A Complex LPA for the Critical and Multicritical Lee-Yang fixed points

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Based on:

- Dario Benedetti, FE, Omar Zanusso [arXiv:2601.15087]

Why study non-unitary theories?

Recent conjecture: [Zambelli, Zanusso, 16'][Lencsés, Miscioscia, Mussardo, Takcks, 24'][Katsevich, Klebanov, Sun, Tarnopolsky, 25'][. . .]

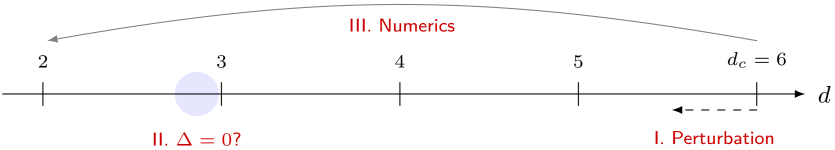
In $d = 2$, the non unitary minimal models $\mathcal{M}(2, 2n+3)$ corresponds to the IR fixed point of

$$S = \int \mathrm{d}^2 x \; \frac{1}{2} \partial_\mu \varphi(x) \partial^\mu \varphi(x) + g \int \mathrm{d}^2 x \; i\varphi^{2n+1}(x)$$

- $\mathcal{PT}$-symmetry $\longrightarrow$ real, bounded from below spectrum
- Lee-Yang theory $\leftrightarrow \mathcal{M}(2,5)$ [Fisher 78'][Cardy, 85']

The idea:

Use the FRG approach to continue from the upper critical dimension to $d = 2$



Wetterich equation [Wetterich, 1993]

$$k\partial_k \Gamma_k = \frac{1}{2}\,\mathrm{Tr}\,\left[k\partial_k \mathcal{R}_k \left(\Gamma_k^{(2)} + \mathcal{R}_k\right)^{-1}\right]$$

Litim cutoff:

$R_k(q) = Z_k\,\left(k^2 - q^2\right)\Theta\left(k^2 - q^2\right)$

Anomalous dimension

$$\eta = -\frac{1}{Z_k}\,k\partial_k Z_k$$

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Local Potential Approximation

$$\Gamma_k[\phi] = \int \mathrm{d}^d x \left(\frac{1}{2} Z_k \left(\partial\phi\right)^2 + V_k(\phi)\right)$$

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- $Z_k = 1$: LPA

  → $\eta = 0,\ \Delta = \dfrac{d-2}{2}$

- $Z_k \neq 1$: LPA'

  → $\eta \neq 0,\ \Delta = \dfrac{d-2+\eta}{2}$

Functional Renormalization Group

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Fixed point equation

$$d\,v(\varphi) - \Delta\,\varphi\,v'(\varphi) = \left(1 - \frac{\eta}{d+2}\right)\frac{c_d}{1+v''(\varphi)}$$

$\eta = 0$ LPA

$\eta = \dfrac{c_d\,v'''(\varphi_0)^2}{\left(1+v''(\varphi_0)\right)^4}$ LPA'

Perturbative treatment

Goal: Construct an ε-expansion in $d = d_c - \varepsilon$

⟶ Linearize by expanding around the Gaussian solution, $v(\varphi) = v_0 + \tilde{v}(\varphi)$

$$d\, v(\varphi) - \Delta\, \varphi\, v'(\varphi) - c_d \left(1 - \frac{\eta}{d+2}\right) \frac{1}{1 + v''(\varphi)} = 0$$

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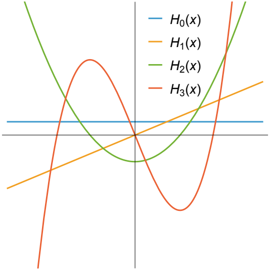
Upper critical dimension

⟶ Polynomialy bounded solution: Hermite polynomials

$$\tilde{v}(\varphi) \simeq \lambda\,H_n(\varphi), \quad n = \frac{2d}{d-2}$$



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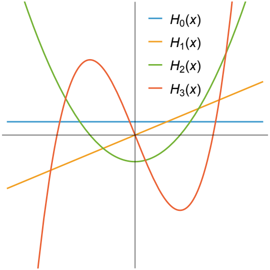

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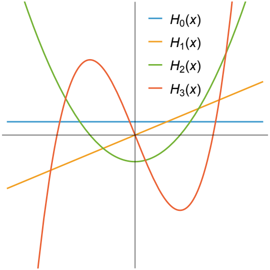

Perturbative potential $(n = 1,\ d = 6 - \varepsilon)$

$$v(\varphi) = v_0 + \varepsilon^{1/2}\lambda\,H_3(\varphi) + \varepsilon\,\lambda^2 \sum_{q=0}^{\infty} \alpha_q H_q(\varphi) + O(\varepsilon^{3/2})$$

$$\simeq 1.4\cdot 10^{-5} + \boxed{i}\,\varepsilon^{1/2}\left(6.1\ \boxed{\varphi^3}\ - 7.6\cdot 10^{-4}\,\varphi\right) - \varepsilon\left(5.6\cdot 10^{-2}\ \boxed{\varphi^2}\ - 3.1\cdot 10^{-5}\right)$$

Complex potential

Coupled fixed point equations

Given $v(\varphi) = u(\varphi) + ih(\varphi)$,

$$u - \frac{\Delta}{d}\,\varphi\, u' = \frac{1 + u''}{(1+u'')^2 + (h'')^2}, \quad h - \frac{\Delta}{d}\,\varphi\, h' = -\frac{h''}{(1+u'')^2 + (h'')^2}$$

