fRG flows and gauge consistency

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Mainly based on
Echigo, Igarashi, KI, Pawlowski and Takahashi, PTEP (2026) 013B01.

Related works

Igarashi, KI and Pawlowski (2016);

Igarashi, KI and Morris, PTEP (2019);

Igarashi and KI, PTEP (2021).

Content

1. Quantum Master Equation (QME) = BRST symmetry + nilpotency
2. An observation: the manner the flow eq. collects quantum corrections
3. Quantum Master Equation including partial quantum corrections
4. Flows with photon qunatum correction for QED with 4-fermi interactions

1. The anti-field formalism a la Batalin-Vilkovisky

For a classical gauge fixed action $S_0[\phi]$ for a gauge theory, define an extended action as

$$S_{\rm cl}[\phi, \phi^*] \equiv S_0[\phi] + \phi^*_A \delta\phi^A$$

Here **antifields** ϕ^*_A are introduced as sources for the BRST transformations $\delta\phi^A$.

The canonical structure via the antibracket for any field variables X and Y, we define

$$(S_{\rm cl}, S_{\rm cl}) = 2(\delta S_0 + \phi^*_A \delta^2 \phi^A)\,, \qquad \text{where} \quad (X, Y) \equiv \frac{\partial^r X}{\partial\phi^A}\frac{\partial^l Y}{\partial\phi^*_A} - \frac{\partial^r X}{\partial\phi^*_A}\frac{\partial^l Y}{\partial\phi^A}$$

Classical master equation (CME): $(S_{\rm cl}, S_{\rm cl}) = 0 \Leftrightarrow$ action invariance *and* the nilpotency.

Generally, we also have the contribution from the path integral measure. We define **the quantum master operator**:

$$\Sigma[\phi, \phi^*] \equiv \frac{\partial^r S}{\partial\phi^A}\frac{\partial^l S}{\partial\phi^*_A} - \frac{\partial^r}{\partial\phi^A}\delta\phi^A = \frac{1}{2}(S,\ S) - \Delta S\,, \qquad \Delta \equiv (-)^{\epsilon_A+1}\frac{\partial^r}{\partial\phi^A}\frac{\partial^r}{\partial\phi^*_A}$$

where $\epsilon_A \equiv \epsilon(\phi^A)$ is the Grassmann parity. The system is BRST invariant quantum mechanically if the two contributions cancel:

$$\Sigma[\phi, \phi^*] = 0\ . \quad \text{(QME)}$$

The quantum BRST transformation is defined as

$$\delta_Q X \equiv (X, S) - \Delta X$$

We have two important algebraic identities without assuming QME:

$$\delta_Q \Sigma[\phi, \phi^*] = 0\,, \qquad \delta_Q^2 X = (X, \Sigma[\phi, \phi^*])\,.$$

The quantum BRST transformation is nilpotent if and only if QME holds.

QME is equivalent to BRST symmetry and its nilpotency.
– The nilpotency is necessary to realize the BRST algebra, its representation and unitarity of non-Abelian gauge theory. (cf. Kugo-Ojima formalism).

QME in terms of 1PI action (in dimensionful form)

Rewriting the QME for Wilson action S, $\Sigma = \frac{1}{2}(S,S) - \Delta S = 0$, in terms of 1PI action, we obtain "The modified Slavnov-Taylor identity" (Ellwanger):

$$\Sigma = \frac{1}{2}(\Gamma,\Gamma) - \underbrace{\mathrm{Str}\left(K\Gamma_{I*}^{(2)}\left[1+\bar{\Delta}^{(0)}\Gamma_I^{(2)}\right]^{-1}\right)}_{\Delta S \quad \text{measure term}} = 0\,,$$

where $\bar{\Delta}^{(0)} = (1-K)\Delta^{(0)}$ is the IR-regulated free propagator,

$$\left(\Gamma_{I*}^{(2)}\right)^A{}_B = \frac{\partial^l\partial^r}{\partial\Phi_A^*\partial\Phi^B}\Gamma_I\,, \qquad \left(\Gamma_I^{(2)}\right)_{AB} = \frac{\partial^l\partial^r}{\partial\Phi^A\partial\Phi^B}\Gamma_I\,.$$

Under a change of the scale $t = \ln\Lambda/\mu$, where Λ is the IR cutoff and μ is arbitrary,

$$\frac{d}{dt}\Sigma \propto \Sigma$$

Starting from a 1PI action that satisfies QME and following the flow eq., we have a 1PI action satisfying QME.

