$2D$ dilaton theories as reduced $d \geq 4$ gravities

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Gravitational theories $S[g] = \int d^d x \sqrt{-g}\, \mathscr{L}(\mathrm{Riem}, \nabla)$ **with second-order equations on** $g_{\mu\nu}(x)dx^\mu dx^\nu = q_{ab}(y)dy^a dy^b + \varphi(y)^2 \gamma_{ij}(\theta)d\theta^i d\theta^j$

$S[g]\big|_{g(q,\varphi)} \overset{!}{=} S_H[q,\varphi] = \int d^2y\sqrt{-q}\left[h_2 - h_3 \Box\varphi + h_4 \mathscr{R}^{2D} - 2\partial_\chi h_4 \left[(\Box\varphi)^2 - \nabla_a \nabla_b \varphi \nabla^a \nabla^b \varphi\right]\right],\ \chi = \nabla_a \varphi \nabla^a \varphi$ $2D$ **Horndeski theory**

$\alpha = h_2 + \chi\partial_\varphi(h_3 - 2\partial_\varphi h_4),\ \beta = \chi\partial_\chi(h_3 - 2\partial_\varphi h_4) - \partial_\varphi h_4,\ \omega = \partial_\chi\alpha - \partial_\varphi\beta$ — *iff* $\omega = 0 \to \alpha = \partial_\varphi\Omega,\ \beta = \partial_\chi\Omega$ [Boyanov & Carballo-Rubio 2025, Carballo-Rubio 2025]

$\mathscr{L}(\mathrm{Riem}, \nabla)$

$\omega = 0 \Leftrightarrow: d \geq 4$ general quasi-topological gravity (QTG)
[JB & Carballo-Rubio 2026, JB 2026 1]

$\omega \neq 0$
[Mazza 2026]

$\supset d \geq 5$ polynomial $\mathscr{L}(\mathrm{Riem})$ QTGs incl. Lovelock (only GR in $d = 4$) — $\Omega(\varphi,\chi) = \varphi^{d-1}\sum_n \alpha_n \left(\frac{k-\chi}{\varphi^2}\right)^n$
[Lovelock 1970, Oliva et al 2010, Myers et al 2010, Dehghani et al 2011, Cisterna et al 2017, Bueno et al 2019–26]

$$\tilde{\mathcal{Z}}_{n+5} = -\frac{3(n+3)}{2(n+1)D(D-1)}\tilde{\mathcal{Z}}_1\tilde{\mathcal{Z}}_{n+4} + \frac{3(n+4)}{2nD(D-1)}\tilde{\mathcal{Z}}_2\tilde{\mathcal{Z}}_{n+3} - \frac{(n+3)(n+4)}{2n(n+1)D(D-1)}\tilde{\mathcal{Z}}_3\tilde{\mathcal{Z}}_{n+2}$$

$\supset d = 4$ non-polynomial $\mathscr{L}(\mathrm{Riem}, \nabla)$ QTGs — $\Omega(\varphi,\chi)$ generic
[Bueno et al 2025, JB & Carballo-Rubio 2026, JB 2026 1, Colleaux 2026]

$$\frac{1}{12}\mathcal{I}_R + \frac{\mathcal{I}_{C^3}}{\mathcal{I}_{C^2}} - \frac{\mathcal{I}_{\hat{R}^3}\mathcal{I}_{C^3}}{\mathcal{I}_{C^2}\mathcal{I}_{R^2C} + 2\mathcal{I}_{\hat{R}^2}\mathcal{I}_{C^3}} \qquad \frac{1}{24\mathcal{I}_{C^2}^2}\left[4\,\mathcal{I}_{C^2}\,\mathcal{I}_{\nabla C\cdot\nabla C} - \mathcal{I}_{\nabla C^2\cdot\nabla C^2}\right]$$

Reversely: [JB & Carballo-Rubio 2026, JB 2026 1, Colleaux 2019]

$\forall\, q,\varphi\ \exists\, d \geq 4\ S[g]$: $S[g]\big|_{g(q,\varphi)} = S_H[q,\varphi]$ → *Generic* $2D$ Horndeski theories are reduced $d \geq 4$ gravities.

$$R_{abcd} = \mathscr{R} q_{a[c}q_{d]b},\ R_{aibj} = -\varphi\nabla_a\nabla_b\varphi\gamma_{ij},\ R_{ijkl} = 2\varphi^2(k-\chi)\gamma_{i[k}\gamma_{l]j}$$

$$\{i_{\mathrm{Riem},\nabla}\}\big|_{g(q,\varphi)} \supset \left\{\mathscr{R}, \frac{\Box\varphi}{\varphi}, \frac{[\nabla_a\nabla_b\varphi]^2}{\varphi^2}, \psi = \frac{1-\chi}{\varphi^2}, \phi = \frac{\chi}{\varphi^2}\right\} \mapsto \{\mathscr{I}_{\mathrm{Riem},\nabla}\} \ni S[g]$$

