Negative Diffusion in the FRG Flow: Hyperdiffusion Solution for the QDM

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I. Abstract

- The low energy sector of strong-interaction matter is described by effective models, which emerge from gluon dynamics. First-principles QCD calculations capture this e.g. via dynamical hadronization. Therefore first-principle calculations require a good understanding of the low energy models.
- A previous study[1] solved the QDM and encountered **strong oscillations** during the flow of the derivative of the effective potential.
- We show that these oscillations are a model feature, which can be traced back to a **negative diffusion coefficient** during the flow.
- Advection-Diffusion equations with **negative diffusion are ill-posed and require specialized methods**, as e.g. adding a regularization term, which converts the flow equations into a well-posed equation.

II. Quark-Diquark Model in the FRG

- Lagrangian of the Quark-Diquark Model (QDM) with a 2SC color superconducting phase[2]

$$\mathcal{L} = \bar\psi\left(\not\partial - \gamma^0\mu_q\right)\psi + \frac{h}{2}\left(\Delta_a^*\psi^T\epsilon^a\tau^2C\gamma^5\psi + \Delta_a\bar\psi\epsilon^a\tau^2C\gamma^5\bar\psi^T\right)$$
$$+\frac{1}{2}(\partial_\mu - 2\,\mu\,\delta_{\mu0})\,\Delta^*\,(\partial^\mu + 2\,\mu\,\delta^{\mu,0})\Delta + U(\Delta^*\Delta)\,.$$

- LPA Flow Equation for the effective potential with the Litim Regulator:

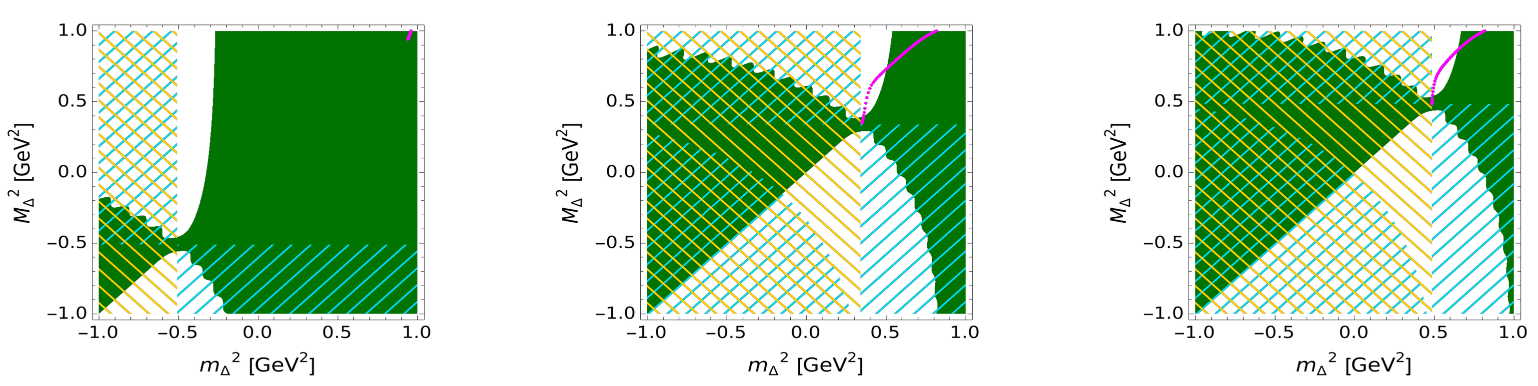
$$\partial_t U = Q + F + S\,,\quad Q = -\frac{k^5}{12\pi^2}\sum_\pm\left(1\pm\frac{8\mu^2}{\chi_D}\right)\frac{\coth(\xi_\pm/2T)}{\xi_\pm}\,,$$
$$\xi_\pm = \xi_\pm(m_\Delta^2,\,M_\Delta^2,\,\mu)\,,\quad M_\Delta^2 = \partial_\Delta^2 U,\quad m_\Delta^2 = \frac{\partial_\Delta U}{\Delta}\,.$$

- $\xi_\pm$ depends on m_Δ^2, M_Δ^2 and μ in a non-trivial way.
- It was found[3] that the flow equation for the effective potential can be written in the form of a conservative advection diffusion equation by taking a Δ derivative and writing it in terms of $u = \partial_\Delta U$:

$$\partial_t u(\Delta) = \frac{\mathrm{d}}{\mathrm{d}\Delta}Q(\Delta,\,u,\,\partial_\Delta u) + \frac{\mathrm{d}}{\mathrm{d}\Delta}F(\Delta,\,u) + S(\Delta)\,.$$

- Q acts as a diffusion term – applying the derivative w.r.t. Δ on Q results in the diffusion coefficient D.