2. Accumulation of quantum corrections in dimensionful flow equation

Under an infinitesimal change of $t = \ln \Lambda/\mu$,

$$\Gamma_I(t+\Delta t) \sim \Gamma_I(t) + \Delta t \cdot \frac{1}{2}\mathrm{Str}\left(\otimes\left[\bar{\Delta}^{-1} + \tau\right]^{-1}\right)(t)\,.$$

Here $\bar{\Delta}$ and τ are the propagator and the vertex at the scale t

$$\bar{\Delta}(t) = \frac{1}{\left(\left(\bar{\Delta}^{(0)}\right)^{-1} + \Gamma_I^{(2)}[0,0]\right)(t)} \quad , \quad \tau(t) = \left(\Gamma_I^{(2)}[\Phi,\Phi^*] - \Gamma_I^{(2)}[0,0]\right)(t)\,,$$

and $\otimes$ contains the derivative of regulator function.

$\Gamma_I^{(2)}(t+\Delta t)$ **contributes to the propagator and the vertex at** $t + \Delta t + \Delta t'$**.**

Accumulation of quantum corrections

At each step of the scale change, propagators and vertices aquire new quantum corrections.

In practice, only approximately we may solve QME as well as flow eq.. Let us consider this question by taking a simple gauge system, **QED with chiral invariant 4-fermi interactions**.

The classical action $\Gamma_{\rm cl} = \Gamma_0 + \Gamma_1 + \Gamma_{2,\rm cl}$**,**

$$\Gamma_0 = \int_x \left[\frac{1}{2}\Big\{(\partial_\mu A_\nu)^2 - (\partial\cdot A)^2\Big\} + \bar\Psi i\partial\!\!\!/\Psi + \big(A^*_\mu - i\partial_\mu \bar C\big)\partial_\mu C + \frac{\xi B^2}{2} + \big(\bar C^* - i\partial\cdot A\big)B\right]$$

$$\Gamma_1 = \int_x \left[-e\bar\Psi A\!\!\!/\Psi - ie\Psi^*\Psi C + ie\bar\Psi\bar\Psi^* C\right],$$

$$\Gamma_{2,\rm cl} = \int_x \left[\frac{G_S}{2}\Big\{\big(\bar\Psi\Psi\big)\big(\bar\Psi\Psi\big) - \big(\bar\Psi\gamma_5\Psi\big)\big(\bar\Psi\gamma_5\Psi\big)\Big\}\right.$$
$$\left. + \frac{G_V}{2}\Big\{\big(\bar\Psi\gamma_\mu\Psi\big)\big(\bar\Psi\gamma_\mu\Psi\big) + \big(\bar\Psi\gamma_5\gamma_\mu\Psi\big)\big(\bar\Psi\gamma_5\gamma_\mu\Psi\big)\Big\}\right],$$

satisfies the Classical Master Equation (CME): $(\Gamma_{\rm cl}, \Gamma_{\rm cl}) = 0\,.$

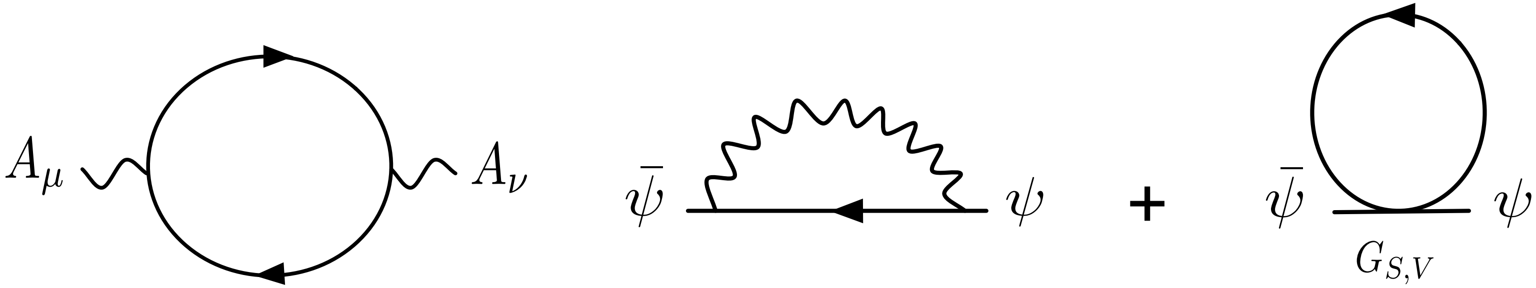
"Str log-formula" for one-loop effective action gives **quantum corrections**

