Singularities in the RG flow

Jakovác Antal
Wigner RCP, Corvinus University Budapest

ERG2026, University of Sussex, Sep 1 – 5, 2026

About renormalizability

common wisdom

- compute scale dependent effective action in an operator basis
- relevance of an operator comes from the rescaled coupling

$$\frac{\partial g_i}{\partial \log k} = -d_i \, g_i + \beta_i(g)$$

- dimension comes from field content (engineering dimension), at small coupling radiative corrections are small (if $d_i \neq 0$)
- at high scales

$$g_i(k) \sim g_{i,UV} \left(\frac{k}{k_{UV}} \right)^{-d_i}$$

- robust argument: renormalizability comes from dimension counting

About renormalizability

consequences

- **renormalizable actions**: anything was the action at large scales, at lower scales only the renormalizable terms remain relevant
- example: 4-fermion operator (in 4D)
  - [fermion operator]=3/2, [4-fermion operator]=6, $d_G = 2$
  - at scale *k* the rescaled coupling behaves as $G \sim k^{-2}$
  - e.g.neutrino interaction is order 1 at 100GeV, and order 10^{-10} at 1MeV
- **lattice QCD** and continuum QCD describes the same IR physics, despite they use different actions and different regularization

About renormalizability

are 4-fermion terms irrelevant in IR?

- **QCD**:
  - at low energy hadron propagators (4 and 6-quark terms) are relevant
  - large couplings, perturbatively an IR Landau pole separates the quark and hadron regime
- **QED:** simpler example
  - consider a model with e^- and p^+ fields
  - standard argumentation $\Rightarrow$ at low energies weakly interacting e-p gas
  - in reality: H-atoms, hardly any free e^- or p^+
  - we are in the weak coupling limit all the time!

Running of the 4-fermion term

let us calculate the running of the 4-fermion term in QED!
(details in A.J. PRD 114 (4), 045001, A.J., A. Patkós, Int.J. Mod.Phys. A 34 (27), 1950154)

- field content: e^-, p^+, γ
- renormalizable terms + e-p interaction term in hermitean formalism

$$L=-\frac{Z_3}{4}F_{\mu\nu}F^{\mu\nu}+\sum_{i=e,p}\bar{\psi}_i\left(iZ_2^{(i)}(i\partial_\mu\pm eA_\mu)\gamma^\mu-m_i\right)\psi_i+\lambda\psi_e^+\psi_p^+\psi_e\psi_p$$

$$\lambda\psi_e^+\psi_p^+\psi_e\psi_p=\lambda_{\alpha\beta\gamma\delta}(p,q,k,l)\psi_{e\alpha}^+(p)\psi_{p\beta}^+(q)\psi_{e\gamma}(k)\psi_{p\delta}(l)(2\pi)^4\delta_{p+q-k-l}$$

- without last term: usual RG evolution

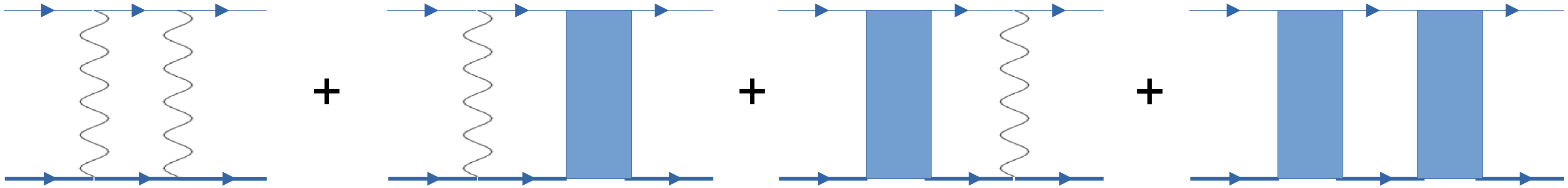
Running of the 4-fermion term

running of the 4-fermion term

- we regulate only the fermion propagators
- explicit contribution from gauge coupling:
  - k^2 evolution from dimension analysis
  - small (finite) perturbative correction
- self-consistent evolution

$$\lambda(k) \sim \frac{k^2}{\Lambda^2}\left(1+\frac{C\,e^4(k)}{k^2+M^2}\right)\lambda(\Lambda)$$