Birkhoff theorem for general QTGs + reconstruction of QTG vacuum solutions

Birkhoff theorem for general QTGs: [Carballo-Rubio 2025, JB & Carballo-Rubio 2026, JB 2026 1]

$$\mathcal{E}_{\mu\nu}\Big|_{g(q,\varphi)} = \frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}}\Big|_{g(q,\varphi)} = \varphi^{2-d}\left[\frac{1}{\sqrt{-q}}\frac{\delta S_H}{\delta q^{ab}}\right]\delta^a_\mu\delta^b_\nu - \frac{1}{2(d-2)}\varphi^{5-d}\left[\frac{1}{\sqrt{-q}}\frac{\delta S_H}{\delta\varphi}\right]\gamma_{ij}\delta^i_\mu\delta^j_\nu = 0 \text{ — offshell Bianchi identity: } \nabla^\mu\mathcal{E}_{\mu\nu} = 0$$

$$g_{\mu\nu}(x)dx^\mu dx^\nu = -n(t,r)^2 f(t,r)dt^2 + \frac{dr^2}{f(t,r)} + r^2 d\Omega^2 \;\rightarrow\; \varphi(y) = r,\;\; \chi(y) = f(t,r) \;\rightarrow\; \text{e.o.m.: } \partial_t f = 0,\;\; \partial_r n = 0,\;\; \frac{d}{dr}\Omega(r, f(r)) = 0$$

General QTGs have $g_{tt}g_{rr} = -1$ static vacuum solutions with $\Omega(r, f(r)) = 2M$ algebraic equation.

Examples of $d = 4$ regular black holes with $\Omega(r,f) \sim r^3 h(\psi)$, $\psi = (1-f)/r^2$

$H(\psi)$	$f(r)$
$\frac{6\psi}{1-l^2\psi}$	$1-\frac{2Mr^2}{r^3+2Ml^2}$
$\frac{6\psi}{\sqrt{1-l^4\psi^2}}$	$1-\frac{2Mr^2}{\sqrt{r^6+4l^4M^2}}$
$-\frac{6}{l^2}\ln\left[1-l^2\psi\right]$	$1-\frac{r^2}{l^2}\left(1-e^{-\frac{2Ml^2}{r^3}}\right)$
$\frac{6}{l^2}\operatorname{arctanh}\left(l^2\psi\right)$	$1-\frac{r^2}{l^2}\tanh\left(\frac{2Ml^2}{r^3}\right)$

Reconstruction of QTG vacuum solutions: [Boyanov & Carballo-Rubio 2025, Carballo-Rubio 2025, JB & Carballo-Rubio 2026, JB 2026 1]

1. $f(r,M) \rightarrow M(r,f) \rightarrow \Omega(\varphi,\chi) := 2M(\varphi,\chi) \rightarrow h_i(\Omega(\varphi,\chi)) \rightarrow S_H[q,\varphi]$ — (q,φ): $2D$ solution to $S_H[q,\varphi]$
2. $S_H[q,\varphi] \rightarrow S_{QTG}[g]$ — g: d-dimensional vacuum solution to $S_{QTG}[g]$

Generic $g_{tt}g_{rr} = -1$ static spacetimes with mass M are d-dimensional vacuum solutions to a general QTG.

Application 1 — **Unified $g_{tt}g_{rr} = -1$ black hole thermodynamics from general QTGs**

$$f(r) = k + \frac{r^2}{\ell^2} - \frac{2M}{r^{d-3}} + \cdots \to M_{ADM} = \frac{(d-2)\Sigma_{d-2}}{16\pi G_0}2M$$ — identification: $\Omega(r,f) := M_{ADM}$ — s.t. $\partial_r\Omega + \partial_f\Omega f' = 0$ onshell

Realisation of functions h_i in $S_H[q,\varphi]$: [Boyanov & Carballo-Rubio 2025, JB & Carballo-Rubio 2026]

$$h_2(\varphi,\chi) = \frac{16\pi G_0}{(d-2)\Sigma_{d-2}}\partial_\varphi\Omega(\varphi,\chi), \quad h_3(\varphi,\chi) = -\frac{32\pi G_0}{(d-2)\Sigma_{d-2}}\partial_\chi\Omega(\varphi,\chi), \quad h_4(\varphi,\chi) = -\frac{16\pi G_0}{(d-2)\Sigma_{d-2}}\int d\varphi\partial_\chi\Omega(\varphi,\chi)$$

