UNIVERSITÄT
HEIDELBERG
ZUKUNFT
SEIT 1386

Funded by
the European Union

European Research Council
Established by the European Commission

Understanding IR divergences in higher derivative theories

Diego Buccio

Why higher derivative?

- Einstein general relativity as a QFT is perturbatively non renormalizable
- Higher derivative operators R^2, $R_{\mu\nu}R^{\mu\nu}$ and $R_{\mu\nu\rho\lambda}R^{\mu\nu\rho\lambda}$ contain a fourth derivative kinetic term for the metric

$$S = \int d^4x\sqrt{-g}\left[+2\Lambda - Z_N R + \frac{1}{2\lambda}C^2 + \frac{1}{\xi}R^2 - \frac{1}{\rho}E\right]$$

$$\Downarrow$$

The theory is now perturbatively renormalizable [Stelle, ‘77]

Problems:

Ghosts, unitarity

IR divergences

IR divergences

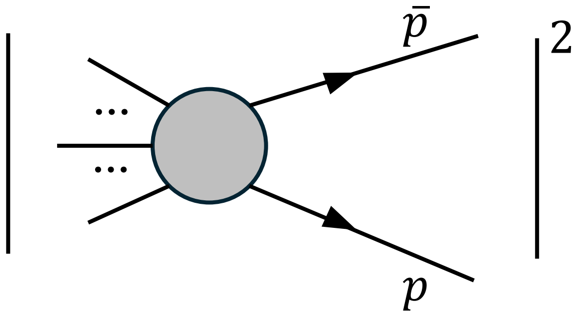
- In theories with massless particles (QED, QCD, gravity,…) loop diagrams can diverge at small momentum
- UV divergences $\Rightarrow$ renormalization
- IR divergences $\Rightarrow$ inclusive IR safe observables (KLN theorem)

Phase space (real) IR divergence $\Leftrightarrow$ Cancellation $\Leftrightarrow$ Loop integral (virtual) IR divergence

2 derivative: virtual divergences

- Theory with massless particles (ex. QED)

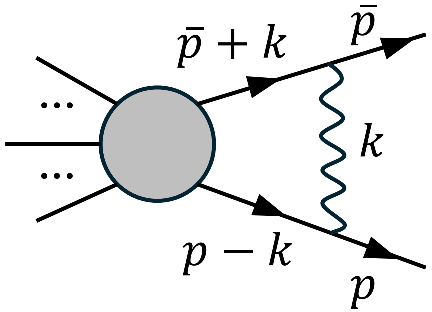
$$n \to 2$$



$$d\sigma_0 \sim \Pi_{p,\bar p} \frac{d^3p^i}{(2\pi^3)2E_i} |A_0(p,\bar p)|^2 \delta^4(p_{in} - p_{fin})$$

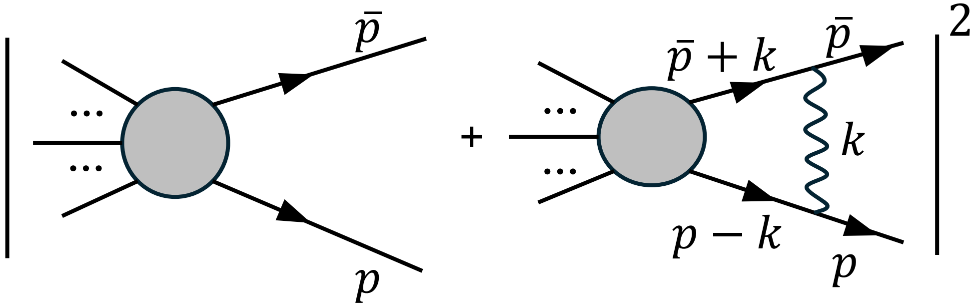
IR divergence only if $p, \bar p$ on-shell