$\partial_k \lambda \rightarrow$

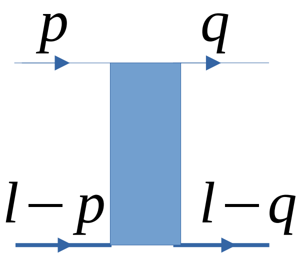


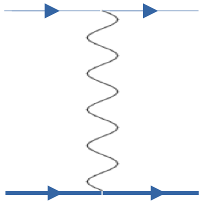
$+$ t-channel + other

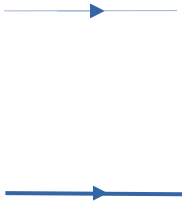
Running of the 4-fermion term

running of the 4-fermion term

- self-consistent evolution in multi index notation
  - fermion-pair propagator $G^{pair(l)}_{p\alpha\sigma,q\beta\delta} = G^{(e)}_{\alpha\beta}(p)\,G^{(p)}_{\gamma\delta}(l-p)\,\delta(pq)$
  - one-photon exchange $V_{p\alpha\sigma,q\beta\delta} = e^2(k)\,\gamma^{\mu}_{\beta\alpha}\gamma_{\mu\delta\sigma}\left((p-q)^2 + i\epsilon\right)^{-1}$
  - 4-fermion 1PI interaction $\lambda^{(l)}_{p\alpha\sigma,q\beta\delta} = \lambda_{\alpha\beta\gamma\delta}(p,q,l-p,l-q)$

$\lambda^{(l)} =$ 

$V =$ 

$G^{pair} =$ 

Running of the 4-fermion term

running of the 4-fermion term

- neglect running of the gauge coupling (important at higher scales)
- in matrix notation

$$\partial_k \lambda = (V+\lambda)\,\partial_k G_k^{pair,(l)}(V+\lambda) + \text{t-channel contribution}$$

- solution from the s-channel term:

$$\lambda(k) = \left(1 + V\, G_k^{pair,(l)}\right)^{-1} V\, G_k^{pair,(l)} V$$

- not surprising: similar to the running of the scalar 4-point function

$$\lambda(k) = \frac{\lambda(k_0)}{1+e^2\beta(k)}$$

Running of the 4-fermion term

running of the 4-fermion term

$$\lambda(k)=\left(1+V\,G_k^{pair,(l)}\right)^{-1} V\,G_k^{pair,(l)}\,V$$

- formally the higher loop corrections are $\sim V\,G_k^{pair,(l)} \sim e^2$
- in scalar case it would mean a finite correction
- but in matrix case it can be **singular** if $\det\left(1+V\,G_k^{pair,(l)}\right)=0$

$$V\,G_k^{pair,(l)}\,u=-u \quad \Rightarrow \quad \textbf{Bethe-Salpeter equation!}$$

- **solutions of Bethe-Salpeter equation show up as poles in the FRG evolution!**

Running of the 4-fermion term

why is a scalar different from a matrix?

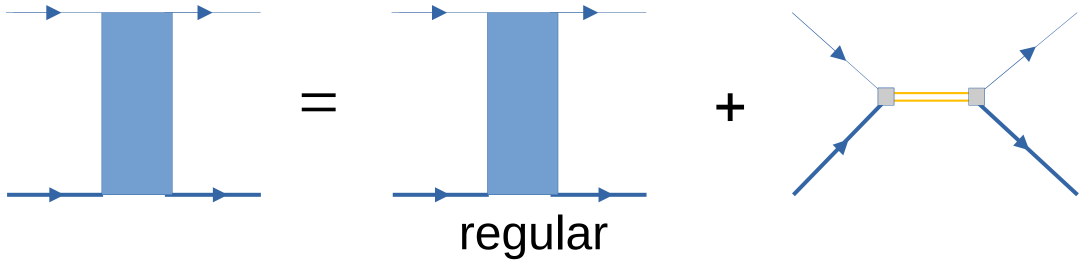
- matrix elements one-by-one are $O(e^2)$
- but we combine multiple (N) matrix elements
- compensation of small coupling is not $\log(k)$, but the number of elements
- consequence: we must combine $N \sim 1/e^2$ modes

Running of the 4-fermion term

how to continue? how to avoid the pole?