Thermodynamic first law: [JB 2026 2]

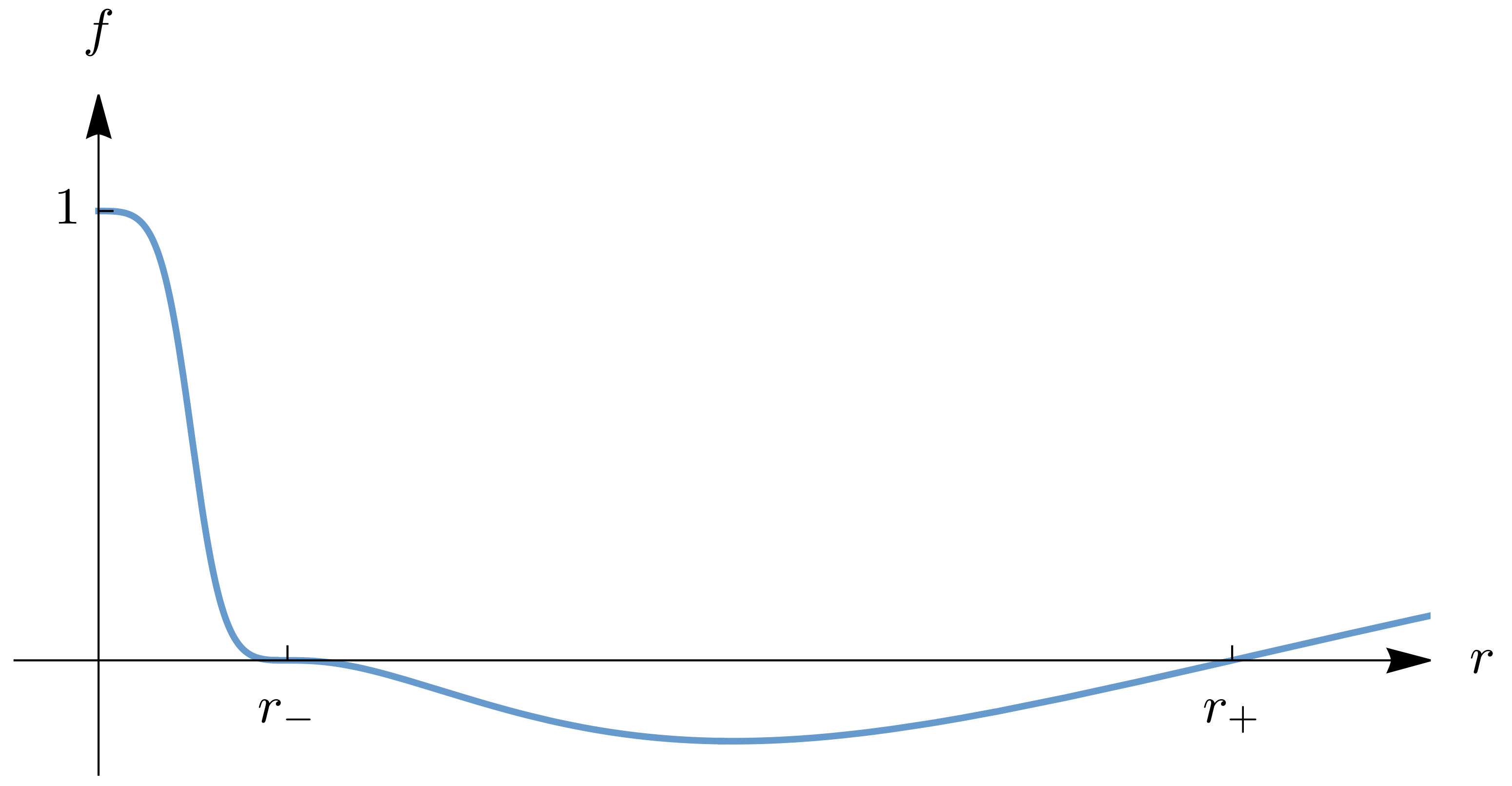
$$M = \Omega(r_+), \quad T = \frac{f'(r_+)}{4\pi} = -\frac{1}{4\pi}\frac{\partial_{r_+}\Omega(r_+)}{\partial_f\Omega(r_+)}, \quad S = -2\pi\oint_{\mathscr{H}} d^{d-2}x\sqrt{h}\left[\frac{1}{16\pi G_0}\frac{\delta\mathscr{L}}{\delta R_{\mu\nu\rho\sigma}}\right]\epsilon_{\mu\nu}\epsilon_{\rho\sigma} = \frac{(d-2)\Sigma_{d-2}}{4G_0}h_4(r_+) = -4\pi\int dr_+\partial_f\Omega(r_+) \to$$

$\delta M = T\delta S$

First law can be derived for generic $g_{tt}g_{rr} = -1$ asymptotically flat or AdS (singular or regular) black holes when realised as vacuum solutions to a general QTG.

