Triviality doesn't (always) mean Gaussianity: A Renormalization Group Perspective

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Based on : Triviality Doesn't mean Gaussianity (To appear soon on arXiv).
In collaboration with Bertrand Delamotte, Adam Rançon and Ivan Balog.

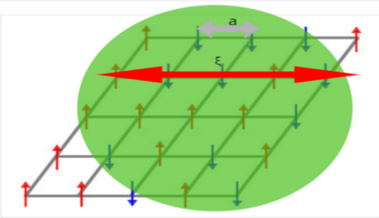
The Central Limit Theorem : A Recap

- Central Limit Theorem: Independent Identically Distributed Random Variables
- Sum $\hat{S} = \sum_{i=1}^{N} \hat{\sigma}_i$, Fluctuations $\to O(\sqrt{N})$, with $N \to \infty$.
- Sum $\frac{\hat{S}}{\sqrt{N}}$, Fluctuations$\to$ Order 1 $\to$ Gaussian Law
- Gaussian Law : $\lim_{N\to\infty} P(\frac{\hat{S}}{\sqrt{N}} = \tilde{s}) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{\tilde{s}-s_0}{\sigma}\right)^2\right]$ with s_0 the mean and σ=variance . ($N = L^d$ for system with size L)

Generalizing the CLT?

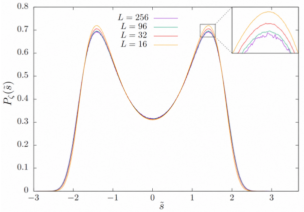
- ~~Independent~~ Correlated Identically Distributed Random Variables
- $G_{ij} = \langle \hat{\sigma}_i \hat{\sigma}_j \rangle$ (or equivalently the correlation length ξ)
- Weakly Correlated ($\sum_i G_{i,j} < \infty$)
- Main Aim: Build a Limiting distribution for strongly correlated variables ($\xi \to \infty$)

Probability Distribution and the Renormalization Group



- Strongly Correlated $\implies \xi \to \infty \implies$ Second-Order Phase Transition
- Framework of Renormalization Group ($d < 4$):
  - Functional Renormalization (already studied: I. Balog , A. Rançon, B. Delamotte, PRL (2022))
  - Perturbative Renormalization (Early but incomplete attempts: e.g. E. Eisenriegler and R. Tomaschitz PRB (1987))

Hints from the Monte-Carlo Simulations



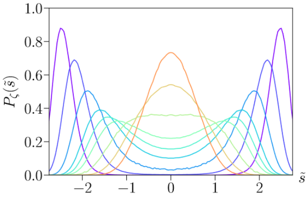


Figure: Different Universal Critical PDFs of the 3d Ising Model [Xu and Landau 2020] (left) and [Balog et. al 2022] (right)

- $\begin{cases} L \to \infty \\ \xi \to \infty \end{cases}$ don't commute!!!
- $\hat{P}(\frac{\Sigma\hat{\sigma}_i}{L^\alpha} = s) = P_\zeta(\frac{\Sigma\hat{\sigma}_i}{L^\alpha} = s)$ with $\zeta = \lim_{L,\xi_\infty \to \infty} sgn(T - T_c)\frac{L}{\xi_\infty}$

$(L^\alpha = \sqrt{\langle s^2 \rangle} = \sqrt{L^{-d}\chi})$

Critical PDF of the Order Parameter

Critical PDF
of the order parameter

Interacting FP
$d < 4$

Gaussian FP
$d \geq 4$

- $d < 4$: universal, non-Gaussian critical PDF $\rightarrow$ Wilson-Fisher Fixed point
- $d = 4$: Triviality $\rightarrow$ Gaussian Fixed point $\rightarrow$But does it mean a Gaussian PDF ?