Complex potential

~~Coupled~~ fixed point equations

Assume $u(\varphi) = 0$,

$\mathcal{PT}$-symmetry $v^*(\varphi) = v(-\varphi)$

$$\boxed{h - \gamma\,\varphi\,h' = -\frac{h''}{1+(h'')^2}}\,, \quad \gamma = \frac{\Delta}{d}, \qquad \boxed{h(0) = 0, \quad h'''(0) = g \in \mathbb{R}}$$

$\longrightarrow$ *Valid* fixed point solution: global solutions (with real part bounded from below)

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Large field $\varphi \to \pm\infty$ behaviour

- $0 < \Delta < d/2$: Two (likely) isolated solutions
  - Gaussian solution (isolated) $h = 0$
  - nontrivial solution (one parameter family)

$$h(\varphi) \sim B\,\varphi^{\frac{1}{\gamma}} - \frac{\gamma^2}{2B(1-\gamma)^2}\varphi^{2-\frac{1}{\gamma}} + \mathcal{O}\left(\varphi^{4-\frac{2}{\gamma}}\right)$$

- $\Delta < 0$: Continuous (two parameters) family

$$h(\varphi) \sim b_1\varphi^{\frac{1}{\gamma}}\left(1+\mathcal{O}\left(\varphi^{-2}\right)\right) + b_2\,e^{\frac{\gamma}{2}\varphi^2}\,\varphi^{-\frac{1}{\gamma}-1}\left(1+\mathcal{O}\left(\varphi^{-2}, e^{\gamma\varphi^2}\,\varphi^{-\frac{2}{\gamma}-2}\right)\right)$$

Newton equation

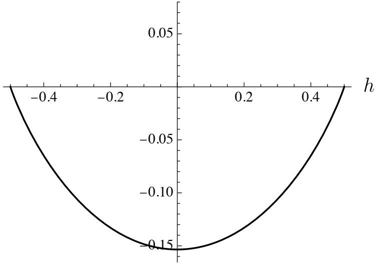
$$h - \gamma\,\varphi\,h' = -\frac{h''}{1+(h'')^2}, \quad h(0)=0, \quad h'''(0)=g$$

Vanishing Δ

Newton equation

$$h'' = -\frac{\mathrm{d}}{\mathrm{d}h}U(h), \quad h(0) = 0, \quad h'''(0) = -h'(0) = g$$

$$U(h) = \frac{\ln\left(1 - \sqrt{1-4h^2}\right) + \sqrt{1-4h^2}}{2}$$



→ Energy conservation:

$$\varphi = \frac{1}{\sqrt{2}} \int_0^h \frac{d\,z}{\sqrt{E - U(z)}}$$

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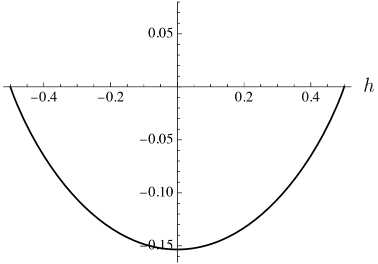

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LPA' upper bound

For any d, there exists $g = g_0(d)$ such that

$$\Delta = \left. \frac{d - 2 + \eta(d,g)}{2} \right|_{g = g_0(d)} = 0$$

$$d_0 = 2\frac{3 - \ln 2}{1 + \ln 2} \sim 2.7249$$

→ At $d = d_0$, $E = 0 \implies g_0(d_0) = \pm\sqrt{1 - \ln 2}$

Numerical resolution

$$h - \frac{\Delta}{d}\,\varphi\,h' = -\frac{h''}{1+\left(h''\right)^2}, \quad h(0) = 0, \quad h'''(0) = g \in \mathbb{R}$$

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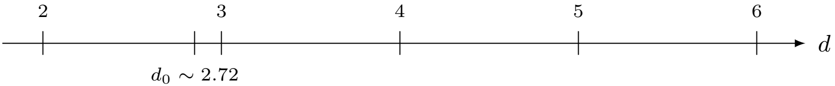
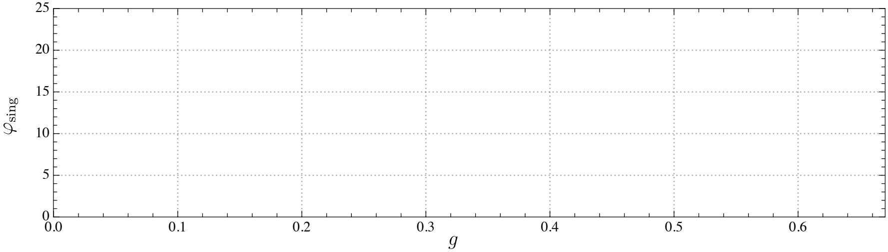
- LPA : $\eta = 0 \longrightarrow \Delta = \dfrac{d-2}{2}$
- LPA' : $\eta = \frac{d+2}{2}\left(1 - \sqrt{1 + \frac{4\,d\,g^2}{(d+2)}}\right) \longrightarrow \Delta = \dfrac{d-2+\eta}{2}$