Application 2 — **Asymptotic formation of inner-extremal regular black holes**

Static inner-extremal regular black holes: [Carballo-Rubio et al 2022]



$$\kappa_- = \frac{1}{2} f'(M,r)\Big|_{r_-} = 0 \;\rightarrow\; \text{no mass inflation}$$

Vaidya solutions of 2D Horndeski theories with $\omega = 0$: [Boyanov & Carballo-Rubio 2025]

$$T_{\mu\nu} = \frac{\dot{M}(v)}{4\pi r^2}\partial_\mu v \partial_\nu v \;\Leftrightarrow\; t_{ab} = \frac{\dot{M}(v)}{8\pi}\partial_a v \partial_b v \;\rightarrow\; \text{e.o.m.: } \beta n \partial_v f = 2\dot{M},\; \partial_r n = 0,\; \frac{d}{dr}\Omega(r, f(v,r)) = 0 \;\rightarrow\; \Omega(r, f(v,r)) = 2M(v)$$

Reversely: Generic $f(M(v), r)$ are Vaidya solutions to general QTGs.

Asymptotic gravitational collapse into inner-extremal regular black hole from horizonless compact object: [JB 2026 3]

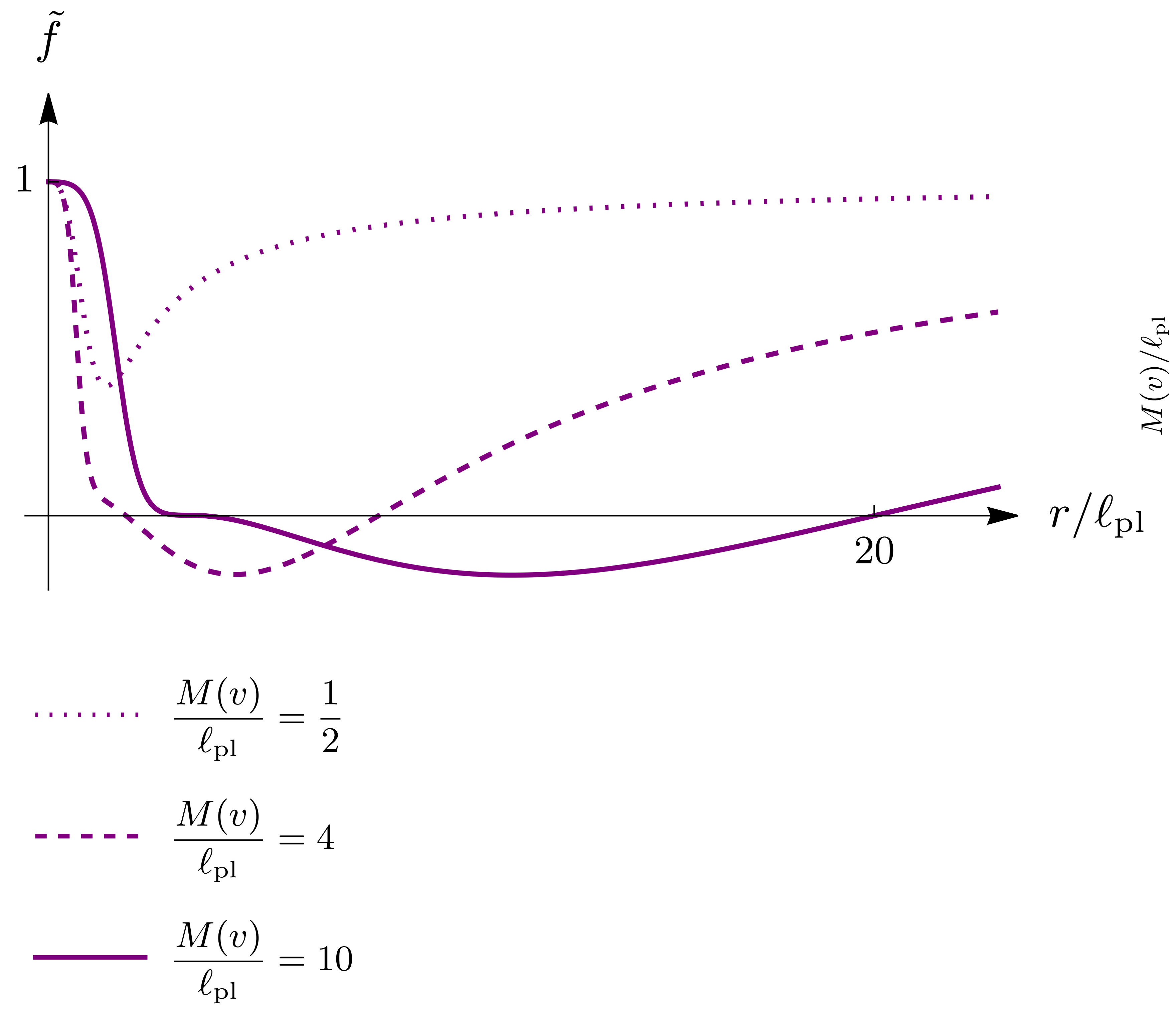
deformed metric: $\tilde{\kappa}_- = \frac{1}{2}\partial_r \tilde{f}(M(v), r)\Big|_{\tilde{r}_-} \rightarrow 0,\; M(v) \rightarrow \infty$ approximate extremality

