Functional Dimensional Regularization for $O(N)$ Models...

... and Lee-Yang

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Recap: How FDR is constructed

At one loop, dimensional regularization generates poles at all critical dimensions

$$d_c = 0, 2, 4, 6, \ldots$$

Instead of selecting only one pole, FDR combines the contribution from all of them.

Master Formula

$$\beta_i^{\mathrm{FDR}}(d) = \sum_{d_c} \mu^{d-d_c}\, \beta_i^{\mathrm{DR}}(d_c)$$

- $\beta_i^{\mathrm{DR}}(d_c)$ is the one-loop beta function obtained from the pole at the critical dimension d_c.

For a general scalar potential, the infinite sum can be performed explicitly:

$$\beta_V^{\mathrm{FDR}}(d) = \sum_{d_c=0,2,\ldots} \mu^{d-d_c} \frac{(-V'')^{d_c/2}}{(d_c/2)!} = \mu^d e^{-V''/\mu^2}.$$

The sum over critical dimensions generates the exponential threshold.

The Ising Wilson-Fisher fixed point

We found excellent results for the Ising universality class:

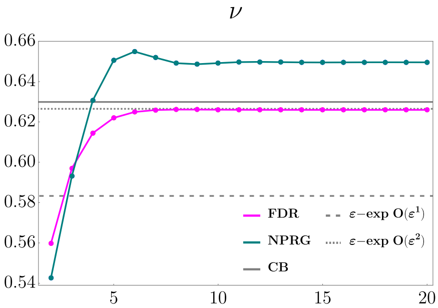


Figure 1: Critical exponent ν vs truncation order N_{tr}.

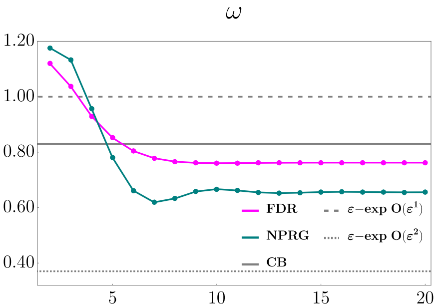
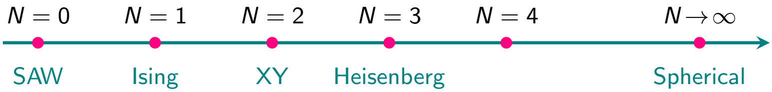


Figure 2: Critical exponent ω vs truncation order N_{tr}.

Is the quality of these results specific to the Ising model, or a systematic feature of FDR?

The $O(N)$ family provides a controlled benchmark for FDR



Why $O(N)$?

The $O(N)$ family spans physically distinct universality classes whose critical properties are accurately known from several independent methods.
For $N > 1$, the longitudinal and transverse sectors provide a non-trivial extension beyond the Ising case.

FDR benchmark

Do the competitive FDR results obtained for Ising extend across the $O(N)$ family?
Can one reach competitive η, ν and ω at DE2 without choosing or optimizing a mass regulator?

We test both questions in $d = 3$, from $N = -2$ to $N = 1000$.

FDR-LPA for the $O(N)$ model

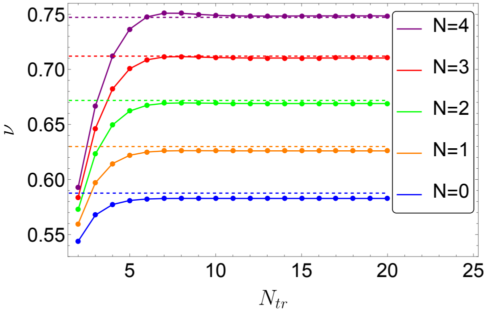
Beta functional for the potential $U(\rho)$

$$\beta_U = \mu^d \exp\left[-\frac{U' + 2\rho U''}{4\pi\mu^2}\right] + (N-1)\,\mu^d \exp\left[-\frac{U'}{4\pi\mu^2}\right]$$

- Dimension-less flow equation

$$\beta_u = -du + (d-2)\rho u' + (N-1)\, e^{-\frac{u'}{4\pi}} + e^{-\frac{u'+2\rho u''}{4\pi}}$$

- Simple field expansion of the form $u(\rho) = \sum_{k=1}^{N_{tr}} \frac{u_k \rho^k}{k!}$
- Convergence is obtained with few number of couplings



Already at LPA, ν is correct to two significant digits

universality class	SAW	Ising	XY	Heisenberg	$O(4)$
N	0	1	2	3	4
$\nu_{\text{FDR-LPA}}$	0.5827	0.6259	0.6689	0.7103	0.7485
benchmark	0.5876	0.6299	0.6717	0.71168	0.7472

What works

The exponential threshold produces rapid polynomial convergence and accurate estimates of ν already at LPA.