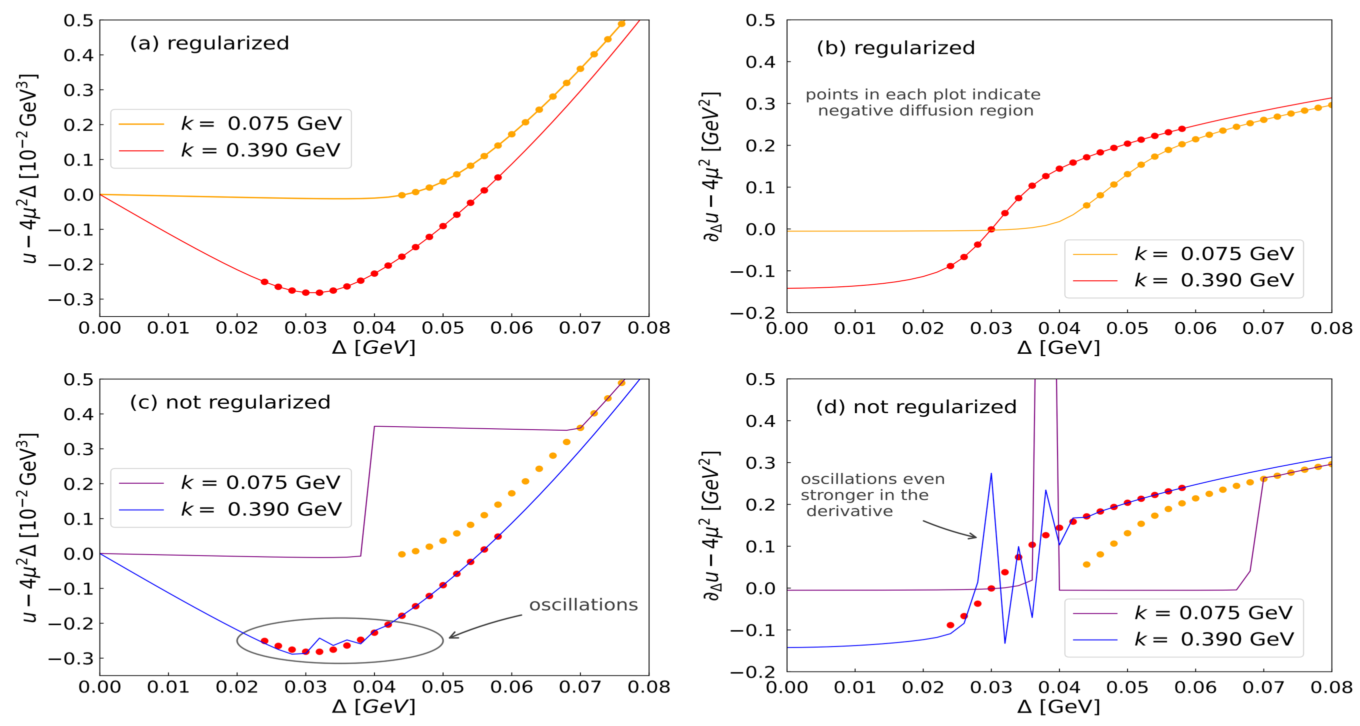
$$\frac{\mathrm{d}Q}{\mathrm{d}\Delta} = D\,\partial_\Delta^2 u + E\,\partial_\Delta u + G\,,\qquad D = \frac{\partial Q}{\partial(\partial_\Delta u)}\,.$$



k = 1 GeV k = 0.39 GeV k = 0.075 GeV

- In the green region, the diffusion coefficient is positive and the hashed regions indicate the regions excluded by convexity restoration. **A white, non-hashed region exists, in which the diffusion is negative. This region is not excluded by convexity restoration.**

IV. Negative Diffusion



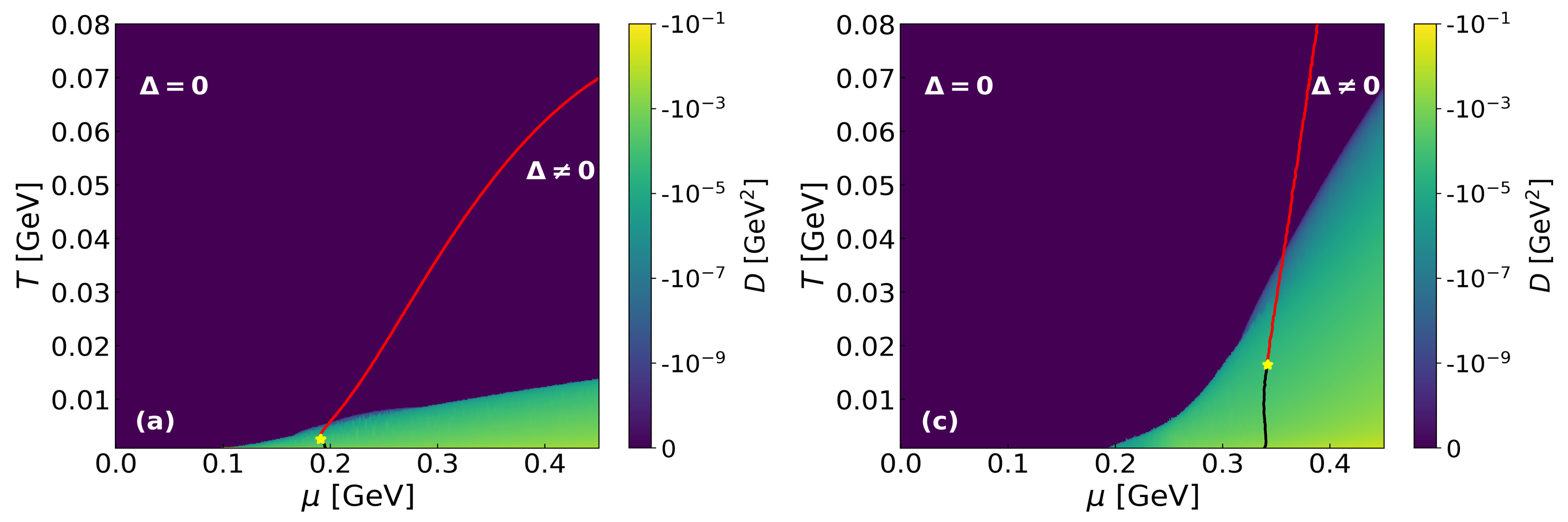
- Negative diffusion can be understood by considering the heat equation with a constant diffusion coefficient for the Fourier transformed $u(\Delta) = \sum_q c_q e^{iq\Delta}$, for a single mode c_q