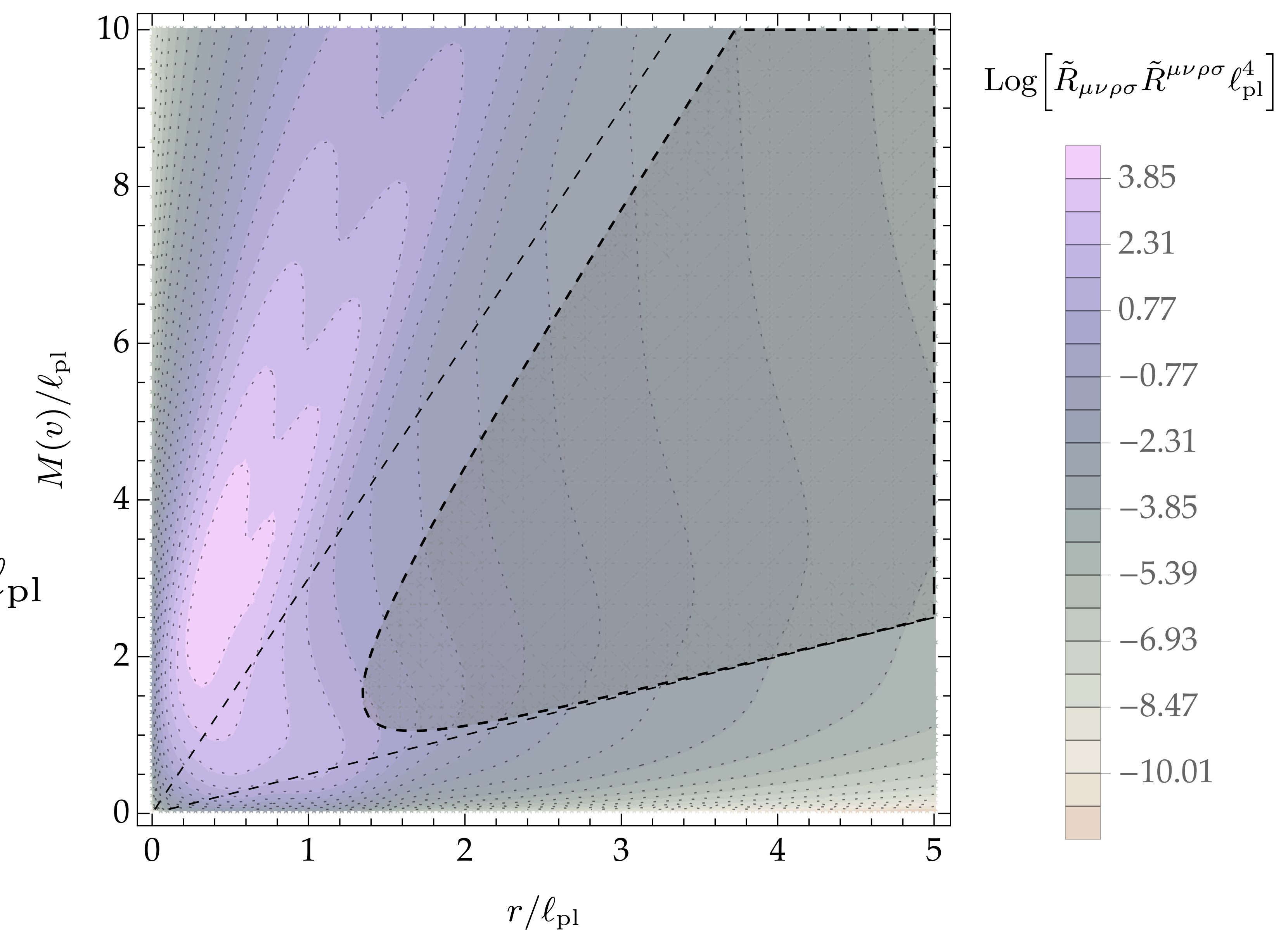
Application 2 — **Asymptotic formation of inner-extremal regular black holes**