Numerical resolution





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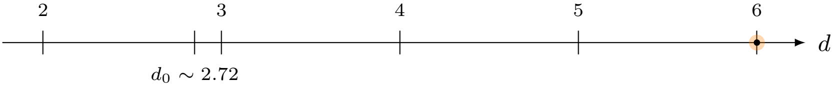
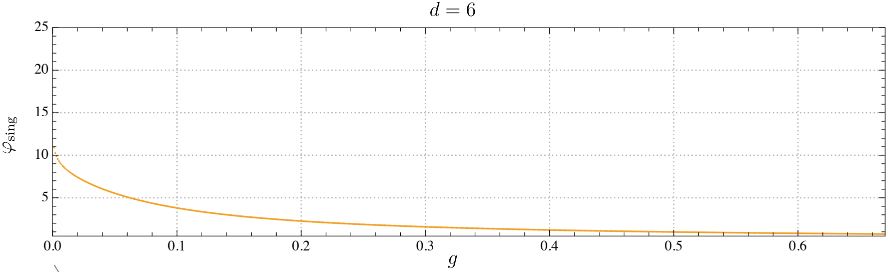
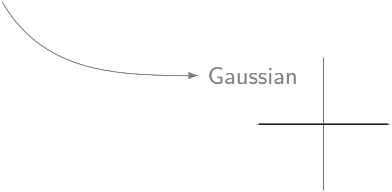
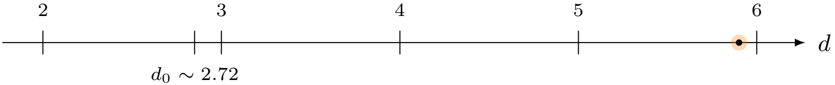


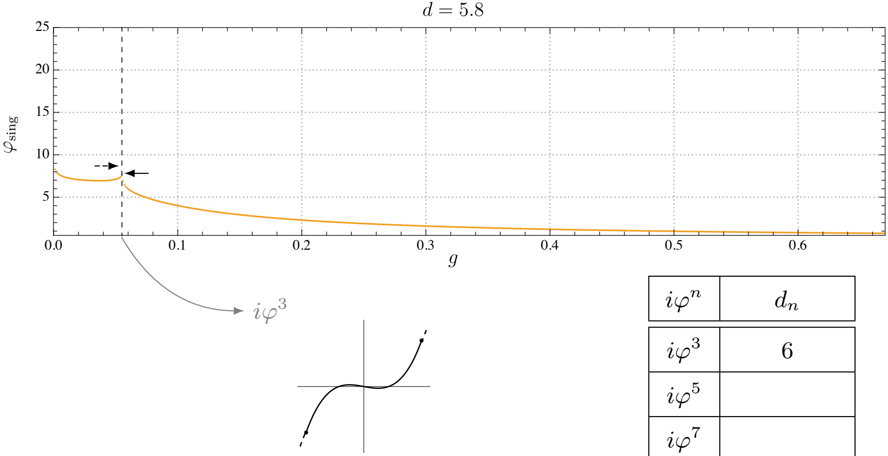




$i\varphi^n$	d_n
$i\varphi^3$	
$i\varphi^5$	
$i\varphi^7$	

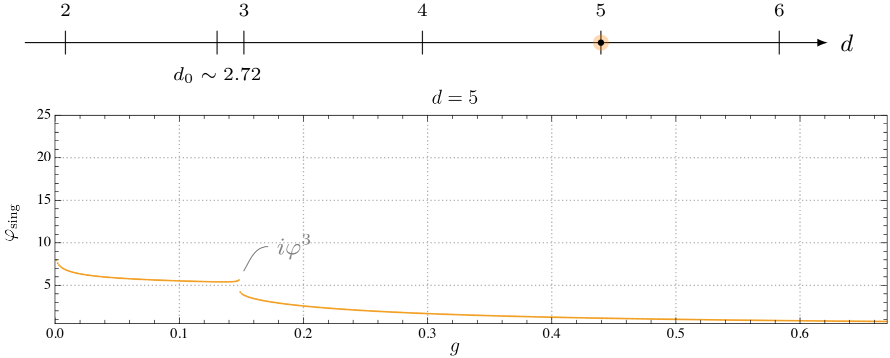
Spike plots LPA





$i\varphi^n$	d_n
$i\varphi^3$	6
$i\varphi^5$	
$i\varphi^7$	

Spike plots LPA



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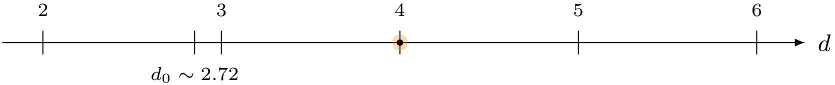

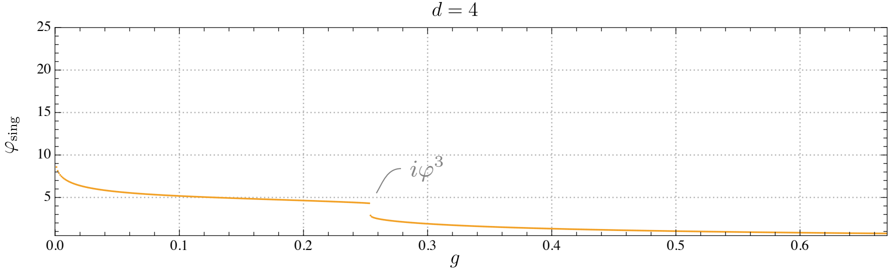

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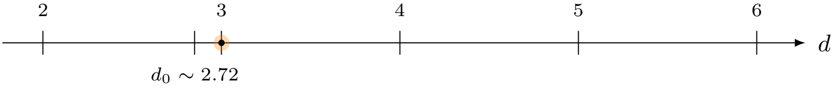