Setting up the Pertubation Theory

- Lattice Hamiltonian : $\mathcal{H} = -J \sum_{\langle ij \rangle} \hat{\sigma}_i \hat{\sigma}_j$
- Universal PDFs $\rightarrow \phi^4$ theory (with periodic boundary conditions).
- PDF :

$$P(\hat{s} = s) \propto \int D\hat{\phi}\, \delta(\hat{s} - s) \exp\left(- \int_V \mathcal{H}[\hat{\phi}] \right), \tag{1}$$

where $\hat{s} = \frac{1}{L^d} \int_V \hat{\phi}(x)$ and

$$\mathcal{H}\left(\hat{\phi}(x)\right) = \frac{1}{2}\left(\nabla\hat{\phi}(x)\right)^2 + \frac{1}{2} t_0\, \hat{\phi}^2(x) + \frac{1}{4!} g_0 \hat{\phi}^4(x). \tag{2}$$

Setting up the Pertubation Theory

- $$P(\hat{s} = s) \propto \int D\hat{\phi}\, \delta(\hat{s} - s) \exp\left(-\int_V \mathcal{H}[\hat{\phi}]\right)$$

  Using $\delta(\hat{s} - s) \propto \lim_{M \to \infty} \exp\left\{\left(-\frac{M^2(\hat{s} - s)^2}{2}\right)\right\}$

  $$P(\hat{s} = s) = \mathcal{N} \int D\hat{\phi} \exp\left(-\int_V \mathcal{H}_M\right) = \mathcal{Z}_M[h = 0], \tag{3}$$

  with:

  $$\mathcal{H}_M\left(\hat{\phi}(x)\right) = \mathcal{H}\left(\hat{\phi}(x)\right) + \frac{M^2}{2}\left(\int_V (\hat{\phi}(x) - s)\right)^2. \tag{4}$$

Setting up the Perturbation Theory

- With a bit of algebra, one can then show:

$$\lim_{M\to\infty} e^{-\Gamma_M[\phi(x)=s]} \propto P(\hat{s}=s), \tag{5}$$

with $\Gamma_M[\phi]$ being the modified Gibbs potential (generator of 1-PIs).

- Rate function (Large Deviation Principle) :

$$P(\hat{s}=s) \propto e^{-L^d \hat{I}(s,\xi_\infty,L)}. \tag{6}$$

and thus

$$\lim_{M\to\infty} \Gamma_M[\phi(x)=s] = L^d \hat{I}(s,\,\xi_\infty,\,L) = I(sL^{\beta/\nu} = \tilde{s}, L/\xi_\infty = \zeta). \tag{7}$$

$$L^{-\beta/\nu} = \sqrt{\langle s^2 \rangle} = \sqrt{L^{-d}\chi}$$

Setting up the Perturbation Theory

- With a bit of algebra, one can then show:

$$\lim_{M\to\infty} e^{-\Gamma_M[\phi(x)=s]} \propto P(\hat{s}=s), \tag{8}$$

with $\Gamma_M[\phi]$ being the modified Gibbs potential (generator of 1-PIs).

- Rate function (Large Deviation Principle) :

$$P(\hat{s}=s) \propto e^{-L^d \hat{I}(s,\xi_\infty,L)}. \tag{9}$$

and thus

$$\lim_{M\to\infty} \Gamma_M[\phi(x)=s] = L^d \hat{I}(s,\,\xi_\infty,\,L) = I(sL^{\beta/\nu}=\tilde{s}, L/\xi_\infty=\zeta). \tag{10}$$

$$L^{-\beta/\nu} = \sqrt{\langle s^2\rangle} = \sqrt{L^{-d}\chi}$$

ϵ-expansion ($d < 4$)

- Rate function at one-loop ($\epsilon = 4 - d$-expansion):

$$I_\zeta(\tilde{x}) = \frac{3}{\epsilon}\left(\frac{1}{2}\zeta^{\frac{1}{\nu}}x^2 + \frac{2\pi^2}{3}x^4\right) + \underbrace{\frac{\pi^2}{4}\left(\frac{\zeta^{\frac{1}{\nu}}}{4\pi^2} + 2x^2\right)^2\left(\gamma_E + \log 2\pi - \frac{3}{2} + \log\left(\frac{\zeta^{\frac{1}{\nu}}}{8\pi^2} + x^2\right)\right)}_{\text{F.P. potential at } \zeta = 0} + \underbrace{\frac{1}{2}\Delta\left(\frac{\zeta^{\frac{1}{\nu}}}{4\pi^2} + 2x^2\right)}_{\text{Finite-size corrections}} + O(\epsilon), \tag{11}$$

.