Why continue to DE2?

At LPA, $\eta = 0$ by construction. Wave-function flows are essential for a genuine test.

FDR-Derivative Expansion for the $O(N)$ model

Derivative-expansion ansatz

$$S = \int \mathrm{d}^d x \left\{ U(\rho) + \frac{1}{2} \left[Z_T(\rho)(\delta_{ab} - P_{ab}) + Z_L(\rho) P_{ab} \right] \partial_\mu \phi^a \partial^\mu \phi^b \right\}$$

- The only invariant in the $O(N)$ model is $\rho = \frac{1}{2}\phi^a \phi_a$
- We aim to compute the β-functions $\beta_U, \beta_{Z_L}, \beta_{Z_T}$
- Flow of the potential @ DE2:

$$\beta_U = \mu^d \, e^{-\frac{U' + 2\rho U''}{4\pi\mu^2 Z_L}} + \mu^d \, (N-1) \, e^{-\frac{U'}{4\pi\mu^2 Z_T}} \,. \tag{1}$$

FDR-Derivative Expansion for the $O(N)$ model

$$
\begin{aligned}
\beta_{Z_L} = \; & \left\{ -\frac{\mu^{d-2}}{4\pi}\frac{Z_L(Z_L' + 2\rho Z_L'') - 3\rho (Z_L')^2}{Z_L^2} + \frac{\mu^{d-4}}{(4\pi)^2}\frac{2\rho Z_L'(-U'Z_L' + U''(3Z_L - 2\rho Z_L') + 2\rho U''' Z_L)}{Z_L^3} \right. \\
& \left. - \frac{\mu^{d-6}}{(4\pi)^3}\frac{\rho(U'Z_L' - U''(3Z_L - 2\rho Z_L') - 2\rho U''' Z_L)^2}{3Z_L^4} \right\} e^{-\frac{U' + 2\rho U''}{4\pi\mu^2 Z_L}} \\
& -(N-1)\left\{ \frac{\mu^{d-6}}{(4\pi)^3}\frac{\rho(U''Z_T - U'Z_T')^2}{3Z_T^4} + \frac{\mu^{d-4}}{(4\pi)^2}\frac{2(U''Z_T - U'Z_T')(Z_T - Z_L + \rho Z_T')}{Z_T^3} \right. \\
& \left. + \frac{\mu^{d-2}}{4\pi}\frac{\rho Z_T(Z_L' + 2Z_T') + Z_T^2 - Z_L(Z_T + 2\rho Z_T') + \rho^2 (Z_T')^2}{\rho Z_T^2} \right\} e^{-\frac{U'}{4\pi\mu^2 Z_T}}
\end{aligned}
$$

$$
\begin{aligned}
\beta_{Z_T} = \; & \left\{ \mu^d \frac{2(Z_T + \rho Z_T')^2}{\rho(Z_T - Z_L)U' + 2\rho^2 Z_T U''} - \frac{\mu^{d-2}}{4\pi}\frac{(N-1)\rho Z_T' + Z_T}{\rho Z_T} \right\} e^{-\frac{U'}{4\pi\mu^2 Z_T}} \\
& - \left\{ \mu^d \frac{2(Z_T + \rho Z_T')^2}{\rho(Z_T - Z_L)U' + 2\rho^2 Z_T U''} + \frac{\mu^{d-2}}{4\pi}\frac{Z_T + 5\rho Z_T' + 2\rho^2 Z_T''}{\rho Z_L} \right\} e^{-\frac{U' + 2\rho U''}{4\pi\mu^2 Z_L}}
\end{aligned}
$$

Compact analytical flow equations!