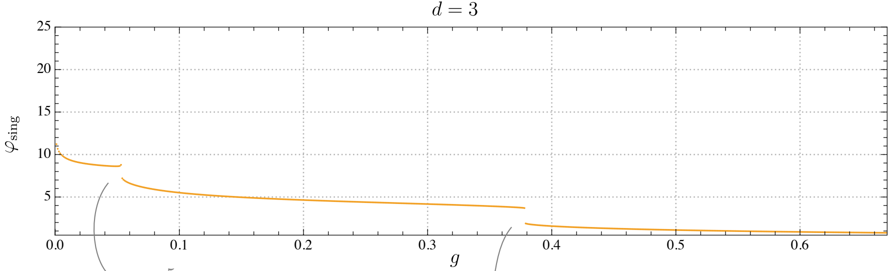

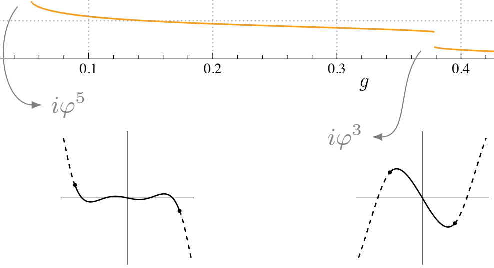

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$i\varphi^3$	6
$i\varphi^5$	$\frac{10}{3} \sim 3.33$
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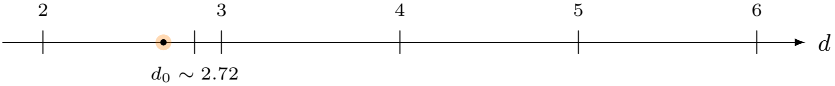
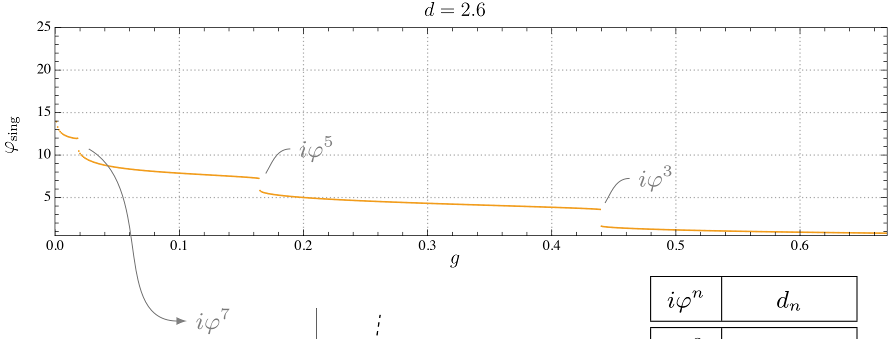
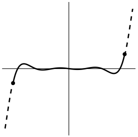
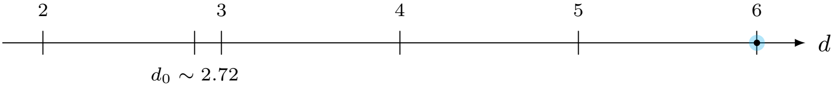


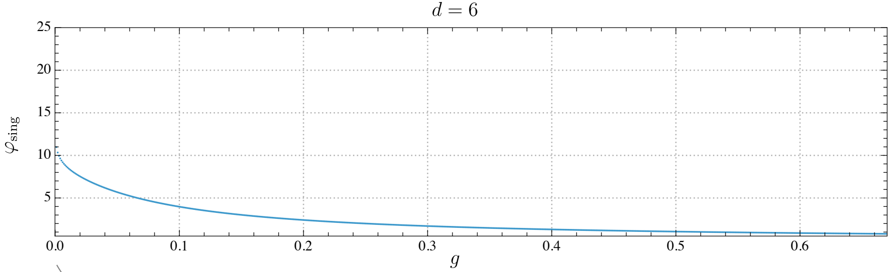




$i\varphi^n$	d_n
$i\varphi^3$	6
$i\varphi^5$	$\frac{10}{3} \sim 3.33$
$i\varphi^7$	$\frac{14}{5} = 2.8$

Spike plots LPA'





Gaussian

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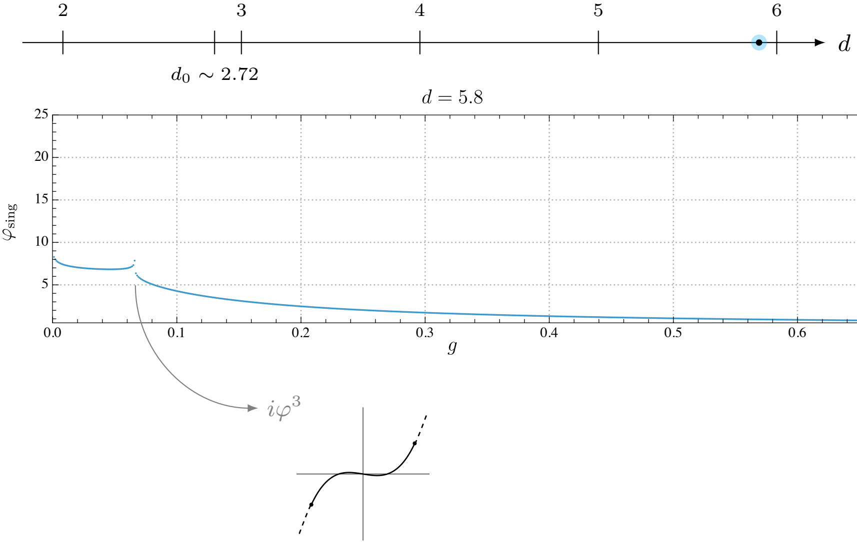