$$\Gamma_q = \frac{1}{2}\mathrm{Str}\log\left((\bar{\Delta}^{(0)})^{-1} + \Gamma_{I,\mathrm{cl}}^{(2)}\right)$$

where $\Gamma_{I,\mathrm{cl}}$ is the classical part of Γ_I. Note that $\Gamma_{I,\mathrm{cl}}^{(2)}$ is field dependent.

Γ_q gives rise all the possible one-loop diagrams.

Contributions of Γ_q to 2-point and 3-point functions ,



(a) Γ_{q}^{AA} (b) $\Gamma_{\mathrm{q}}^{\bar{\psi}\psi}$ The 2nd term turns out to be zero.

Figure 1: Contributions to 2-point functions. Internal propagators $\bar{\Delta}^{(0)}$ are IR-regularized.

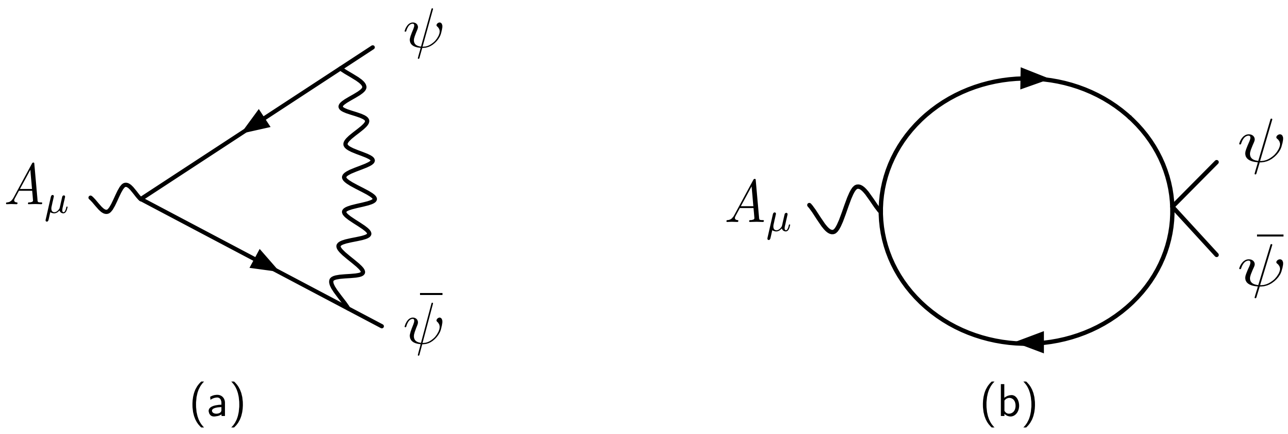


Figure 2: Contributions to 3-point vertex. Internal propagators $\bar{\Delta}^{(0)}$ are IR-regularized.

3. Quantum Master Equation

Consider the one-loop improved 1PI effective action

$$\Gamma = \Gamma_{\rm cl} + \Gamma_q = \Gamma_{\rm cl} + \frac{1}{2}{\rm Str}\log\left((\bar{\Delta}^{(0)})^{-1} + \Gamma^{(2)}_{I,{\rm cl}}\right) ,$$