A field expansion turns the flow into a fixed-point problem

Dimension-less flow equations:

$$
\begin{aligned}
\beta_u &= -du + (d-2+\eta)\rho u' + \mu^{-d}\beta_U \\
\beta_{z_L} &= \eta z_L + (d-2+\eta)\rho z_L' + \mu^{\eta}\beta_{Z_L} \\
\beta_{z_T} &= \eta z_T + (d-2+\eta)\rho z_T' + \mu^{\eta}\beta_{Z_T}
\end{aligned}
$$

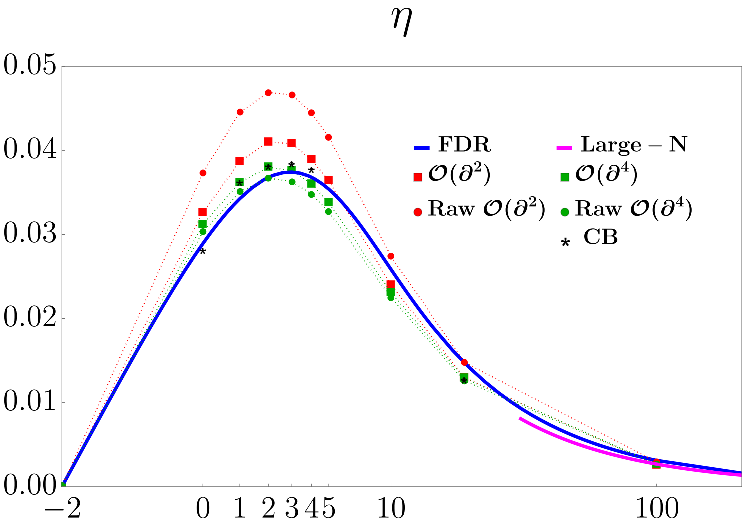
We expand around the running minimum κ:

$$
u(\rho) = \sum_{n=2}^{n_u} \frac{u_n}{n!}(\rho-\kappa)^n, \qquad z_L(\rho) = 1 + \sum_{n=1}^{n_z} \frac{\zeta_n}{n!}(\rho-\kappa)^n, \qquad z_T(\rho) = 1 + \psi_0 + \sum_{n=1}^{n_z} \frac{\psi_n}{n!}(\rho-\kappa)^n.
$$

$$
\boxed{\text{functional flows}} \longrightarrow \boxed{\beta_i(g) = 0} \longrightarrow \boxed{M_{ij} = \partial_j \beta_i|_*} \longrightarrow \boxed{\eta, \quad \nu, \quad \omega}
$$

- Fast convergence: $n_u \sim 10$ already stabilizes the main estimates
- The scan over N is lightweight enough to generate smooth exponent functions.

FDR-DE2 gives a competitive anomalous dimension across N

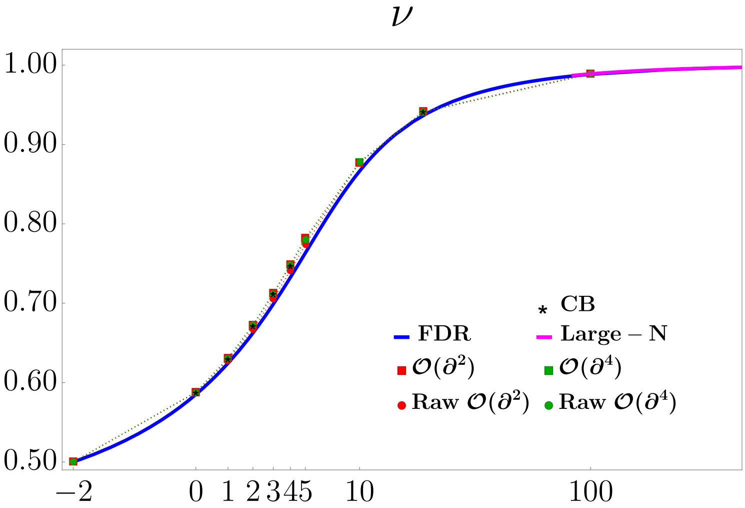


Main result

The FDR-DE2 curve tracks DE4 and conformal-bootstrap estimates more closely than standard NPRG-DE2 over the benchmark range.

- Ising: $\eta_{\mathrm{FDR}} = 0.03426$.
- XY: $\eta_{\mathrm{FDR}} = 0.03679$.
- Heisenberg: $\eta_{\mathrm{FDR}} = 0.03742$.
- Large-N behavior is recovered from $N \sim 100$.