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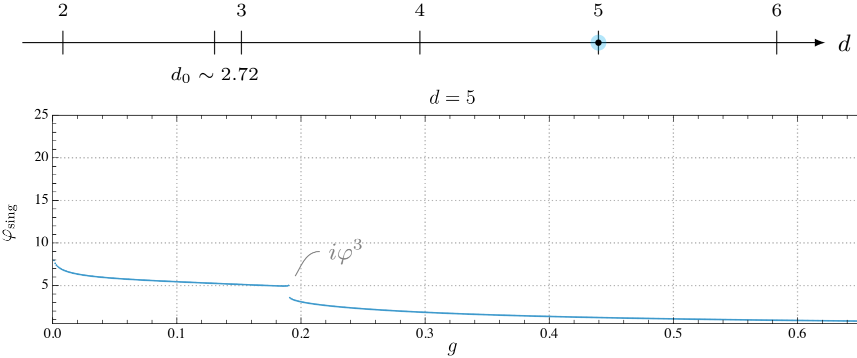

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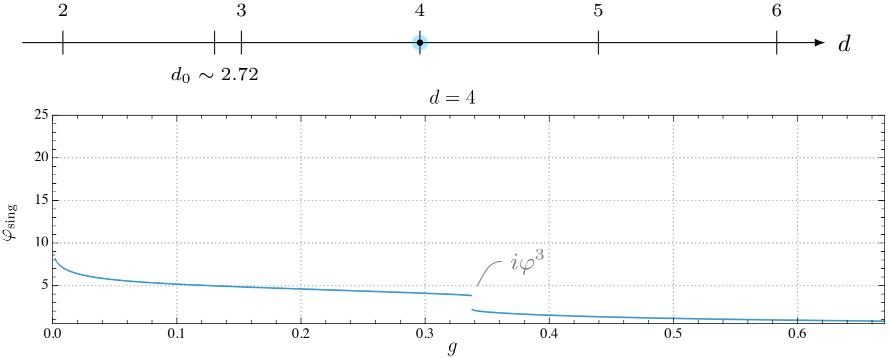

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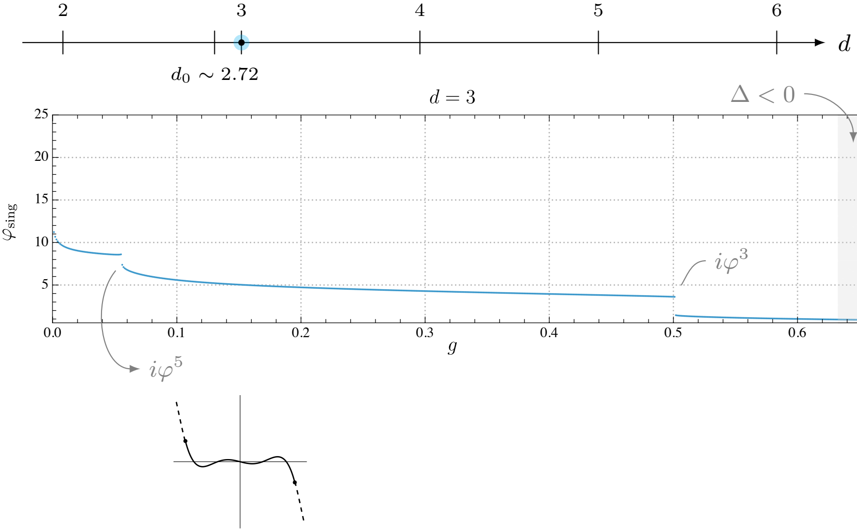

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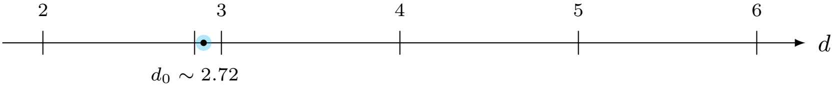

