Istituto Nazionale di Fisica Nucleare

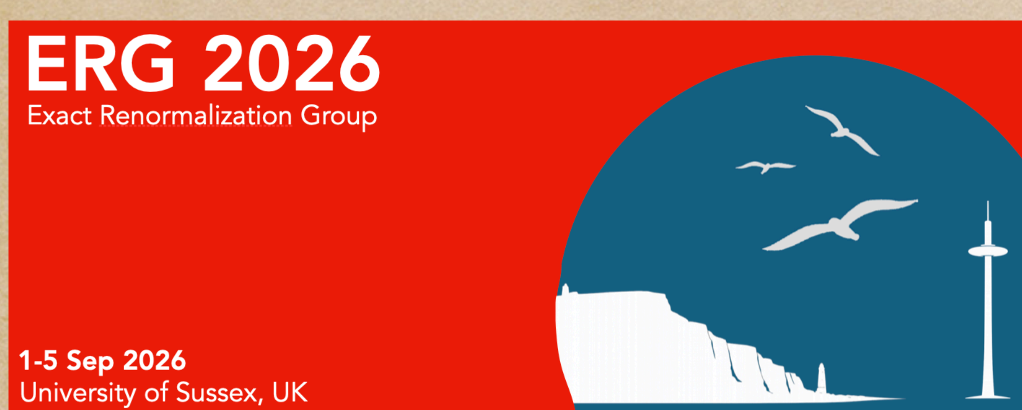
Theory and Phenomenology of Fundamental Interactions

UNIVERSITY AND INFN · BOLOGNA

Exploring multi parameter optimization in FRG

Gian Paolo Vacca

2/9/2026



In collaboration with
Alessandro Codello and *Davide Zarrilli*

Goal

The idea is to push the study of minimizing the regulator dependence in FRG analysis of critical theories, allowing its functional form to change within a given function space. This form will be parametrized by a set of parameters to be tuned according to the principle of minimal sensitivity (PMS).

Outline

- Context
- Regulators
- Test in 3D Ising model for DE2 and DE4 truncations
- Conclusions and outlook

1PI effective average action and Its RG flow

$$e^{-W_k[J]} = Z_k[J] = e^{-\Delta S_k[\frac{\delta}{\delta J}]} Z_k[J] = \int [d\varphi]\; e^{-S[\varphi] - \Delta S_k[\varphi] + J\cdot\varphi}$$

Coarse-graining infrared regulator: $\Delta S_k[\varphi] = \frac{1}{2}\varphi \cdot R_k \cdot \varphi$

$$R_k(p^2) > 0 \text{ for } p^2 \ll k^2$$
$$R_k(p^2) \to 0 \text{ for } p^2 \gg k^2$$
$$R_k(p^2) \to \infty \text{ for } k \to \Lambda \ (\to \infty)$$

Legendre transform

$$\Gamma_k[\phi] = \operatorname*{extr}_J \left(J\cdot\phi - W_k[J]\right) - \Delta S_k[\phi]$$

$$\boxed{e^{-\Gamma_k[\phi]} = \int [d\varphi]\; e^{-S[\varphi] + \frac{\delta \Gamma_k}{\delta \phi}\cdot(\varphi - \phi) - \Delta S_k[\varphi - \phi]}}$$

Wetterich-Morris (RG flow) equation

$$\boxed{\partial_t \Gamma_k = \frac{1}{2}\,\mathrm{Tr}\left[\left(\Gamma_k^{(2)} + R_k\right)^{-1} \partial_t R_k\right]} \qquad t = \ln k/k_0$$

It defines directly QFTs/SFTs given field content and symmetries

Approximations schemes

Derivative Expansion: include family of operators with an increasing number of derivatives. # functions: LPA (1), DE2 (+1), DE4 (+3), DE6 (+8) …

If Z_2 symmetry, to have all functions even, we write

$$\Gamma_k[\phi] = \int d^D x \left[V_k(\phi) + \frac{1}{2} Z_k(\phi)(\partial\phi)^2 + \frac{1}{2} W_k^a(\phi)(\partial_\mu\partial_\nu\phi)^2 + \frac{1}{2}\phi W_k^b(\phi)(\partial^2\phi)(\partial\phi)^2 + \frac{1}{2} W_k^c(\phi)\big((\partial\phi)^2\big)\big)^2 + \cdots \right]$$

