isometric 1p rescaling

$$\tilde{\mathscr{V}}_{1,L} = W \mathscr{V}_{1,L} W^\dagger, \quad W = \sum_i \left( \frac{1}{\sqrt{m_i}} \sum_{a \in B_i} |\varepsilon_a\rangle \right) \langle \varepsilon_i |$$

$$W = \int_p |p\rangle\langle e^{-s} p|$$

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4. Double/relative commutant → low/high-E algebra

$$\mathscr{A}_L = (\tilde{\mathscr{V}}_{1,L} \cup \tilde{\mathscr{V}}^\dagger_{1,L})'' \qquad \mathscr{A}_H = \mathscr{A}'_L \cap \mathscr{A}$$

$$O_< \otimes \mathbf{1}_> \in \mathscr{A}_L \qquad \mathbf{1}_< \otimes O_> \in \mathscr{A}_H$$

RG is realized as a conditional expectation on $\mathscr{A}_L$.

# Algebraic RG (ARG)

✓ Applicable to any systems

- No tensor factorization assumed. Not limited to $\infty$ dim or continuum.
- Definition of 'particle' can be arbitrary. e.g. single-site Pauli op.
- 'Hamiltonian' need not be a physical one. e.g. $\sqrt{H^2 - \mathbf{P}^2}$ (Lorentz invariant)

# Resource theory

Resource theory clarifies *which* operations can make use of *what* resources of states.

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Given an operation $\Delta$, which we call the resource-destroying map (RDM),

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$$\Delta(\rho) = \rho \quad \text{(free states have no resource to be destroyed)}$$

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- (A broader class of) free operations are those commuting with $\Delta$:

$$\mathscr{E} \circ \Delta = \Delta \circ \mathscr{E} \quad \text{(free operations do not generate any resource that can be destroyed)}$$

If $\rho$ is a free state, $\Delta(\mathscr{E}(\rho)) = \mathscr{E}(\Delta(\rho)) = \mathscr{E}(\rho)$ so free operations does not generate any resources for free states.

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- Resource of free states should be zero $\mathscr{R}(\mathscr{F}) = 0$ and resource of states never increase under free operations

$$0 \leq \mathscr{R}(\mathscr{E}(\rho)) \leq \mathscr{R}(\rho), \quad \forall \rho$$

$\Delta \in \mathscr{E}$

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<u>Example</u>: Entanglement as resource. Local Operations and Classical Communication (LOCC) is free operations. Separable states are free.

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Quantify resource $\mathscr{R}$ given $\Delta$?

<u>Example</u>: Entanglement as resource. Local Operations and Classical Communication (LOCC) is free operations. Separable states are free.

# Resource theory of RG

Consider the Cesàro mean of each RG step $\mathscr{G}$:

$$\boxed{\mathscr{G}_\infty = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathscr{G}^n \qquad \text{(long time average)}}$$

This becomes the IR limit when it exists. Even if the IR limit does not exist (e.g. limit cycles), the RDM exists.

One can show that $\mathscr{G}_\infty(\rho) = \rho$ iff $\rho \in \mathscr{F}(\mathscr{G})$. So $\mathscr{G}_\infty = \Delta$ (RDM).

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One can show that $\mathscr{G}_\infty(\rho) = \rho$ iff $\rho \in \mathscr{F}(\mathscr{G})$. So $\mathscr{G}_\infty = \Delta$ (RDM).

With this, one can information theoretically identify the resource monotone of RG:

$$\boxed{\mathscr{R}(\rho) := D(\rho \| \mathscr{G}_\infty(\rho)) \qquad \text{relative entropy from the IR limit}}$$

# RG resource monotone = RG monotone

- This RG monotone is rigorously non-increasing under RG flow.
- It can be related to Zamolodchikov metric and central charge near the IR fixed point:

Let $\rho \propto e^{-\beta H}$ then near the IR f.p. $\rho_*$,

$$\boxed{D(\rho(g)\|\rho_*) = \frac{C_*}{2} g^{\mathrm{Zam}}_{ij}(g_*)\,\delta g^i \delta g^j + O(\delta g^3)}$$

For a single linearized RG Eigen-direction $\beta = -\omega\delta g$,

$$\boxed{D(\rho(g(s))\|\rho_*) = \frac{C_*}{\kappa\omega}(c(s) - c_{\mathrm{IR}}) + O(\delta g^3).}$$

$$\langle \mathcal{O}_i(x)\mathcal{O}_j(y)\rangle = \frac{g^{\mathrm{Zam}}_{ij}}{|x-y|^{2\Delta}}$$

Zamolodchikov c-theorem:

$$\frac{dc}{ds} = -\kappa\, g^{\mathrm{Zam}}_{ij}\beta^i\beta^j, \quad \kappa > 0$$

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Bogoliubov-Kubo-Mori (BKM) metric