- $x = \sqrt{g_*}\tilde{s}$, $(g_* = O(\epsilon))$
- ill-defined at $\epsilon \to 0 \implies$ PDF in $d \geq 4$?

Results at two-loop order (S. Sahu J. Stat. Mech. (2025))

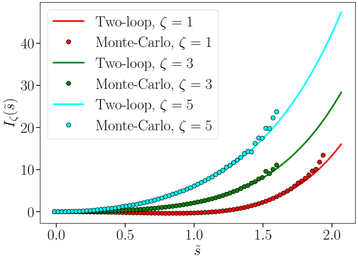


Figure: Rate function $I_\zeta(\tilde{s})$ at two loops

- Further Generalization to $O(n)$-models: S.Sahu J. Stat. Mech. (2026).

The story at and beyond the upper critical dimension....

- Rigorous Mathematical Results by Aizenman [PRL '81] and Duminil-Copin [Annals of Mathematics '21].
- Consider the β-function for the ϕ^4 coupling :

$$\beta(g_R) = -(d-4)g_R + \frac{3}{16\pi^2} g_R^2 + \mathcal{O}(g_R^3),$$

- Hence, in $d \geq 4$

$$\implies \quad g_R \to 0,$$

as the UV cut-off $\Lambda \to \infty$ (*Continuum limit*).

In fact, the fixed-point effective potential in the Wetterich-formalism reduces to (after usual-rescaling):

$$\boxed{\tilde{U}_* = 0} \tag{12}$$

In terms of correlation functions this means in a zero external momentum configuration (at criticality):

$$\Gamma_*^{(n)}(0, 0 \cdots, 0) = 0. \tag{13}$$

Hints from one-loop

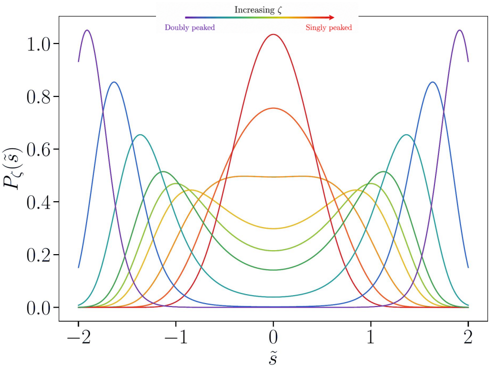


Figure: We define another universal quantity ζ_c, such that $I''_{\zeta_c}(\tilde{s} = 0) = 0$

Hints from one-loop

As show before, obviously the $\epsilon-$expansion fails.....but

- On Triviality of ϕ^4 theory, early hints from the one-loop result:

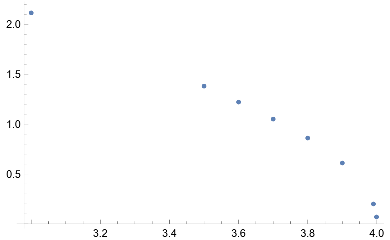


Figure: Value of ζ_c as dimension is varied from $d = 3$ to $d = 4$

- Need for a different approach: *Large N*

Large N → Exactly solvable limit of $O(N)$-models

Large N → Beginning of a perturbative series in $1/N$

Finite-size scaling

Finite-size scaling	$2 < d < 4$	$d > 4^*$
χ	$\chi_L \sim L^{2-\eta}$	$\chi_L \sim L^{d/2}$
Scaling variable	$\tilde{s} = sL^{\beta/\nu}(= L^{\frac{d-2+\eta}{2}})$	$\tilde{s} = sL^{d/4}$
Correlation length	$\xi_L \sim L$	$\xi_L \sim L^{d/4}$
ζ	$\zeta = \lim_{L,\xi_\infty \to \infty} sgn(T - T_c)\frac{L}{\xi_\infty}$	$\zeta = \lim_{L,\xi_\infty \to \infty} \frac{L^{d/4}}{\xi_\infty}$