In general one has, defining $G_k[\phi] = \left(\Gamma_k^{(2)}[\phi] + R_k\right)^{-1}$

$$\boxed{\begin{aligned} &\partial_t \Gamma_k[\phi] = \frac{1}{2} Tr\left(G_k[\phi]\partial_t R_k\right) \\ &\partial_t \Gamma_k^{(1)}[\phi] = -\frac{1}{2} Tr\left(G_k[\phi]\Gamma_k^{(3)}[\phi]G_k[\phi]\partial_t R_k\right) \\ &\partial_t \Gamma_k^{(2)}[\phi] = Tr\left(G_k[\phi]\Gamma_k^{(3)}[\phi]G_k[\phi]\Gamma_k^{(3)}[\phi]G_k[\phi]\partial_t R_k\right) - \frac{1}{2} Tr\left(G_k[\phi]\Gamma_k^{(4)}[\phi]G_k[\phi]\partial_t R_k\right) \\ &\dots \end{aligned}}$$

Vertex expansion: e.g.

$$\Gamma_k[\phi] = \sum_{n=0}^{\infty} \int \mathrm{d}^D x_1 \cdots \mathrm{d}^D x_n \, \Gamma_k^{(n)}(x_1, \cdots, x_n)\, \phi(x_1)\cdots\phi(x_n)$$

Universal quantities

Critical quantities are encoded in the expansion coefficients describing the flow around the scale invariant fixed point:

$$\beta^i(g_*) = 0$$

Typically universal critical exponents are

- anomalous dimension η

- eigenvalues of the stability matrix or variational operators at functional level

$$M^i{}_j \equiv \left.\frac{\partial \beta^i}{\partial g^j}\right|_*$$

Popular regulators

$$x = \frac{q^2}{k^2}$$

$$W_k(q^2) = aZ_{0,k}\frac{k^2 x}{e^x - 1}$$

$$\Theta_k^m(q^2) = aZ_{0,k}k^2(1-x)^m\theta(1-x), m \in \mathbb{N}$$

$$E_k(q^2) = aZ_{0,k}k^2 e^{-x}$$

- Wetterich

- $m = 1$ corresponds to the optimized Litim's regulator

The parameter a was introduced to reduce the dependence of a critical exponent on the regulator according to the Principal of Minimal Sensitivity (PMS)

Connection between PMS and conformal invariance

De Polsi, Balog, Tissier, Wschebor (2020)
Ibañez, Tissier, De Polsi (2026)

See De Polsi 's talk

Recent 1-paramenter systematic study

Balog, Chaté, Delamotte, Marohnić, Wschebor (2019)

Analysis in the derivative expansion for the Ising universality class in $D=3$ have been pushed up to order DE6 (with some approximations), solving the system of differential equations.

Different regulators were used.

All with an overall normalization as a sigle parameter to be optimized according to PMS.

Optimization of critical exponents η and ν done independently.

TABLE III. Ising critical exponents in $d = 3$ obtained with various regulators. The numbers in parentheses give the distance of the results to the conformal bootstrap values taken here as the exact ones.

	regulator	ν	η
LPA	W	0.65059(2062)	0
	Θ^1	0.64956(1959)	0
	Θ^3	0.65003(2006)	0
	Θ^4	0.65020(2023)	0
	Θ^8	0.65056(2059)	0
	E	0.65103(2106)	0
$O(\partial^2)$	W	0.62779(218)	0.04500(870)
	Θ^2	0.62814(183)	0.04428(798)
	Θ^3	0.62802(195)	0.04454 (824)
	Θ^4	0.62793(204)	0.04474(844)
	Θ^8	0.62775(222)	0.04509(879)
	E	0.62752(245)	0.04551(921)
$O(\partial^4)$	W	0.63027(30)	0.03454(176)
	Θ^3	0.63014(17)	0.03507(123)
	Θ^4	0.63021(24)	0.03480(150)
	Θ^8	0.63036(39)	0.03426(204)
	E	0.63057(60)	0.03357(272)
$O(\partial^6)$	W	0.63017(20)	0.03581(49)
	Θ^4	0.63013(16)	0.03591(39)
	Θ^8	0.63012(15)	0.03610(20)
	E	0.63007(10)	0.03648(18)
conf. boot.		0.629971(4)	0.0362978(20)

Some works on optimization of regulator dependence

Long story… here are few selected references.