1-loop



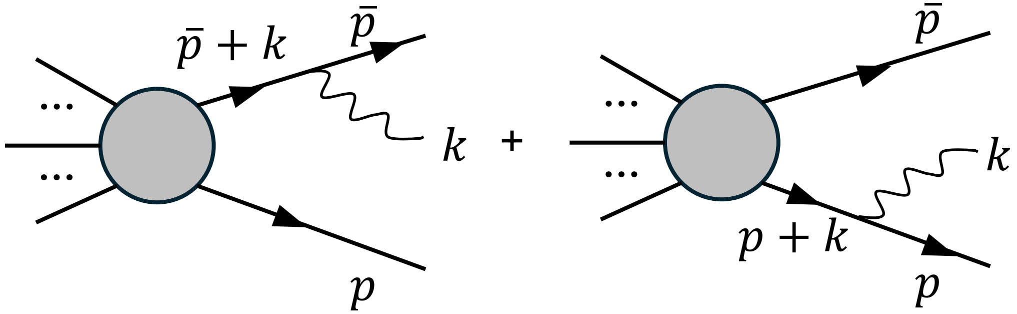
$$A_{virt}(p,\bar p) \sim iA_0 g^2 \int \frac{d^4k}{(2\pi)^4}(p\cdot\bar p)\boxed{\frac{1}{(\bar p+k)^2}\frac{1}{(p-k)^2}\frac{1}{k^2}}$$

$$d\sigma_{1\,loop} \sim d\sigma_0\left[1+2g^2\, Im\left(\boxed{\int \frac{d^4k}{(2\pi)^4}\frac{(p\cdot\bar p)}{(k\cdot\bar p)(p\cdot k)}\frac{1}{k^2}}\right)+O(g^3)\right] \propto$$

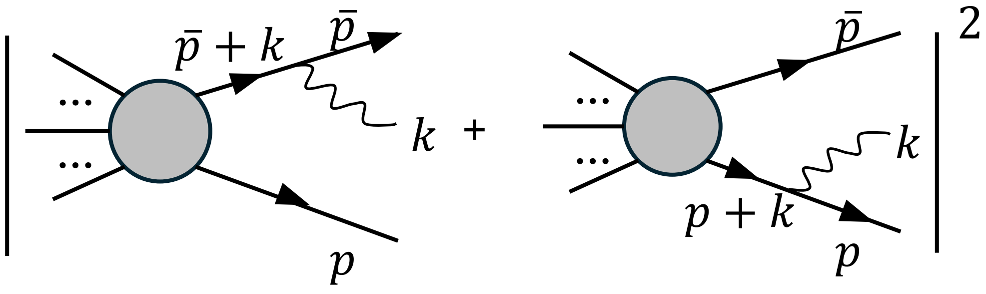


Real IR divergent amplitudes

$n \to 3$



Soft limit $k^\mu \to 0$ $\qquad$ $A_3(p,\bar{p},k) \sim g A_0(p,\bar{p}) \left(\frac{p\cdot\epsilon}{p\cdot k} - \frac{\bar{p}\cdot\epsilon}{\bar{p}\cdot k}\right)$



$\propto$

$$d\sigma_3 = 2 d\sigma_0\, g^2 \boxed{\frac{(p\cdot\bar{p})}{(k\cdot\bar{p})(p\cdot k)} \frac{d^3k}{(2\pi^3)2k_0}}$$

Divergent soft differential CS

IR-safe observables

- Real instruments have minimal sensitivity E_0
- Soft particles with $k < E_0$ cannot be observed

Processes with soft emission indistinguishable from process without

$$d\sigma_{incl} = d\sigma_{1\ loop} + \int_{k<E_0} d\sigma_3 + O\left(g^3\right) = d\sigma_0 + g^2 \ln\left(\frac{f(p,\bar{p})}{E_0}\right)$$

IR region regulated by the sensitivity threshold!

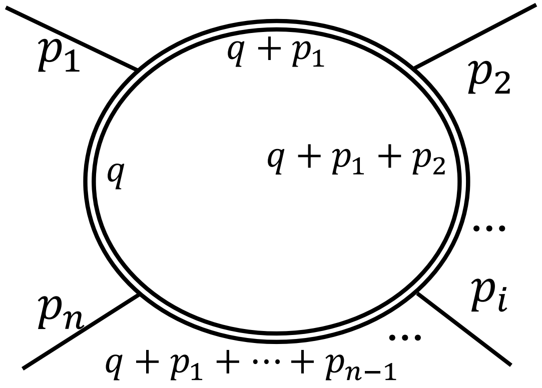
$$\boxed{d\sigma_0 \left[1 - 2g^2 \int \frac{(p\cdot\bar{p})}{(k\cdot\bar{p})(p\cdot k)} \frac{d^3k}{(2\pi^3)2k_0}\right]}$$

$$\boxed{2d\sigma_0\, g^2 \int_{k<E_0} \frac{(p\cdot\bar{p})}{(k\cdot\bar{p})(p\cdot k)} \frac{d^3k}{(2\pi^3)2k_0}}$$