*Berche and Kenna '12, Brezin and Zinn-Justin '85, retrieved in our-large-N analysis

Large N calculations for $d > 4$ case

In the limit of large N, the gap Eqn for rate function $I(\rho)$ is given by:

$$\rho = \underbrace{\frac{3N}{u_R}(I'(\rho) - \delta r)}_{\text{Mean-field contribution}} - \underbrace{\frac{N}{2L^{d-2}} \sum_{n \neq 0}^{R} \frac{1}{4\pi^2 n^2 + I'(\rho) L^2}}_{\text{Infrared fluctuations}} \tag{14}$$

Using the fact that $\chi \sim L^{d/2}$, we have:

$$\tilde{\rho} = \rho L^{\frac{d}{2}}, \quad (\rho = \frac{s^2}{2}) \tag{15}$$

$$\zeta = (\delta r)^{1/2} L^{\frac{d}{4}} \quad (\xi_\infty \propto \delta r^{-\frac{1}{2}}) \tag{16}$$

and consequently:

$$\tilde{I}'(\tilde{\rho}) = L^{\frac{d}{2}} I'(\rho), \tag{17}$$

since ($I(\rho)$ scales as L^{-d}).

Large N calculations for $d > 4$ case

Rewriting the gap equation Eq. (14) in terms of these new scaling variables we thus have:

$$\tilde{\rho} = \frac{3N}{u_R}\left(\tilde{I}'(\tilde{\rho}) - \zeta^2\right) - \cancel{\frac{N}{2L^{\frac{d}{2}-2}} \sum_{n \neq 0} \frac{1}{4\pi^2 n^2 + \frac{\tilde{I}'(\tilde{\rho})}{L^{\frac{d}{2}-2}}}} \tag{18}$$

Now we want to study Eq.(18) in the thermodynamic limit or $L \to \infty$ to find out the universal rate function. Of course in the limit $L \to \infty$, the *fluctuations* die out and we recover:

$$\boxed{\tilde{I}_\zeta(\tilde{s}) = \mathrm{sgn}(T - T_c)\,\frac{\zeta^2 \tilde{s}^2}{2} + \frac{\tilde{s}^4}{24}} \tag{19}$$

$(\tilde{\rho} = \tilde{s}^2/2)$.

Triviality doesn't mean Gaussianity !!

- All the infrared fluctuations are killed and what survives are the gaussian fluctuations or the mean field theory.
- However, *Mean field theory* doesn't mean a Gaussian PDF
- Rather, what one finds is a family of $non-$Gaussian PDFs indexed by ζ with a $\zeta_c = 0$.
- Fixed Point potential is no longer closely related to the critical rate function.
- Holds true even at $d = 4$.

Future plans and Perspectives

- Development of other Non-perturbative methods to tackle this problem
  - Is there anything to be learnt about the PDFs at or above the critical dimension from the FRG methods ?
  - An unified FRG framework that works in all dimensions.
- Works beyond large-N.
- $d = 4$ is unphysical from a Stat-Mech viewpoint, however our results extend beyond the ϕ^4-theory.

Thank you for your attention!

Failure of the Usual Perturbation Theory

- The usual $4 - \epsilon$ expansion is organized in terms of *dimensionless* coupling constant g with $g \sim O(\epsilon)$, which becomes of order ϵ in $2 < d < 4$.
- However for $d > 4$, this *dimensionless* coupling becomes exactly 0.
- *This is why the ϵ-expansion fails at $d \geq 4$.*