LPA: 1-par optimization with smooth regulators — Liao, Polonyi, Strickland (1999)

Litim's work on optimal criteria aimed at improving convergence. — Litim (2000)

Studies for DE2, PMS, 1-par — Canet, Delamotte, Mouhanna (2003)

General discussion — Pawlowski (2005)

LPA: Compactly Supported Smooth (CSS) regulators: — Márián, Jentschura, Nándori (2014)

Loops in $k-R_k$ space, measures of RG trajectories. — Pawlowski, Scherer, Schimdt, Wetzel (2016)

Multiparameter regulators

Here we shall focus on families of compact regulators, which can change their shape in the momentum dependence within a polynomial basis. We use a Chebyshev basis because of it good convergence properties in a space of compact functions.

Setting $Z_{0,k} = 1$ we consider regulators with a zero of order m at $x = 1$. The a p-parameter approximation of the regulators is

$$\Theta_p^m(x) \equiv -k^2(1-x)^{m-1} \sum_{i=1}^{p} a_i \left(T_i(2x-1) - 1 \right) \theta(1-x)$$

For $p = 1$ one recovers the form already used in the literature:

$$\Theta_1^m(x) = 2k^2 a_1 (1-x)^m \theta(1-x)$$

We compute numerically all the complicated threshold integrals using a gaussian quadrature not based on a Chebyshev grid because of the integrand behavior around the origin would give a poor convergence.

Polynomial expansion

In DE2 / DE4 (2 / 5 functions) we use a polynomial expansion around the origin starting from the minimal case of truncations up to dimension of operators $n=4$ / $n=6$ respectively in $D=3$.

From this basic truncation ($t=0$) we shall add to all functions new terms in the polynomials in ϕ^2 up to $t=t_{max}$ where convergence can be inferred.

$$\rho = \phi^2/2$$

$$V(\rho) = \sum_{i=0}^{n+t} v_{2i}\,\rho^i \qquad Z(\rho) = 1 + \sum_{i=1}^{n-3+t} z_{2i}\,\rho^i$$

DE2 case

$$W^a(\rho) = \sum_{i=0}^{n-5+t} w^a_{2i}\,\rho^i \qquad W^b(\rho) = \sum_{i=0}^{n-6+t} w^b_{2i}\,\rho^i \qquad W^c(\rho) = \sum_{i=0}^{n-6+t} w^c_{2i}\,\rho^i$$

DE4 case

As usual one works with all rescaled dimensionless field (φ) and functions and their flow equations. η is extracted from the (rescaled) fixed point equation for $\beta_z(\varphi=0) = 0$ together with the β functions of all the couplings.

Optimization of η according to PMS

In general the dependence on the parameters a inside R_k leads to

$$\partial_t g_i = \beta_i(g,\eta;a) \qquad \eta(g;a)$$

$$g \equiv (g_i) \quad i = 1,\cdots,N$$

$$a \equiv (a_i) \quad i = 1,\cdots,p$$

At fixed point $\beta_i(g,\eta;a)=0$ one has implicitly $g = g_*(a)$ and $\eta = \eta_*(g_*;a)$

Path 1:

$$\text{PMS:} \quad 0 = \left[\frac{d\eta}{da_i}\right]_* = \left[\frac{dg_j}{da_i}\partial_{g_j}\eta + \partial_{a_i}\eta\right]_*$$

$$0 = \left[\frac{d\beta_j(g,\eta;a)}{da_i}\right]_* = \left[\frac{dg_{*l}}{da_i}\partial_{g_l}\beta_j + \cancel{\frac{d\eta}{da_i}}\partial_\eta g_j + \partial_{a_i} g_j\right]_*$$