4 derivative

On-shell cross sections well-defined [Salvio, Strumia '18]

In higher derivative (HD) theories there are off-shell IR divergences too:



$$\sim \int d^4q \frac{N(p, q_i)}{q^4 (q+p_1)^4 \times \cdots \times (q + p_1 + \cdots + p_{n-1})^4}$$

$q \ll p_i$

$q \gg p_i$

$$\frac{1}{p_1^4 \times \cdots \times (p_1 + \cdots + p_{n-1})^4} \int d^4 q \boxed{\frac{N(q, p_i)}{q^4}} + \cdots$$

$$\int d^4 q \boxed{\frac{N(p, q_i)}{q^{4n}}} + \cdots$$

Independent of the particular kinematical configuration

Shift invariant case

[Holdom '22, '24;
D.B., Donoghue, Percacci, '23;
DB, Parente, Zanusso '24;
Anderson, Bateman, Hezog, Turok '26]

- Special case: shift-invariant theories

All legs of interaction vertices have a derivative

$$\Downarrow$$

$$N(q, p_i\ ) \propto \Pi_i\left(q + \Sigma_{j=0}^{i} p_j\right)^2$$

$$\Downarrow$$

No IR divergences

With shift-invariance, only $\partial_\mu \phi$ in the action

[Delzescaux, Duclut, Mouhanna and Tissier, '23]

New variable $A_\mu = \partial_\mu \phi$ with only longitudinal mode propagating

HD diagrams rewritten as ordinary 2 derivative diagrams

Question:

Can I have well-defined HD off-shell processes in the massless limit?

Indefinite norm space and asymptotic states

- First of all, we must define asymptotic states for the HD field

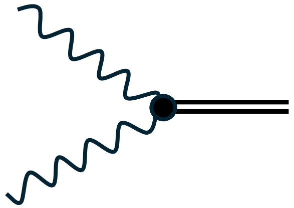
- Convention: Feynman prescription for the ghost and the normal mode
- The ghost has negative norm, but the S matrix preserves the inner product (unitarity but not positive probability)

- In the massless limit, we take as asymptotic states quantum superpositions of the ghost and the normal mode [Holdom ‘22, ‘24]
- It might be that the negative norm state is not observable because of hidden symmetries or indistinguishability from other composite fields

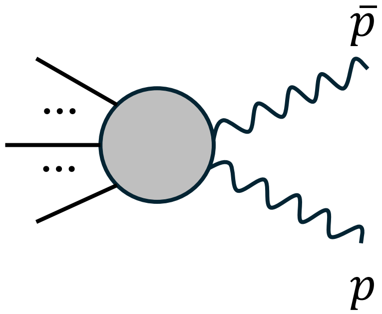
[Boninfante ’26;
Holdom ’24,’26; Bateman, Turok 26]

Example: three-point vertex

 $= \dfrac{1}{p^4 + m^2 p^2} = \dfrac{1}{m^2}\left(\dfrac{1}{p^2 - i\epsilon} - \dfrac{1}{p^2 + m^2 - i\epsilon}\right) =$ +

 $= \quad g$

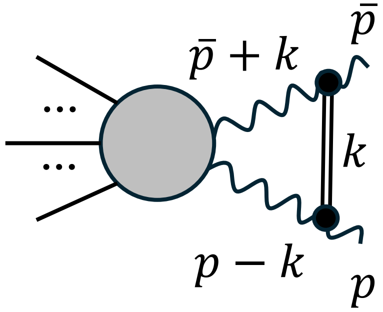
$n \to 2$



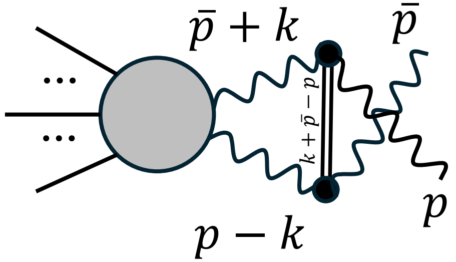
$= A_0(p, \bar{p}),$ with $p^2, \bar{p}^2 \gg m^2$