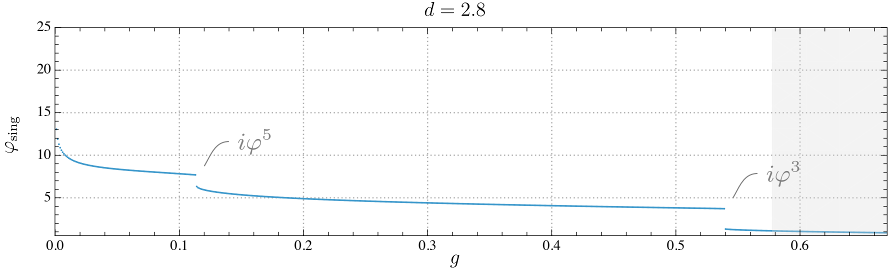

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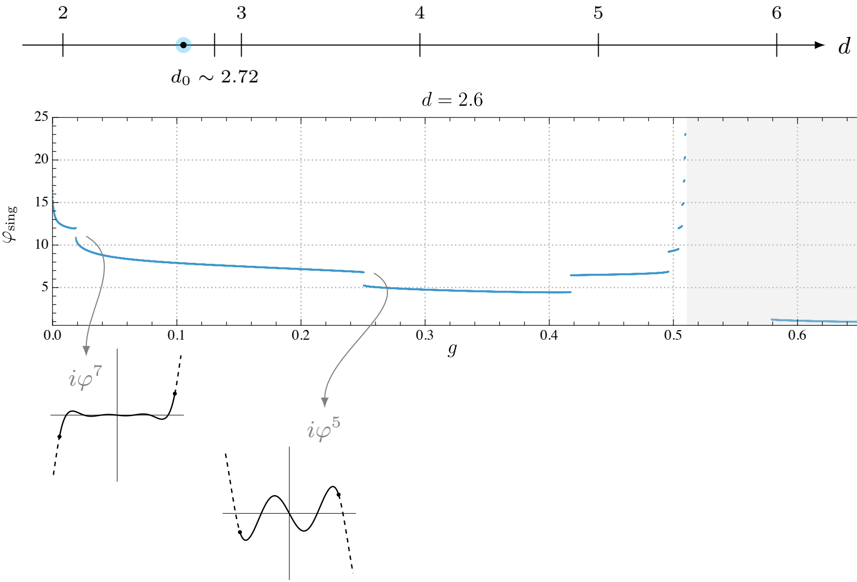

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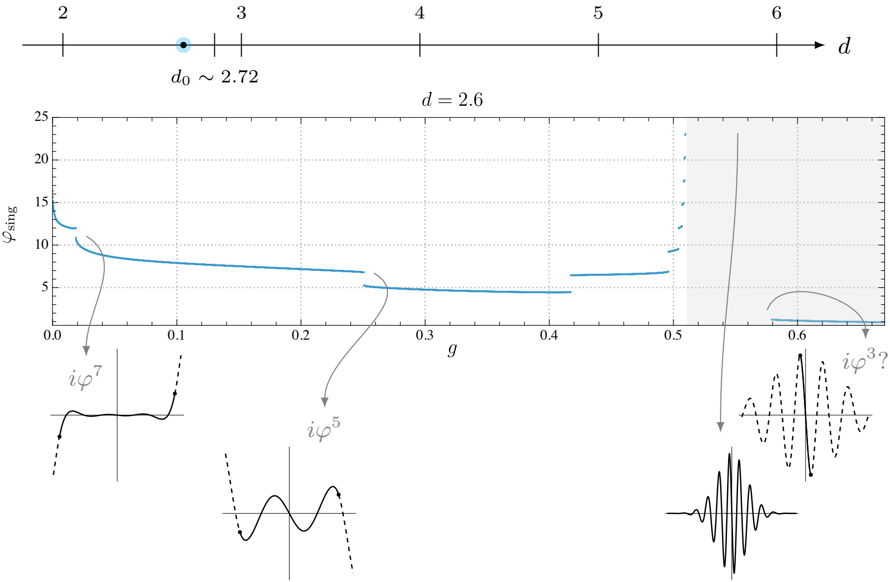

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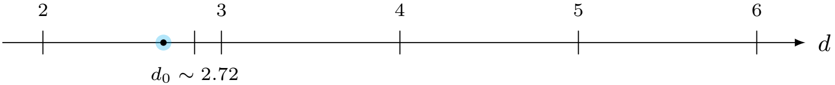