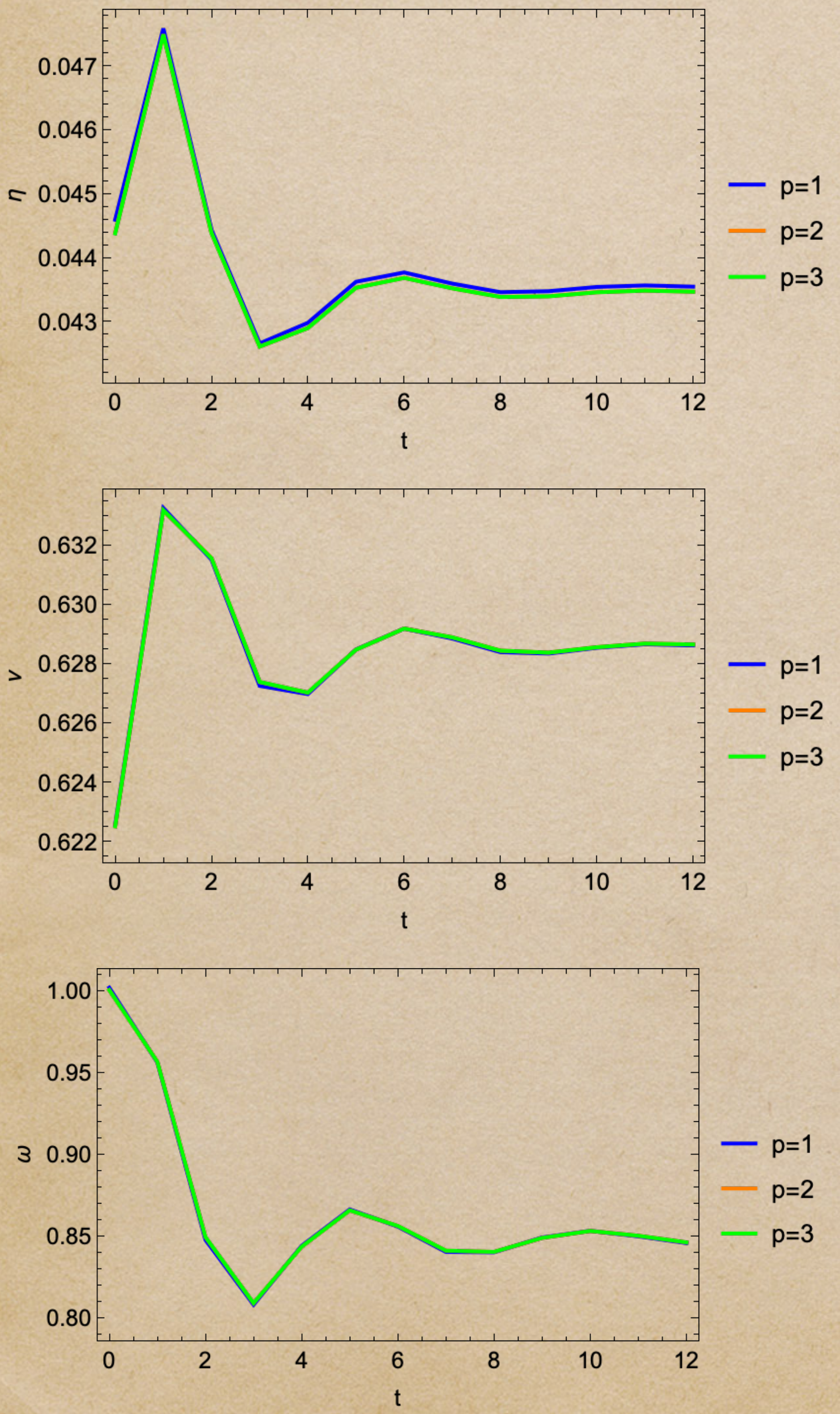
$(N+1)(p+1)-1$ equations for the unknowns $\quad g_{*j}\,, \bar{a}_i\,, \left[\frac{dg_{*j}}{da_i}\right]_{a_i=\bar{a}_i}$

Path 2: preferred numerical optimization at hand!

DE2 (minimum), DE4 (maximum), DE6 (minimum)

Results for DE2

$\Theta^2_{k,p}$



The change with p in the critical exponents is small. For η, $p_1 \to p_3$: change of ~0.18%.