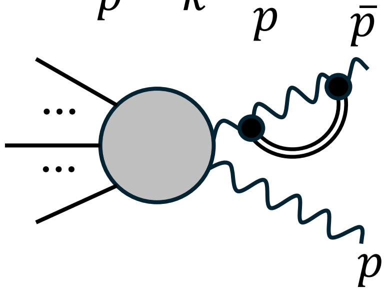
$$d\sigma_0 \sim \Pi_{p,\bar{p}} \frac{d^4 p^i}{(2\pi^4)} |A_0(p, \bar{p})|^2 \delta^4\left(p_{in} - p_{fin}\right)$$

Virtual divergences

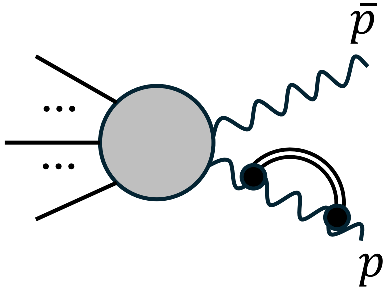


$$= A \sim i\frac{1}{2} g^2 A_0 \frac{1}{p^2\bar{p}^2}\frac{1}{m^2}\int \frac{d^4k}{(2\pi)^4}\left(\frac{1}{k^2-i\epsilon}-\frac{1}{k^2-m^2-i\epsilon}\right) =$$