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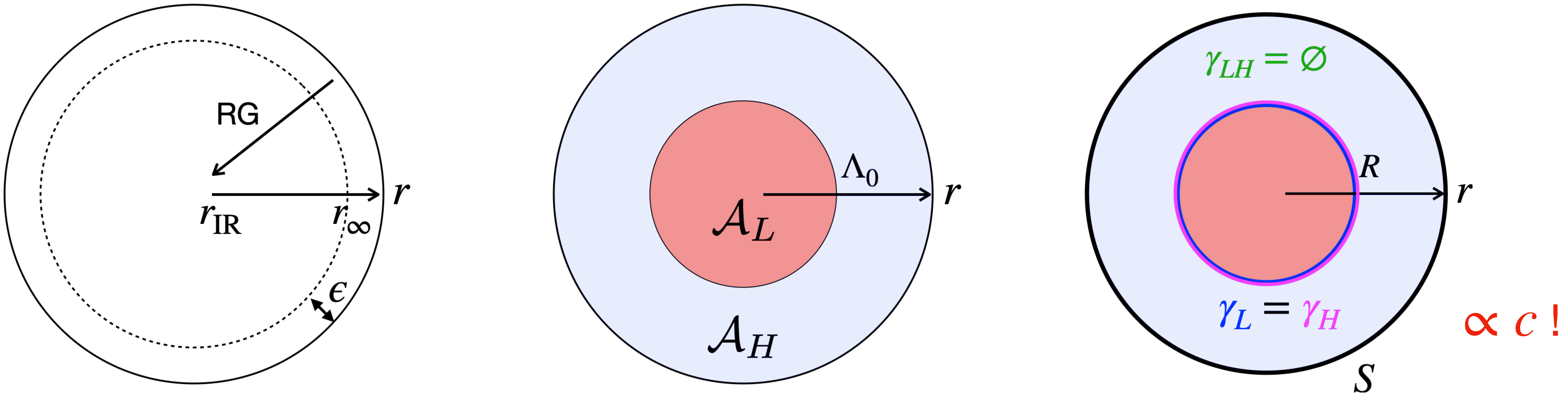
# Holographic calculation

- AdS/CFT identifies the radial direction as energy scale.
- The subalgebra associated to a bulk subregion = low/high-energy algebra
- Algebraic relative entropy resource of coarse graining can be computed by Ryu-Takayanagi formula:



# Summary

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@QFTlover →

- IRG: Wilsonian RG as quantum channel. Interpolating pure and mixed-state RG.
- ARG allows us to apply the methodology to generic systems, generic notion of particles, and generic notion of energy.
- RG monotone can be understood operationally through resource theory.
- Its relation to Zamolodchikov metric and central charge can be shown.
- Algebraic formulation allows us to compute the monotone holographically.

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- Algebraic formulation allows us to compute the monotone holographically.

Other monotones?

Type II or III von Neumann algebra?

Relation to quantum diffusion/generative model, Bayesian RG?

Relation between theory and state RG?

Discrete flow equation?

# Appendix

# Informational RG (IRG)

Why quantum information? … It provides a universal formulation!

- To extend WRG to discrete and/or finite systems
- To formulate RG monotones systematically
- Just for fun!

# Informational RG (IRG)

[Costa-Brink-Nogueira-Klein ’22, ’23] reformulated the WRG as

- Partial trace = integrating out: $\mathrm{Tr}_{>} = \int \mathscr{D}\phi_{>}$
- Unitary = basis transformation = relabeling: $U_s = \int \mathscr{D}\phi \, |\tilde{\phi}(p)\rangle\langle\tilde{\phi}(e^{-s}p)|$

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However, it is not clear how to extend to more generic systems…

🤔 What do we integrate out?

🤔 Rescaling is not implemented unitary?

🤔 Infinitesimal RG (→ flow eq.)?

# Recap: Wilsonian RG

$$Z_\Lambda = \int \mathscr{D}\phi_< \int \mathscr{D}\phi_> \, e^{-S_\Lambda[\phi_<, \phi_>]} = \int \mathscr{D}\phi_< \, e^{-S'_{\Lambda(s)}[\phi_<]} \to \int \mathscr{D}\phi' \, e^{-S^{(s)}_\Lambda[\phi']}$$

**Integrating out high-momentum modes**

$$\phi_> := \tilde{\phi}(p_i \geq e^{-s}\Lambda)$$

**Relabel each mode**

$$\phi'(p'_I = e^s p_i \leq \Lambda)$$
$$:= \phi_<(p_I \leq e^{-s}\Lambda)$$

**Resample to bring back to the original dimensions**

$$\{\phi'(p'_I)\}_{I=1}^N \to \{\phi'(p'_i)\}_{i=1}^M$$

# Algebraic RG (ARG)

✓ RG is realized as a conditional expectation on $\mathscr{A}_L$.

✓ Works with finite-dimensional, discrete systems.

✓ Infinitesimal RG as a single-site isometry (spontaneously breaking translational invariance in energy spectrum).

✓ Approximate uniform rescaling is characterized by the Kullback-Leibler divergence. In thermodynamic limit, this gives the dilation $\psi'(e^{-s}p) = e^{s/2}\psi(p)$.

✓ Averaging over isometric configurations recovers uniform rescaling.

✓ Applicable to any systems

- No tensor factorization assumed. Not limited to ∞ dim or continuum.
- Definition of 'particle' can be arbitrary. e.g. single-site Pauli op.
- 'Hamiltonian' need not be a physical one. e.g. $\sqrt{H^2 - \mathbf{P}^2}$ (Lorentz invariant)