For ν, FDR-DE2 stays competitive but does not uniformly improve LPA



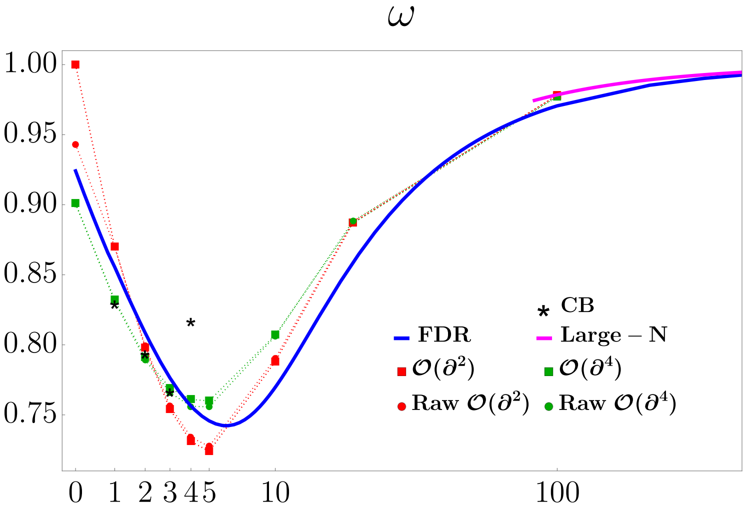
Truncation dependence of ν

FDR-DE2 captures the overall N-dependence. However, for several small values of N, the LPA estimates remain slightly closer to the reference results.

$$N = 1: \quad \nu_{\mathrm{LPA}} = 0.6259, \quad \nu_{\mathrm{DE2}} = 0.6244, \quad \nu_{\mathrm{CB}} = 0.6299.$$

$$N = 3: \quad \nu_{\mathrm{DE2}} = 0.6988, \quad \nu_{\mathrm{CB}} = 0.71168.$$

ω is harder, but FDR captures the full N dependence



- The minimum around $N \simeq$ 4–5 is reproduced.
- Agreement is less uniform than for η and ν.
- The large-N regime sets in later: approximately $N \sim 1000$ for ω.

Interpretation

Subleading directions are a more demanding probe of the truncation.

The one-loop FDR flow reproduces the two-loop ε-expansion

Near $d = 4 - \varepsilon$, the consistent counting is

$$\lambda_2, \lambda_4 = \mathcal{O}(\varepsilon), \qquad \psi_2, \zeta_2 = \mathcal{O}(\varepsilon^2), \qquad \lambda_4^* = \frac{(4\pi)^2}{N+8}\,\varepsilon.$$

Transverse flow

$$\beta_{\psi_2} = 2\psi_2 - \frac{2\lambda_4^2}{3(4\pi)^3} + \mathcal{O}(\varepsilon^3) = 0$$

$$\Longrightarrow \qquad \psi_2^* = \frac{\lambda_4^{*2}}{3(4\pi)^3}.$$

Longitudinal flow

$$\beta_{\zeta_2} = 2\zeta_2 - \frac{(N+8)\lambda_4^2}{3(4\pi)^3} + \mathcal{O}(\varepsilon^3) = 0$$

$$\Longrightarrow \qquad \zeta_2^* = \frac{(N+8)\lambda_4^{*2}}{6(4\pi)^3}.$$

Anomalous dimension

$$\eta = \frac{(N-1)\psi_2^* + \zeta_2^*}{4\pi} = \boxed{\frac{N+2}{2(N+8)^2}\,\varepsilon^2} + \mathcal{O}(\varepsilon^3).$$

This is exactly the standard two-loop $O(N)$ result.

Beyond $O(N)$: FDR for the Lee–Yang universality class

Does FDR retain its efficiency outside the unitary $O(N)$ family?

The Lee–Yang universality class provides a qualitatively different test of FDR:

- It describes a non-unitary critical theory.
- Its field-theory description involves an imaginary cubic interaction,

$$V(\phi) \sim ig\phi^3,$$

  with upper critical dimension $d_c = 6$.
- It allows us to test the same FDR construction continuously from $d = 6$ down to $d = 3$.

Ongoing joint work with C. A. Sánchez–Villalobos and A. Codello

FDR for the Lee–Yang model in a nutshell

The Lee–Yang model

At order DE2, the theory is described by

$$S[\phi] = \int \mathrm{d}^d x \left[V(\phi) + \frac{1}{2} Z(\phi)(\partial_\mu \phi)^2 \right],$$

where V and Z are generally complex functions.