[JB 2026 3]

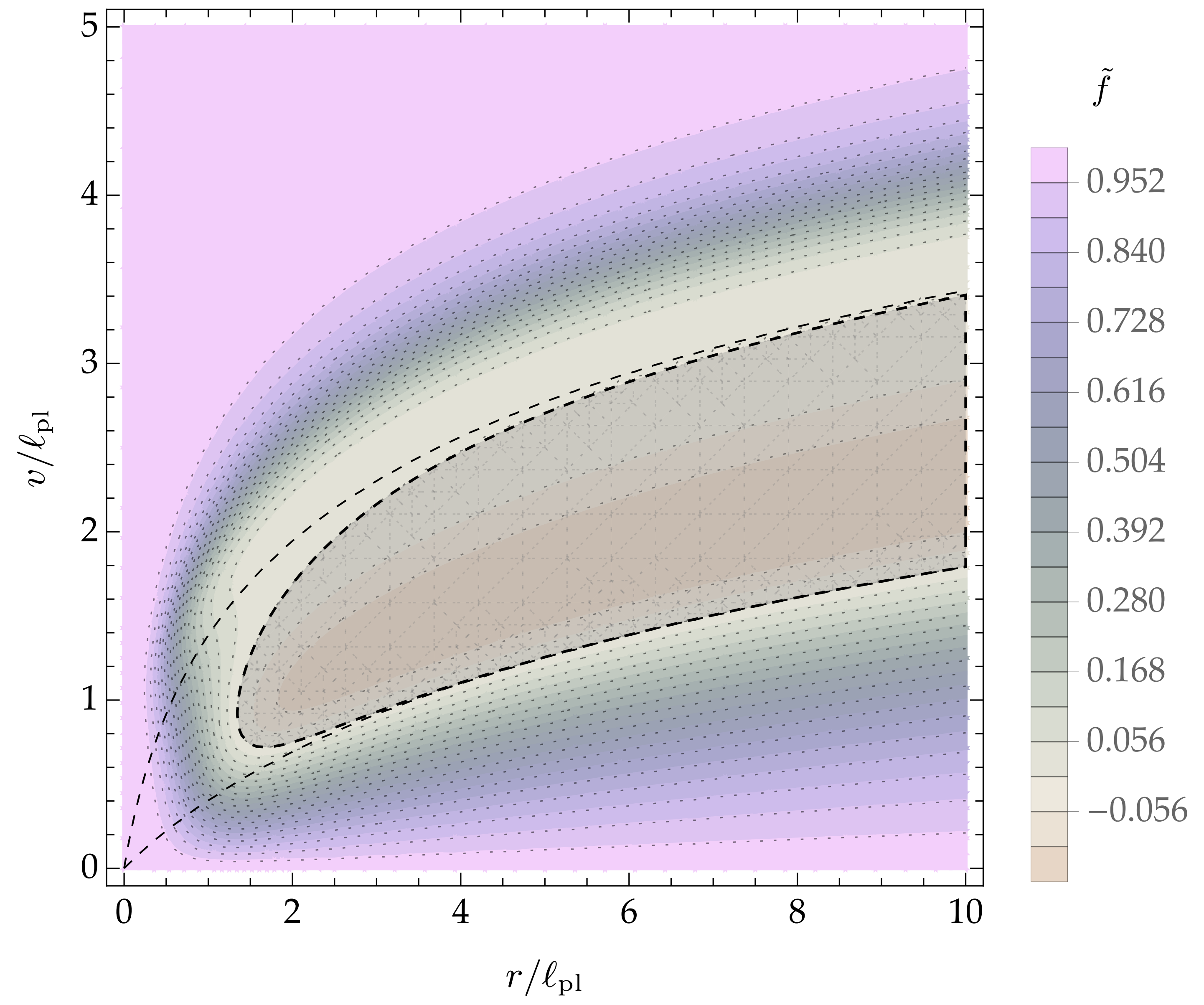
$$\tilde{f}(M(v),r) = \frac{(r-\frac{1}{3}M(v))^3(r-2M(v)) + M(v)^{4-q}L^q}{r^4 - M(v)r^3 + \frac{127}{54}M(v)^2r^2 - \frac{19}{27}M(v)^3 r + \frac{2}{27}M(v)^4 + M(v)^{4-q}L^q}, \quad \tilde{r}_- = r_- + \mathcal{O}(M^{1-q/3}), \quad \tilde{\kappa}_- \sim -\frac{CL^{2q/3}}{M^{2q/3+1}} + \mathcal{O}(M^{-q-1})$$