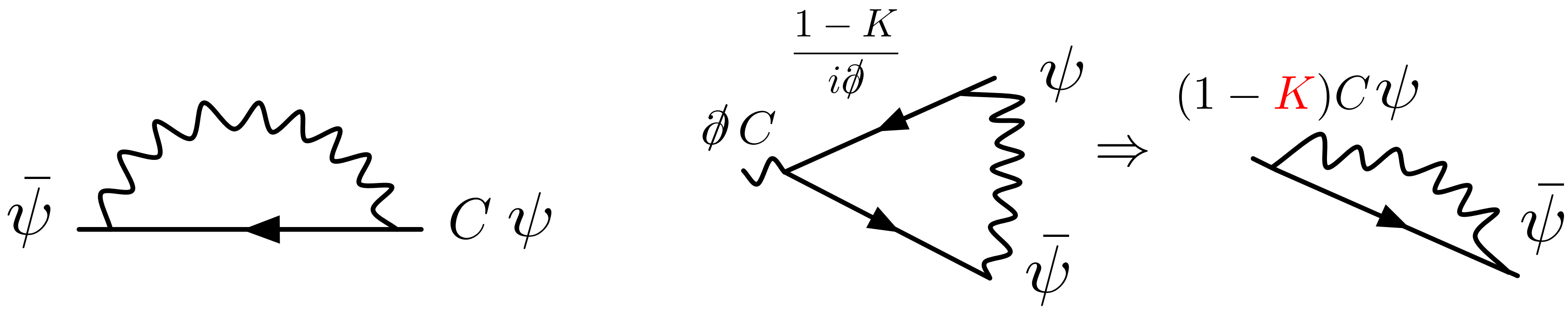
We would like to confirm the vanishing of Σ up to the following order

$$\Sigma = \frac{1}{2}(\Gamma_{\rm cl}, \Gamma_{\rm cl}) + (\Gamma_{\rm cl}, \Gamma_q) - {\rm Str}\left(K\Gamma^{(2)}_{I*}\left[1 + \bar{\Delta}^{(0)}\Gamma^{(2)}_{I,{\rm cl}}\right]^{-1}\right)$$

We have shown the vanishing of $A_\mu C$ and $\bar{\psi}\psi C$ terms of Σ, $\Sigma^{AC} = 0$ and $\Sigma^{\bar{\psi}\psi C} = 0$.

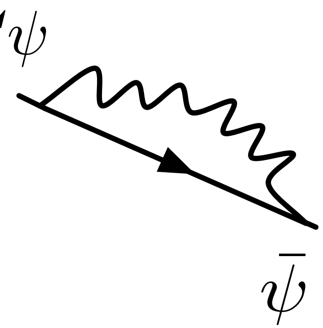
- $\Sigma^{AC} = 0$ determines the longitudinal part of photon two point function.
- $\Sigma^{\bar{\psi}\psi C} = 0$. Let us explain how it is realized.

- $\delta_{\rm BRST}\psi \propto C\psi$ contribution in $(\Gamma_{\rm cl}, \Gamma_q)$



- Measure term contribution

$$-\mathrm{Str}\left(K\Gamma^{(2)}_{I*}\left[1+\bar\Delta^{(0)}\Gamma^{(2)}_{I,\mathrm{cl}}\right]^{-1}\right)$$



$$\Rightarrow \qquad -\mathrm{Str}\left(K\Gamma^{(2)}_{I*}\bar\Delta^{(0)}\Gamma^{(2)}_{I,\mathrm{cl}}\bar\Delta^{(0)}\Gamma^{(2)}_{I,\mathrm{cl}}\right)$$

$\frac{\partial}{\partial\psi^*_\alpha}\frac{\partial^r}{\partial\psi_\beta}\Gamma_I = ie\delta_{\alpha\beta}C$ fermion prop. $\bar\psi$ photon prop. ψ

Figure 3: $\bar\psi - C\psi$ term from measure term contribution.

In the measure term, $K\Gamma^{(2)}_{I*} \sim KC$ and the measure term produce the term that exactly cancels the remaining term from $\bar\psi\psi C$ terms in $(\Gamma_{\rm cl}, \Gamma_q)$: therefore, $\Sigma^{\bar\psi\psi C} = 0$.

- An important observation: the proof above does not depend on the details of the photon propagator. This allows us to use the photon propagator with one-loop quantum correction:

$$\Gamma = \Gamma_{\rm cl} + \frac{1}{2}\mathrm{Str}\log\left((\bar{\Delta}^{(0)})^{-1} + \left(\Gamma_{I,\rm cl} + \Gamma_q^{AA}\right)^{(2)}\right) ,$$

where Γ_q^{AA} is the one-loop quantum correction to the photon propagator. The condition $\Sigma^{\bar{\psi}\psi C} = 0$ still holds with the above RG-improved 1PI effective action.

Up to now, our discussions are on the dimensionful flow equation and QME.