$\mathcal{PT}$-symmetry

Under the combined transformation $\phi \to -\phi$, $i \to -i$ the action remains invariant:

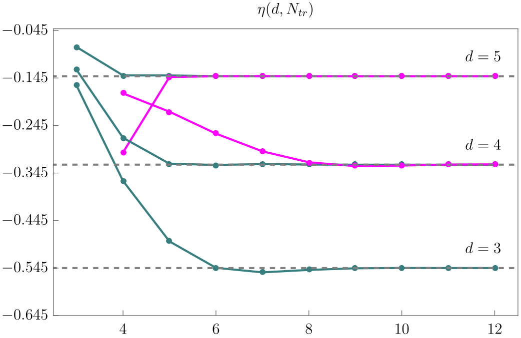
$$V(\phi) = V^*(-\phi), \qquad Z(\phi) = Z^*(-\phi).$$

The FDR-DE2 flow

The scalar FDR beta functionals β_v and β_z were already derived and used for the Ising universality class:

$$\begin{aligned}
\beta_v &= -d\, v + \frac{d-2+\eta}{2}\varphi\, v' + e^{-\frac{v''}{z}} \\
\beta_z &= \eta\, z + \frac{d-2+\eta}{2}\varphi\, z' + \left\{ -\frac{(v''')^2}{6z^2} - \frac{z''}{z} + \left[\frac{1}{z^2} + \frac{v''}{3z^3} \right] v''' z' + \left[\frac{3}{2z^2} - \frac{v''}{z^3} - \frac{(v'')^2}{6z^4} \right] (z')^2 \right\} e^{-\frac{v''}{z}}
\end{aligned}$$

FDR-DE2 gives competitive results for the Lee–Yang anomalous dimension



Anomalous dimension

d	FDR–DE2	6 loops	Fuzzy sphere
5	−0.1414	−0.152(1)	−0.15
4	−0.328	−0.354(6)	−0.346
3	−0.545	−0.578(5)	−0.564

- FDR follows the Lee–Yang fixed point continuously from $d = 6$ down to $d = 3$.
- The zero-mass expansion converges rapidly in all three dimensions.
- The relative difference with the six-loop and fuzzy-sphere estimates remains at % level.

Ongoing joint work with C. A. Sánchez–Villalobos and A. Codello

Conclusions

$O(N)$ universality classes

- The DE2 beta functionals remain compact analytical combinations of only two exponential thresholds.
- The approach reproduces the leading two-loop ϵ-expansion result for η.
- In $d = 3$, FDR provides competitive critical exponents across the $O(N)$ family. The estimate of η is particularly accurate, while ν is more sensitive to the derivative truncation.

Lee–Yang extension

- The same scalar FDR flow can be continued to complex, $\mathcal{PT}$-symmetric functions.
- Polynomial expansions allow us to follow the fixed point continuously from $d = 6$ down to $d = 3$.
- The resulting η remains within 8% of six-loop and fuzzy-sphere estimates.

FDR provides a simple functional framework for both unitary and non-unitary universality classes.

Thank you `arXiv:2604.26207`

Questions ? :)

Backup: FDR estimates reported in the paper

	FDR-LPA			FDR-DE2		
N	η	ν	ω	η	ν	ω
0	0	0.5827	0.7818	0.02891	0.5851	0.923
1	0	0.6259	0.7622	0.03426	0.6244	0.8555
2	0	0.6689	0.7457	0.03679	0.6624	0.8088
3	0	0.7103	0.7345	0.03742	0.6988	0.7767
4	0	0.7485	0.7296	0.03680	0.7326	0.7562
5	0	0.7825	0.7313	0.03540	0.7634	0.7453
10	0	0.8867	0.7916	0.02585	0.8664	0.7697

Backup: references used in the comparison

- P. Beretta and A. Codello, *Functional Dimensional Regularization for $O(N)$ Models*, arXiv:2604.26207.
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- F. Kos, D. Poland, D. Simmons-Duffin and A. Vichi, *Precision Islands in the Ising and $O(N)$ Models*, JHEP 08, 036 (2016).
- S. M. Chester et al., *Carving out OPE space and precise $O(2)$ model critical exponents*, JHEP 06, 142 (2020).
- M. V. Kompaniets and E. Panzer, *Minimally subtracted six-loop renormalization of $O(n)$-symmetric ϕ^4 theory and critical exponents*, Phys. Rev. D 96, 036016 (2017).