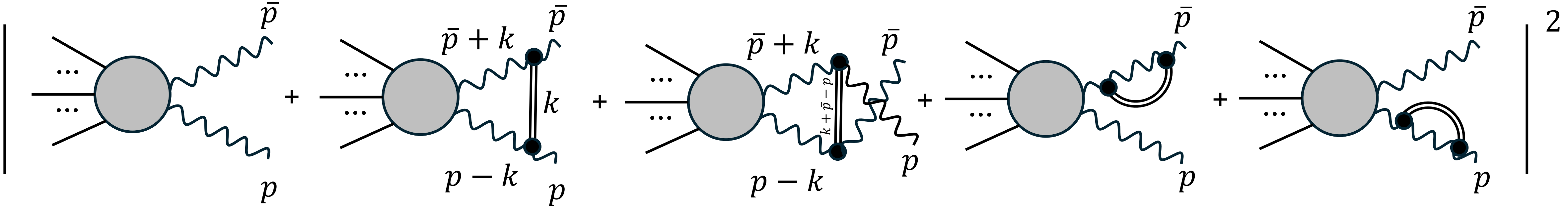




$$= A\left(\frac{1}{p^2\bar{p}^2}\to\frac{1}{\bar{p}^4}\right)$$

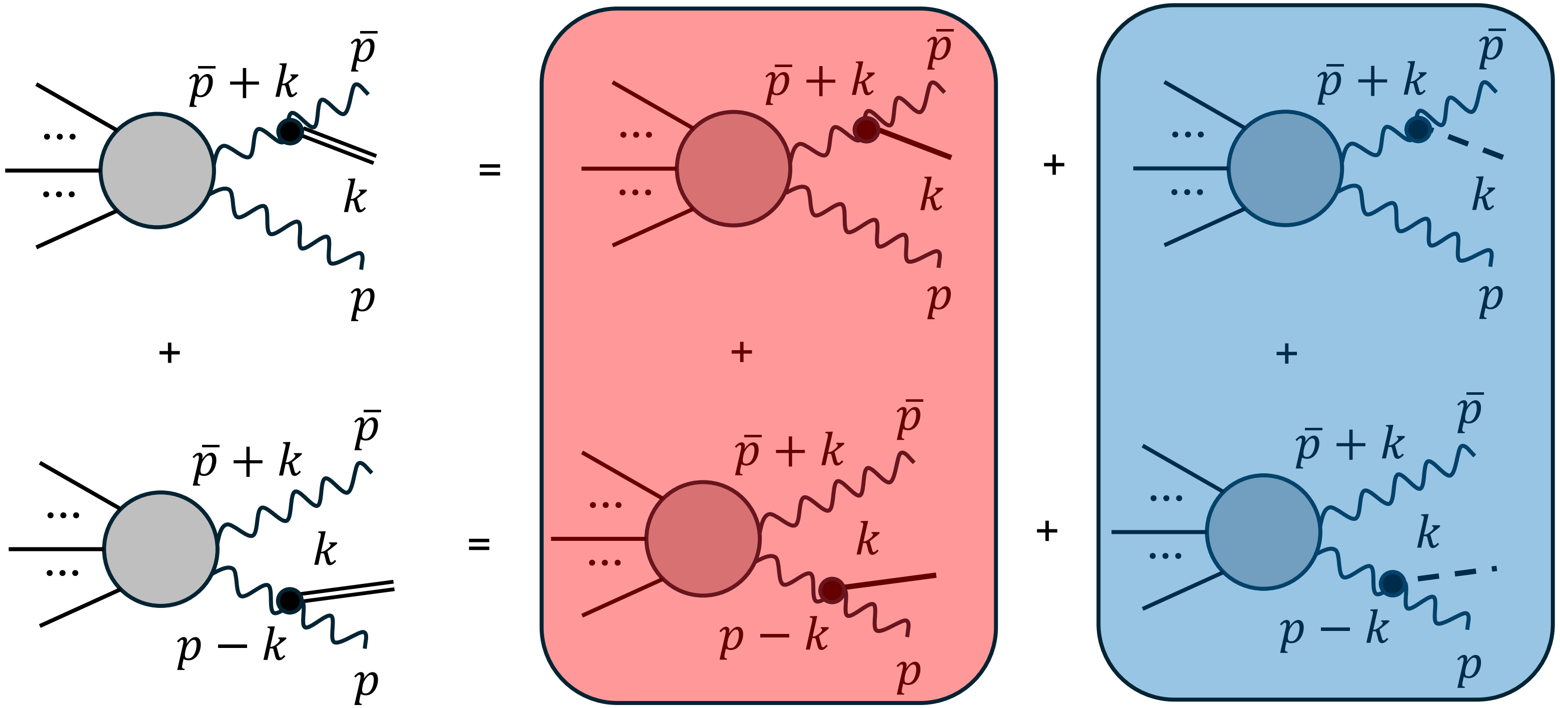


$$= A\left(\frac{1}{p^2\bar{p}^2}\to\frac{1}{p^4}\right)$$



$$d\sigma_{1\,loop} \sim d\sigma_0\left[1+g^2\left(\frac{1}{p^4}+\frac{1}{\bar{p}^4}+\frac{2}{p^2\bar{p}^2}\right) Im\left(\frac{1}{m^2}\int\frac{d^4k}{(2\pi)^4}\left(\frac{1}{k^2-i\epsilon}-\frac{1}{k^2-m^2-i\epsilon}\right)\right)+O\left(g^3\right)\right]\propto \ln\left(\frac{p_i}{m}\right)$$

Scattering amplitudes with a soft composite external state



$$A_3 = gA_0\left(\frac{i}{(p-k)^2} + \frac{i}{(\bar{p}+k)^2}\right)$$

$$d\sigma_3 \sim d\sigma_0 g^2 |A_0|^2 \left(\frac{1}{p^4} + \frac{1}{\bar{p}^4} + \frac{2}{p^2\bar{p}^2}\right)\left(\boxed{\frac{d^3k}{(2\pi^3)2|k|}} - \boxed{\frac{d^3k}{(2\pi^3)2\sqrt{k^2-m^2}}}\right) \sim \frac{d^3k\, m^2}{(2\pi^3)|k|^3}$$

Off-shell IR divergences cancellation

- While taking $m \to 0$, we introduce an instrumental sensitivity on the composite state $k > E_0 > m$

$$d\sigma_{incl} = d\sigma_{1\,loop} + \int_{k<E_0} d\sigma_3 + O\left(g^3\right) = d\sigma_0 + g^2 \ln\left(\frac{f(p,\bar{p})}{E_0}\right)$$

$$d\sigma_0\left[1 - g^2\left(\frac{1}{p^4} + \frac{1}{\bar{p}^4} + \frac{2}{p^2\bar{p}^2}\right)\int \frac{d^3k}{(2\pi)^3}\frac{1}{2|k|^3} + O\left(g^3\right)\right]$$