4. Ansatz to 1PI action for dimensionless flow eq.

(Here we ignored the antif-field dependence.)

$$\Gamma[\Phi] = \frac{1}{2}\Phi^A\Big(\frac{1}{\Delta^{(0)}}\Big)_{AB}\Phi^B + \Gamma_I[\Phi]\,,$$

$$\Gamma_I[\Phi] = \frac{1}{2}\int_p A(-p)_\mu\ \mathcal{A}_{\mu\nu}(t,p)\ A(p)_\nu - e(t)\ \bar\Psi \not{A}\Psi + G_{S,V}(t)(\bar\Psi\Psi\bar\Psi\Psi)$$

$$\Big(\ \mathcal{A}_{\mu\nu}(t,p) = P^T_{\mu\nu}\mathcal{T}(t,p^2) + P^L_{\mu\nu}\mathcal{L}(t,p^2)\ \Big)$$

$$\Gamma[\Phi] = \frac{1}{2}\int_p A_\mu(-p)\Big[P^T_{\mu\nu}\Big\{p^2 + \mathcal{T}(t,p^2)\Big\} + P^L_{\mu\nu}\Big\{\xi^{-1}p^2 + \mathcal{L}(t,p^2)\Big\}\Big]A_\nu(p)$$

$$+\int_p\Big[\bar C(-p)ip^2C(p) + \bar\Psi(-p)\Big] - e(t)\int_{p,q}\bar\Psi(-p)\not{A}(p-q)\Psi(q)\not{p}\Psi(p)$$

$$+\frac{1}{2}\int_{p_1,\cdots,p_4}(2\pi)^4\delta^4(p_1+p_2+p_3+p_4)$$

$$\times\Big[G_S(t)\Big\{\big(\bar\Psi(p_1)\Psi(p_2)\big)\ \big(\bar\Psi(p_3)\Psi(p_4)\big) - \big(\bar\Psi(p_1)\gamma_5\Psi(p_2)\big)\ \big(\bar\Psi(p_3)\gamma_5\Psi(p_4)\big)\Big\}$$

$$+G_V(t)\Big\{\big(\bar\Psi(p_1)\gamma_\mu\Psi(p_2)\big)\ \big(\bar\Psi(p_3)\gamma_\mu\Psi(p_4)\big) + \big(\bar\Psi(p_1)\gamma_5\gamma_\mu\Psi(p_2)\big)\ \big(\bar\Psi(p_3)\gamma_5\gamma_\mu\Psi(p_4)\big)\Big\}\Big]\,,$$

In our Ansatz, we find:

- $\mathcal{A}_{\mu\nu}$ is a quantum correction determined by the flow eq. and QME.
- $\mathcal{A}_{\mu\nu}$ contributes to the photon propagator.

$$\bar{\Delta}_{\mu\nu} = \left(\left(\bar{\Delta}^{(0)} \right)^{-1} + \mathcal{A} \right)_{\mu\nu}$$

- We use the IR regulated free propagator for fermion.
- We have running couplings $e(t)$ and $G_{S,V}(t)$.

The one-loop structure of the dimensionless flow equation consists of the above propagators and coulplings. In the lowest order of the running couplings, their contributions are the same as those found in dimensionful formulations.

Hierarchical Phase Structure: a funnel halved by the $\alpha = 0$ plane.