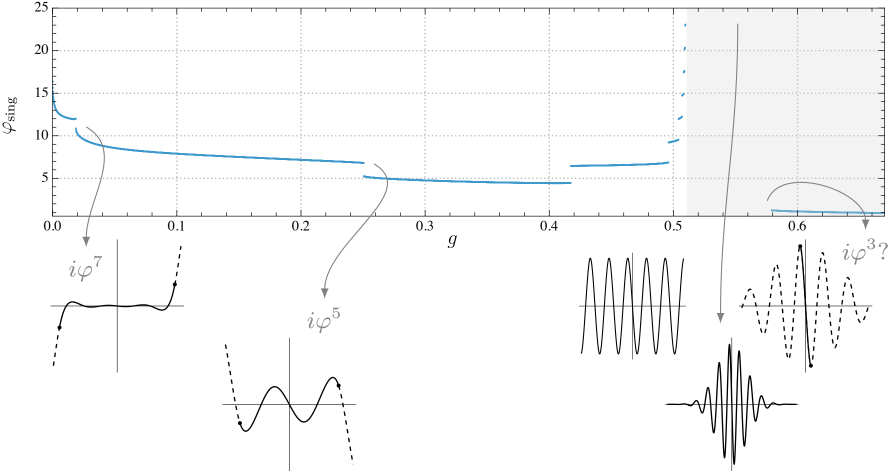

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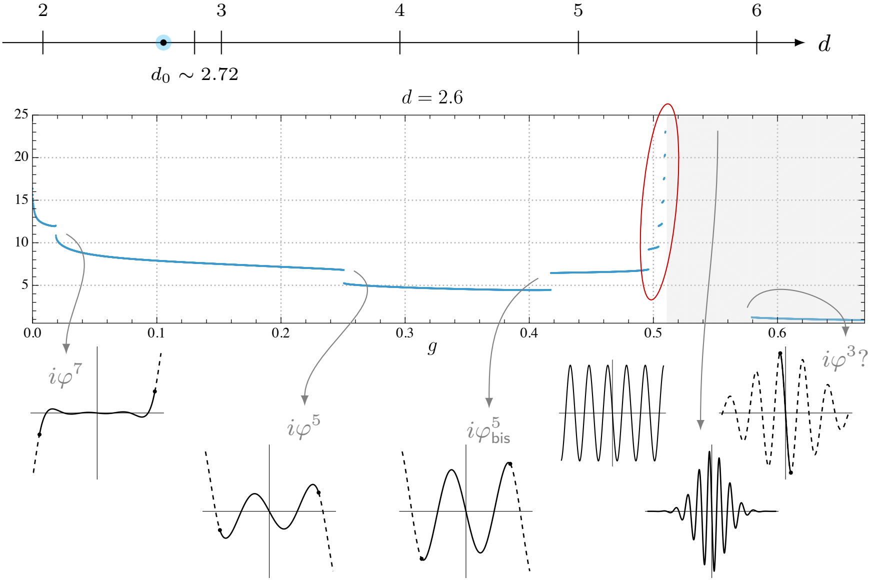
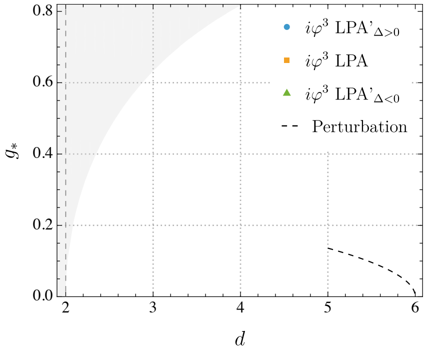
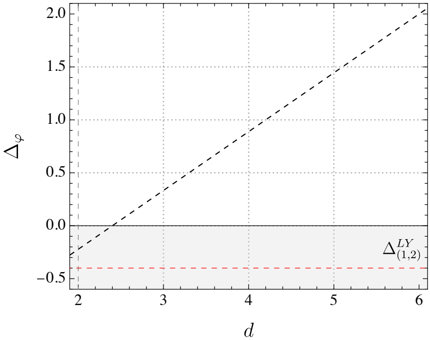
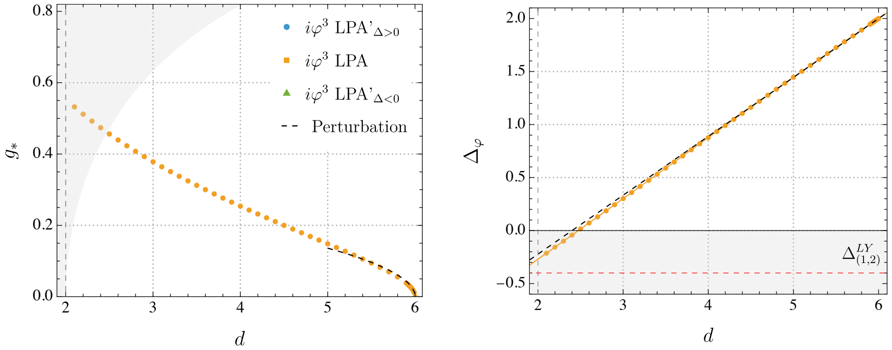






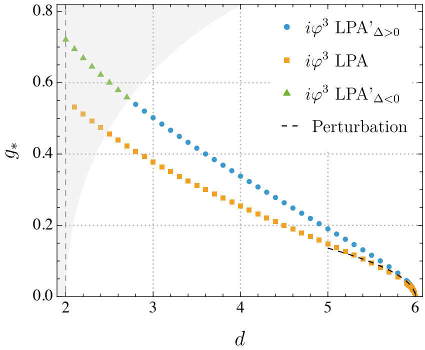
Approximation	$\Delta_\varphi^{d=5}$	$\Delta_\varphi^{d=4}$	$\Delta_\varphi^{d=3}$	$\Delta_\varphi^{d=2}$
5-loops [Borinsky et al., 21']	$1.4245(10)$	$0.827(6)$	$0.215(10)$	$-0.390(15)$
CFT $\mathcal{M}_{(2,5)}$				-0.4

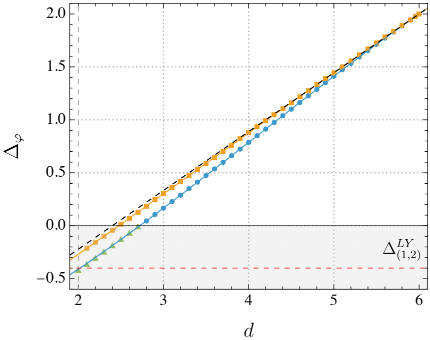
$i\varphi^3$ theory



Approximation	$\Delta_\varphi^{d=5}$	$\Delta_\varphi^{d=4}$	$\Delta_\varphi^{d=3}$	$\Delta_\varphi^{d=2}$
LPA at $\varphi = 0$	1.44587	0.87621	0.30122	-0.27004
5-loops [Borinsky et al., 21']	$1.4245(10)$	$0.827(6)$	$0.215(10)$	$-0.390(15)$
CFT $\mathcal{M}_{(2,5)}$				-0.4