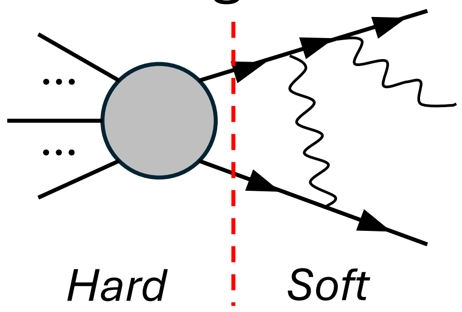
$$d\sigma_0\left[1 - g^2\left(\frac{1}{p^4} + \frac{1}{\bar{p}^4} + \frac{2}{p^2\bar{p}^2}\right)\int_{k<E_0} \frac{d^3k}{(2\pi)^3}\frac{1}{2|k|^3}\right]$$

IR safe cross section in the massless limit!

Hard-soft separation

Up to 2 derivative

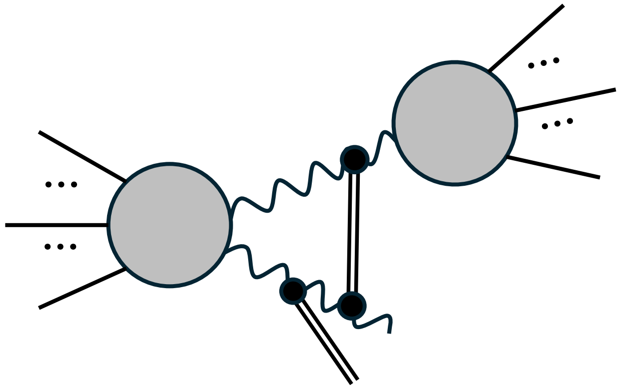
IR divergences only with almost on-shell external legs



IR effect only in soft region, separated from hard process

Higher derivative

IR divergences from all loops



IR enhancement potentially everywhere

IR divergences associated to every HD vertex/line ➡ can I reabsorb them in g?

[DB, Percacci, Menezes, Donoghue '24; DB, Parente, Zanusso '24]

YES, but gauge dependent!

[Salvio, Strumia, Vitti '25; DB, De Brito, Parente '25]

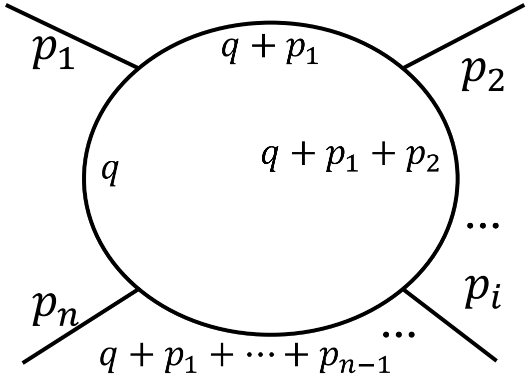
Conclusions

- It is possible to define inclusive IR safe HD off-shell cross sections
- No separation between hard and soft part of the process in HD
- Is it possible for n-point interaction vertices?
- The IR logarithmic enhancement at large external momenta is still there. Resulting running gauge dependent
- Full on-shell cross sections?

Thank you!

2 derivatives theories

Generic off-shell amplitudes cannot be IR divergent ($\sum_{i=1}^{k} p_i \neq 0 \ \forall\, k,\ |p_i| \neq m^2 \ \forall\, i$)



$$\sim \int d^4q \, \frac{N(p,q_i)}{q^2(q+p_1)^2 \times \cdots \times (q+p_1+\cdots+p_{n-1})^2}$$

$q \ll p_i$

$q \gg p_i$

$$\frac{1}{p_1^2 \times \cdots \times (p_1+\cdots+p_{n-1})^2} \int d^4q \, \frac{N(p,q_i)}{q^2} + \cdots$$

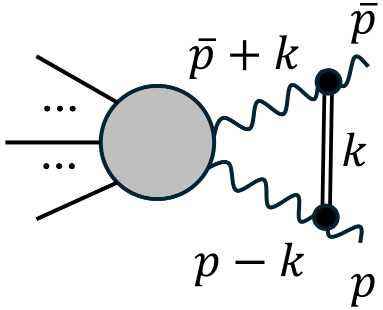
$$\int d^4q \boxed{\frac{N(p,q_i)}{q^{2n}} + \cdots}$$