$$\partial_t c_q = -D\,q^2 c_q$$

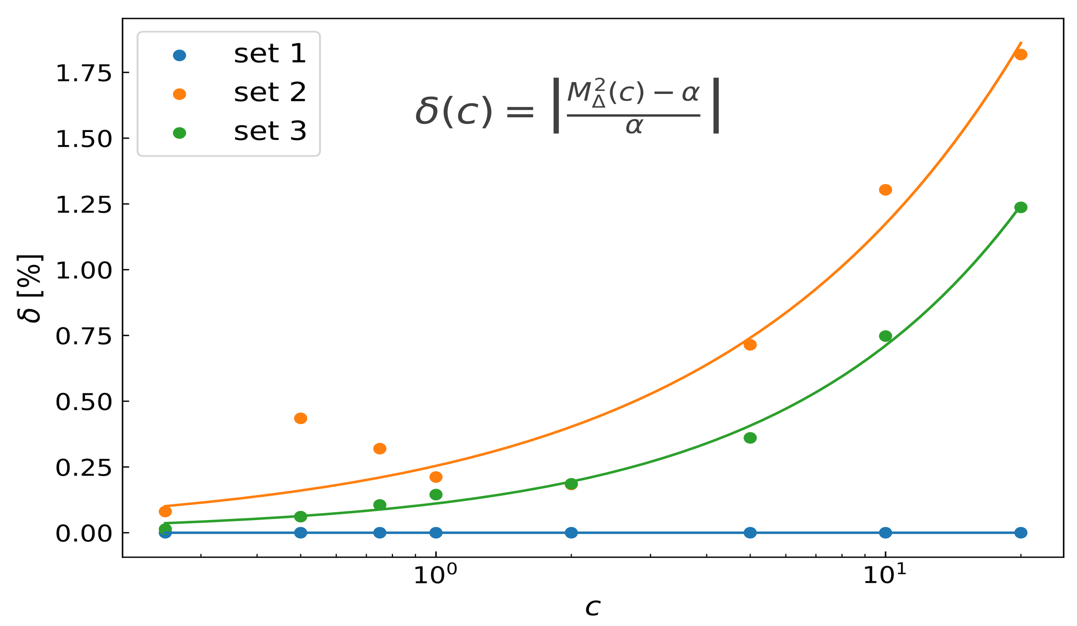
- For a **negative** D, all modes c_q will **grow exponentially**, causing the problem to be ill-posed.
- This **exponential growth** can be **regularized by introducing a hyperdiffusion-like term** $C\partial_\Delta^4 u$, which damps the high q modes[4]

$$\partial_t c_q = -D\,q^2 c_q - Cq^4 c_q\,.$$

- **With the regularization, the oscillations, during the flow, vanish.**



- Negative diffusion occurs in high-μ low-T region and grows with the Yukawa coupling: (a) compared to (c).



- The regularization term can be systematically removed with a fit:

$$M_\Delta^2(c) = \alpha + \beta\,\sqrt{c}\,.$$

References

- [1] J. Stoll, N. Zorbach, and J. Braun, arXiv:2510.01066
- [2] F. R. Sattler, The Phase Diagram of QCD at High Densities
- [3] E. Grossi and N. Wink, arXiv:1903.09503
- [4] T. Witelski, Shocks in nonlinear diffusion, Applied Mathematics Letters 8, 27 (1995)

Based on

arxiv:2608.20196

V. Applications

- Models with complex scalar field and a chemical potential
- Especially truncations in these models which require higher derivatives of u, e.g. LPA'
- high density, low temperature sector of QCD (2SC, CFL phases)

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