Example: $q = 4$





limiting-curvature spacetimes



Application 3 — **Non-singular cosmologies**

[JB & Magueijo 2026]

FLRW ansatz:

$$q_{ab}(y)dy^a dy^b = -d\tau^2 + a(\tau)^2 \frac{dr^2}{1-kr^2}, \quad \varphi(y) = a(\tau)r, \quad \chi = \nabla_a\varphi\,\nabla^a\varphi = 1 - \left(k^2 + \dot{a}^2\right)r^2 \quad \text{—-} \quad \text{here: } k = 0, \ \psi = \frac{1-\chi}{\varphi^2} = \left(\frac{\dot{a}}{a}\right)^2 = H^2$$

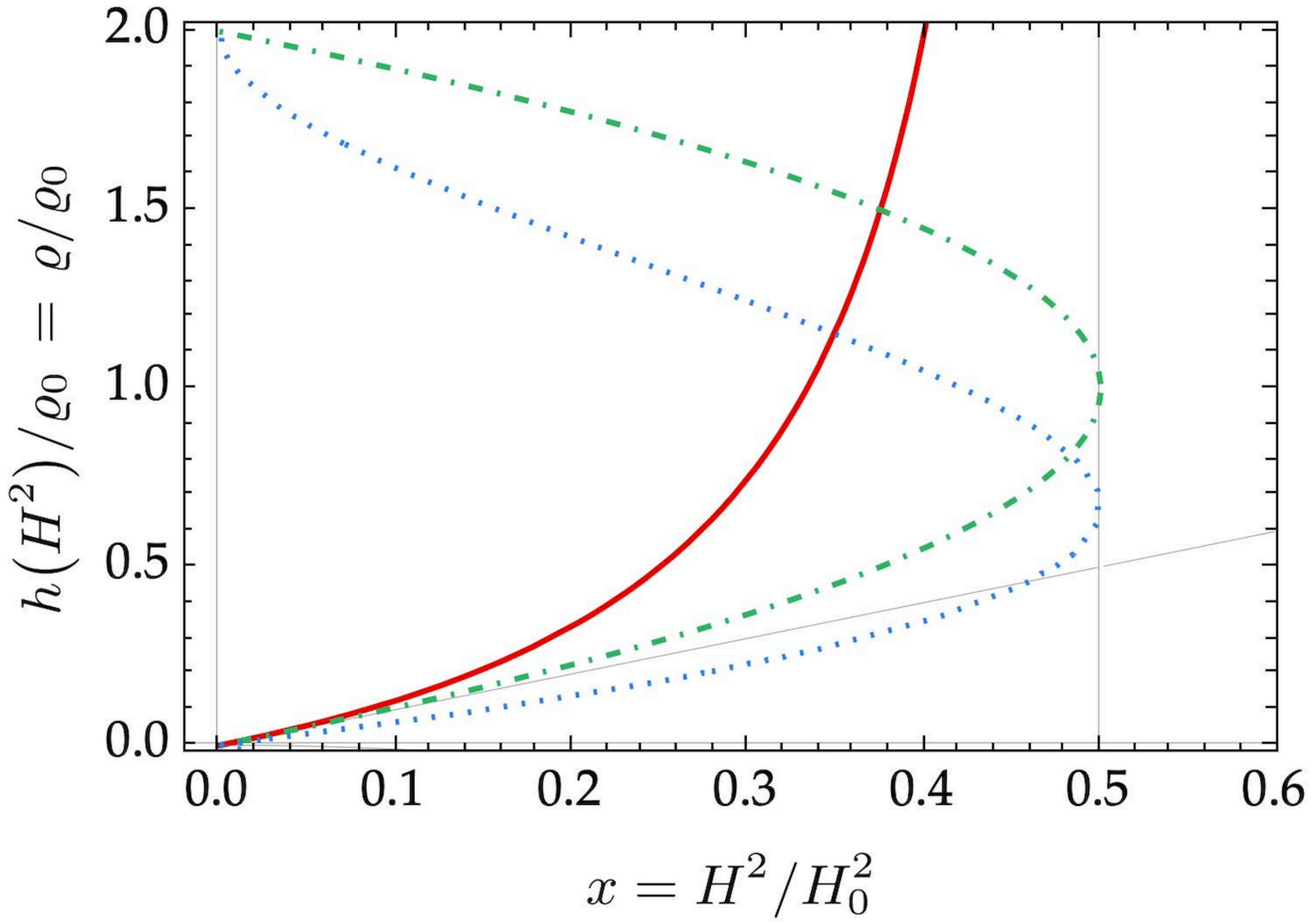
equations of motion: $\Omega(\varphi,\chi)\varphi^{-(d-1)} = h(\psi) = \dfrac{16\pi G}{(d-2)(d-1)}\rho = \varrho, \ \dot{\rho} = -(d-1)(\rho+p)H, \ p = w\rho$