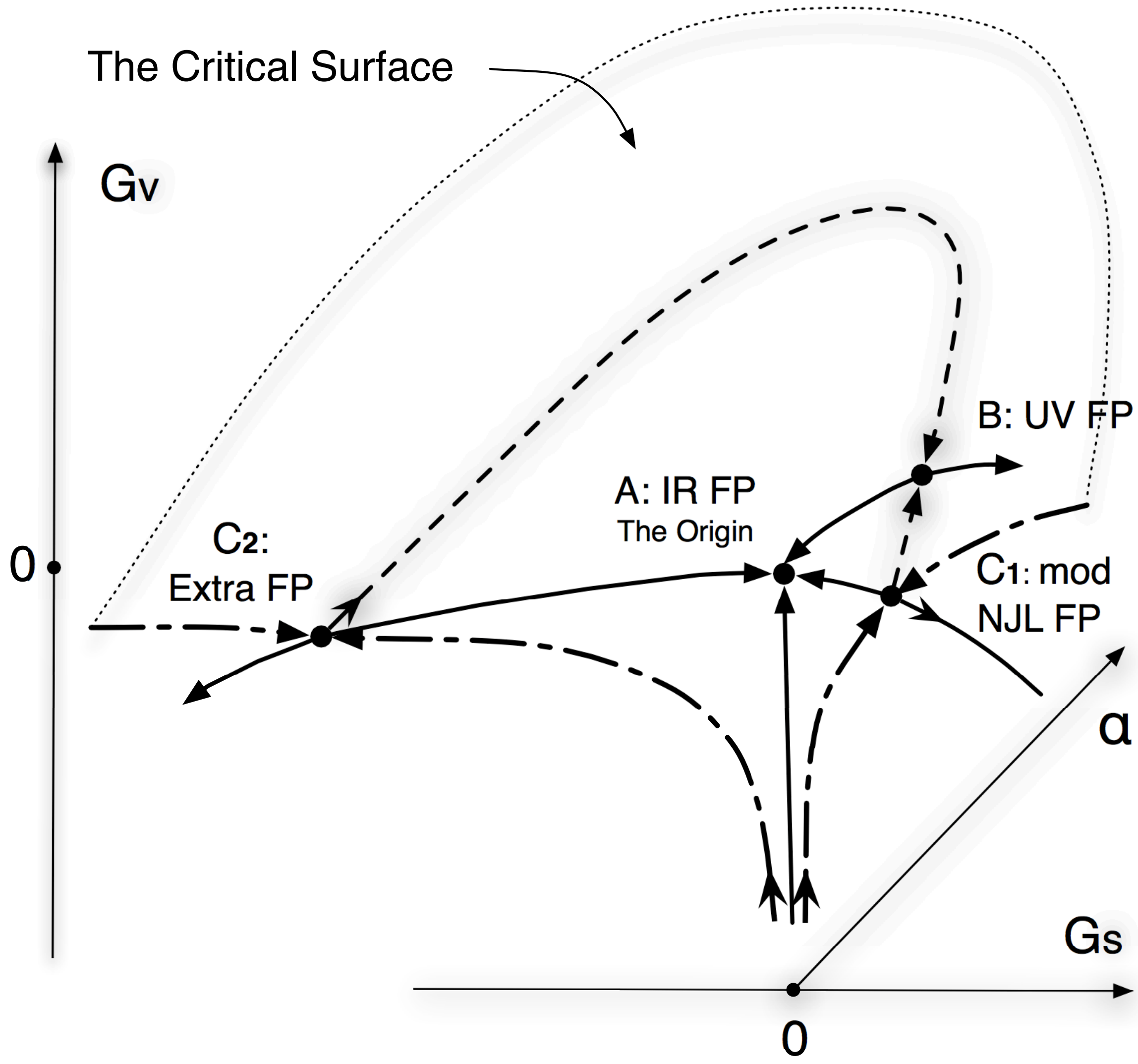


Figure 4: Four fixed points and critical surface and critical lines

Four fixed points

Fixed Point	# of Rel. ops.	(α, G_S, G_V)
A: I.R.	0	The origin
B: U.V.	1	$(13.5,\ 6.99,\ 0.573)$
C_1: mod. NJL	2	$(0,\ 26.3,\ -3.29)$
C_2: extra	2	$(0, -105,\ -52.6)$

- U.V. fixed point is in Landau gauge though not shown in the above figure.
- Aoki et. al. (1997) studied the same system with different regularisation functions and approximations. Some higher order terms are also included here.

Puzzle of e and e'

$$\Gamma_1 = \int_x \left[-e\bar{\Psi}\not{A}\Psi - ie'\Psi^*\Psi C + ie'\bar{\Psi}\bar{\Psi}^* C \right] ,$$

- The couplings e and e' are renormalized differently in the presence of 4-fermi couplings.
- Yet, QME holds when we take $e' = e$.
- This could be a generic problem when couplings other than gauge coupling are introduced.

Summary and Discussion

1. Quantum Master Equation (QME) = BRST symmetry + nilpotency

2. An observation: the manner the flow eq. collects quantum corrections

3. QME including partial quantum corrections

4. Flows with photon qunatum correction for QED with 4-fermi interactions

Possible improvement: include quantum corrections to fermion propagator and interaction vertices.