If $\quad \exists\, k \,|\, \sum_{i=1}^{k} p_i = 0$, (2 on-shell internal lines)

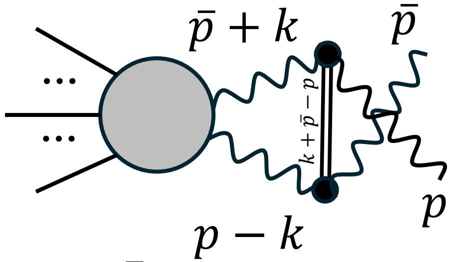
$$\Rightarrow \quad \frac{1}{p_1^2 \times \cdots \times (p_1+\cdots+p_{n-1})^2} \int d^4q \boxed{\frac{N(p,q_i)}{q^4} + \cdots}$$

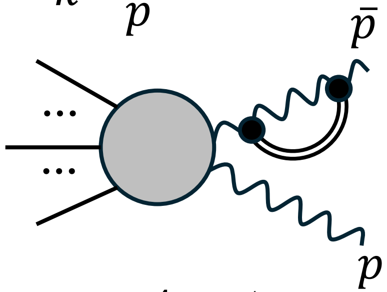
Virtual IR divergence

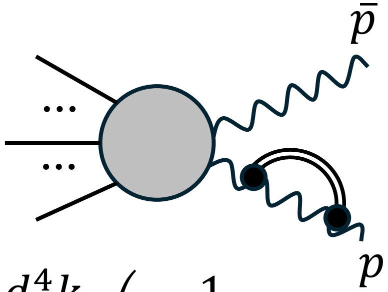
Virtual divergences



$$A \sim i\frac{1}{2}g^2 A_0 \frac{1}{p^2\bar{p}^2}\frac{1}{m^2}\int \frac{d^4k}{(2\pi)^4}\left(\frac{1}{k^2 - i\epsilon} - \frac{1}{k^2 - m^2 - i\epsilon}\right)$$

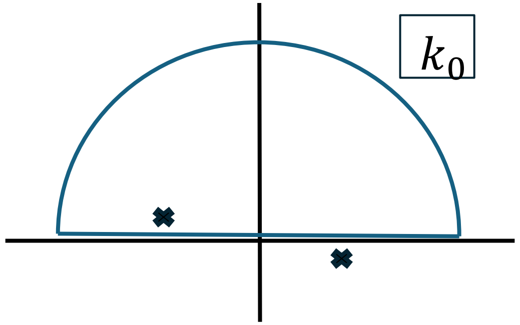






$$i\frac{1}{2}g^2 A_0 \frac{1}{\bar{p}^4}\frac{1}{m^2}\int \frac{d^4k}{(2\pi)^4}\left(\frac{1}{k^2 - i\epsilon} - \frac{1}{k^2 - m^2 - i\epsilon}\right) \qquad i\frac{1}{2}g^2 A_0 \frac{1}{p^4}\frac{1}{m^2}\int \frac{d^4k}{(2\pi)^4}\left(\frac{1}{k^2 - i\epsilon} - \frac{1}{k^2 - m^2 - i\epsilon}\right)$$

$$d\sigma_{1\,loop} \sim d\sigma_0\left[1 + g^2\left(\frac{1}{p^4} + \frac{1}{\bar{p}^4} + \frac{2}{p^2\bar{p}^2}\right) Im\left(\frac{1}{m^2}\int \frac{d^4k}{(2\pi)^4}\left(\frac{1}{k^2 - i\epsilon} - \frac{1}{k^2 - m^2 - i\epsilon}\right)\right) + O(g^3)\right]$$



$$= d\sigma_0\left[1 - g^2\left(\frac{1}{p^4} + \frac{1}{\bar{p}^4} + \frac{2}{p^2\bar{p}^2}\right)\frac{1}{m^2}\int \frac{d^3k}{(2\pi)^3}\left(\frac{1}{2|k|} - \frac{1}{2\sqrt{k^2 + m^2}}\right) + O(g^3)\right] \propto \ln\left(\frac{p_i}{m}\right)$$