three qualitatively different non-singular scenarios in UV:

$\dfrac{x}{1-2x}$ (asymptotic de Sitter)

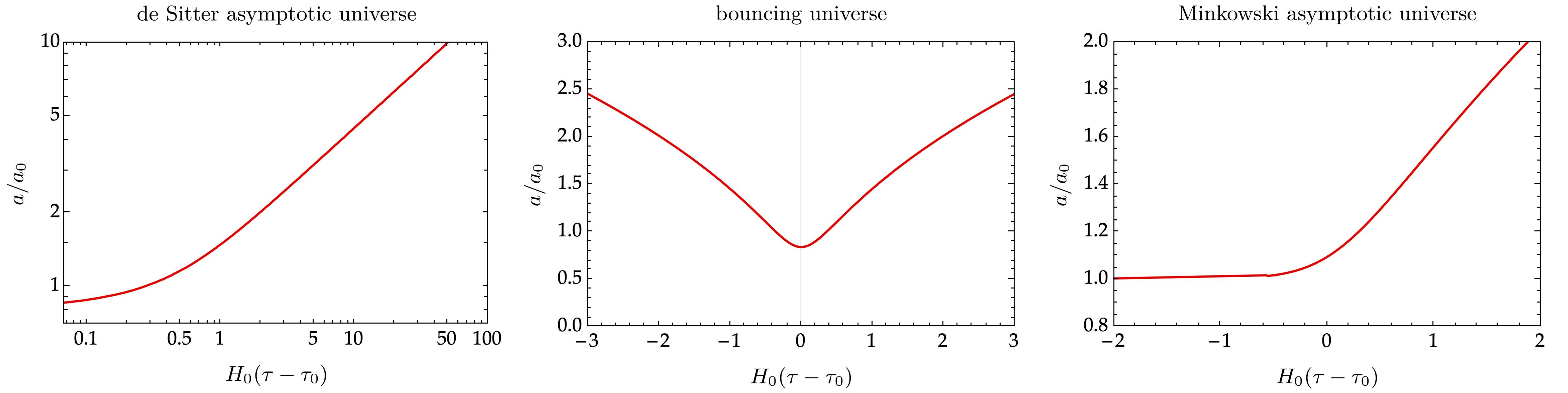
$1 \pm \sqrt{1-2x}$ (bounce)

$x = \dfrac{27}{64}\dfrac{h}{\varrho_0}\left(\dfrac{h}{\varrho_0} - 2\right)^2$ (asymptotic Minkowski)



Application 3 — **Non-singular cosmologies**

[JB & Magueijo 2026]



Summary

1. Generic $2D$ Horndeski theories are reduced $d \geq 4$ gravities $\rightarrow$ Solving $2D$ Horndeski theory = solving d-dimensional vacuum gravity.

2. QTGs (a.k.a. gravitational theories whose reduction is a $2D$ Horndeski theory satisfying $\omega = 0$) have $g_{tt}g_{rr} = -1$ static vacuum solutions with $g_{tt} = -f(r)$ satisfying $\Omega(r, f(r)) = 2M$ algebraic equation.

3. Any $g_{tt}g_{rr} = -1$ static spacetime with mass M is (and can be reconstructed explicitly as) a d-dimensional gravitational vacuum solution.

4. The first law of thermodynamics $\delta M = T\delta S$ can be derived for any $g_{tt}g_{rr} = -1$ asymptotically flat or adS singular or regular black hole when realised as QTG vacuum solution.

5. Generic dynamical $g_{tt}g_{rr} = -1$ spacetimes of the form $g_{tt} = f(M(v), r)$ are exact Vaidya solutions of a general QTG.