- 4-fermion interaction is singular, so this form is not appropriate to go below the corresponding scale
- we extract the singular part near the divergence

$$\lambda^{(l)} = \lambda^{(l)}_{regular} + \lambda^{(l)}_{singular}, \quad \text{where} \quad \lambda^{(l)}_{singular} \sim \frac{v^{(l)} \otimes v^{(l)}}{k - K^{(l)}_{sing}}$$



Running of the 4-fermion term

effective fields for bound states

- the singular part can be represented with an auxiliary field in the Lagrangian as $\frac{1}{2} H \left(K_{sing} - k \right) H + \psi_e \psi_p v H + \text{h.c.}$
- for all singularities we need to introduce a new H-mode (excited H-atoms)
- H-propagator is arbitrary up to the pole behaviour

Conclusion and consequences

running of the 4-fermion term – main steps

- running is similar to the usual 1-loop running $\lambda(k)=\frac{\lambda(k_0)}{1+e^2\beta(k)}$
- the correction in the denominator is finite and perturbative (e^2 is small)
- it still can lead to singularities where the Bethe-Salpeter equation is satisfied

$$V\,G_k^{pair,(l)}u=-u$$

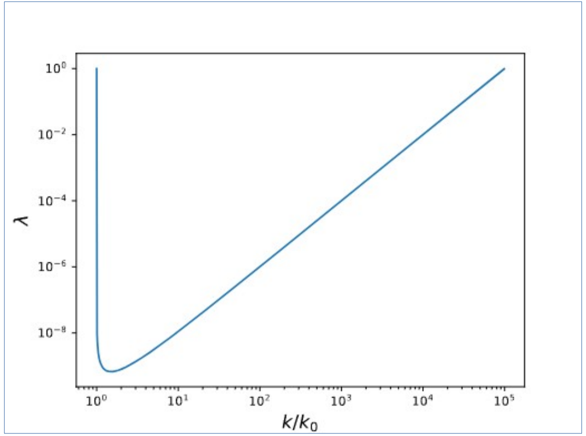
- transfer singularities to the propagators of auxiliary fields (bound states)

bound states ➡ **singularities of the RG flow**

Conclusion and consequences

direct consequence

- four fermion (4f) interaction can be **relevant** in IR
- large number of modes compensate e^2 suppression
- for a long time 4f seems to be irrelevant, anomalous behaviour shows up only close to the singularity



Conclusion and consequences

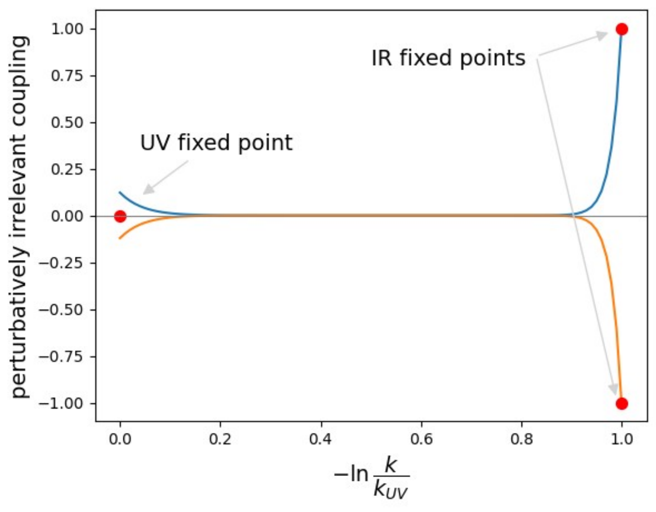
how predictive is QED?

- QED without 4f is completely self-consistent, **but** does not describe real IR physics
- **are there other perturbatively irrelevant operators becoming relevent in IR?**
- how can we reveal true IR physics? shall we incorporate all potentially relevant terms? how do we know which are they?

Conclusion and consequences

do exact calculations (full FRG or MC simulations) help?

- they start from an action without 4-fermion interaction
- but it is **generated later** (exact action) – the contribution is small, but is serves as a seed for the bound state formation
- **caveat**: what if there are several IR fixed points? the exact method chooses one



even exact methods can not describe the complete IR reality!