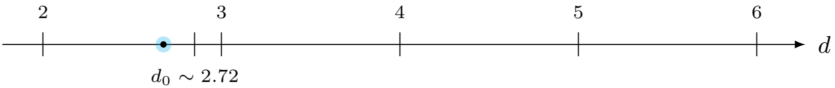
$i\varphi^3$ theory

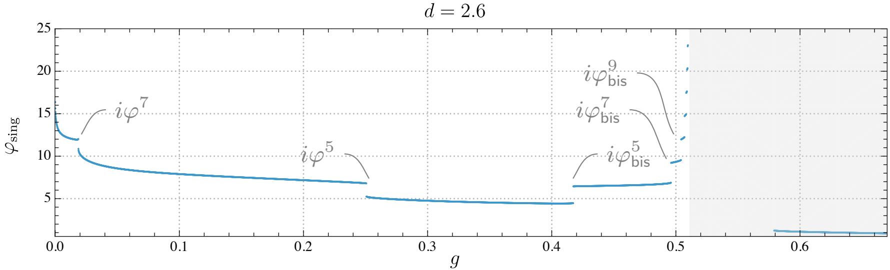




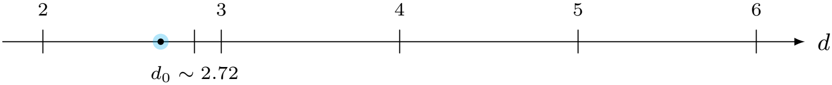
Approximation	$\Delta_\varphi^{d=5}$	$\Delta_\varphi^{d=4}$	$\Delta_\varphi^{d=3}$	$\Delta_\varphi^{d=2}$
LPA at $\varphi = 0$	1.44587	0.87621	0.30122	-0.27004
LPA'$_{\Delta>0}$	1.41178	0.78675	0.16659	-0.41035
LPA'$_{\Delta<0}$				-0.42809
5-loops [Borinsky et al., 21']	$1.4245(10)$	$0.827(6)$	$0.215(10)$	$-0.390(15)$
CFT $\mathcal{M}_{(2,5)}$				-0.4

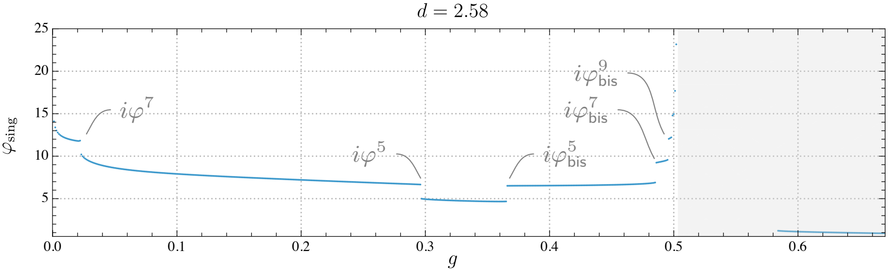
Back to spike plots (LPA')



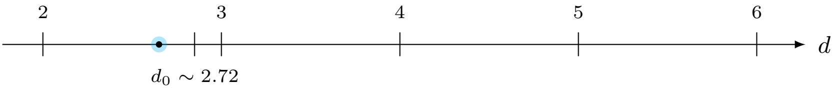


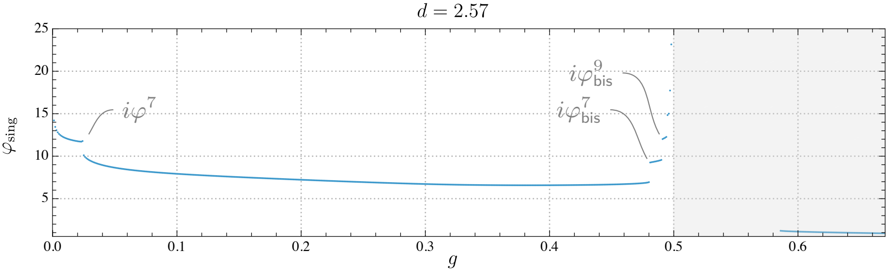
Back to spike plots (LPA')



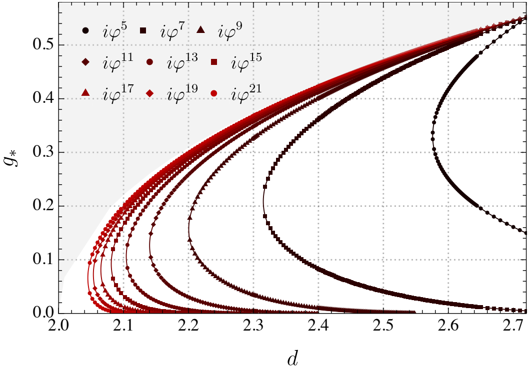


Back to spike plots (LPA')





Fixed points $d < d_0$



$i\varphi^n$	d_c
$i\varphi^3$	/
$i\varphi^5$	2.5758
$i\varphi^7$	2.315
$i\varphi^9$	2.200
$i\varphi^{11}$	2.1399
$i\varphi^{13}$	2.104
$i\varphi^{15}$	2.080

Conclusion

$$h - \frac{\Delta}{d}\,\varphi\,h' = -\frac{h''}{1+(h'')^2}, \quad h(0)=0, \quad h'''(0) = g \in \mathbb{R}$$

- **Perturbation theory**: Identified weakly coupled $i\varphi^{2n+1}$ theories near $d = d_{2n+1} - \varepsilon$
- Classified **global solutions** of the flow equation
  - $\Delta > 0$: isolated global solutions
  - $\Delta < 0$: continuum of solutions
- **Numerical results**:
  - $\Delta_\varphi|_{d=2}$ with an error of $2.6\% - 7\%$ compared to $\mathcal{M}(2,5)$ for $i\varphi^3$
  - $d = d_0$, the value at which $\Delta = 0$ for $i\varphi^3$
  - infinite sequence of non-perturbative fixed points annihilating with the $i\varphi^{2n+1}$, $n > 1$ fixed points

Outlook

- Improve the benchmark for $i\phi^3$ theory
- Understand the non-perturbative fixed points. . .