$$t = 12$$

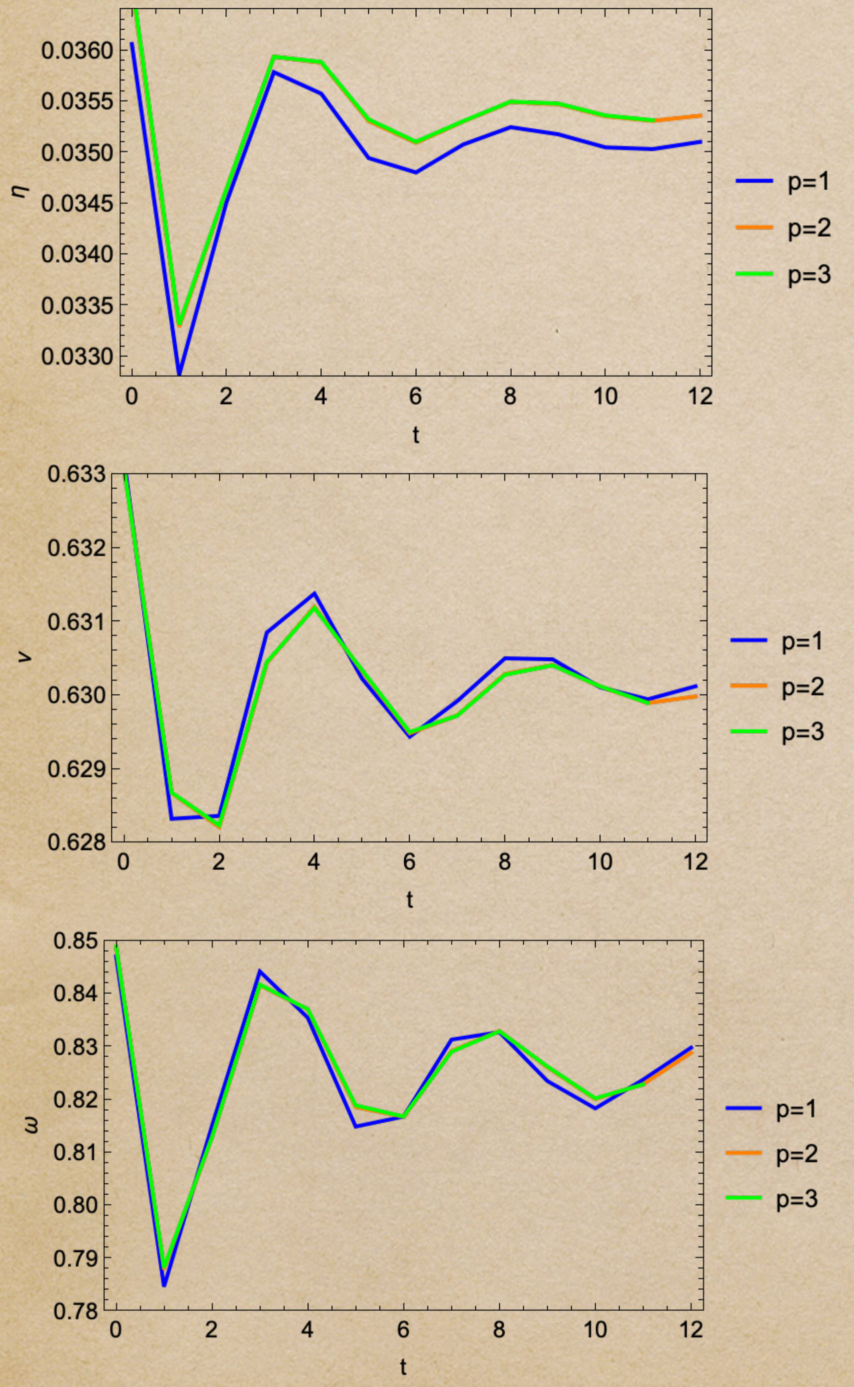
p	1	2	3
η	0.043543	0.043466	0.043464
ν	0.62861	0.62864	0.62864
ω	0.84561	0.84600	0.84604

This was expected, for a truncation which depends at maximum on 2-derivative operators, since $p > 1$ controls higher orders.

For p large enough essentially same results using $\Theta_{k,p}$ and $\Theta^2_{k,p}$

Results for DE4

$\Theta^3_{k,p}$



Curves at $p = 2,3$ essentially overlap.

The change in η with p is appreciable!
$p_1 \to p_3$: change of ~0.81%.

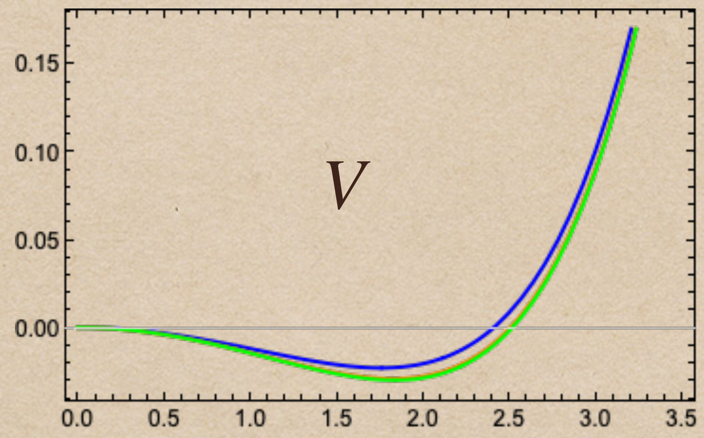
One can estimate the asymptotic values of the critical exponents for $t \to \infty$
Methods: local harmonic center, fit of dumped oscillators, autoregressive predictions/recurrences.
They all essentially agree.

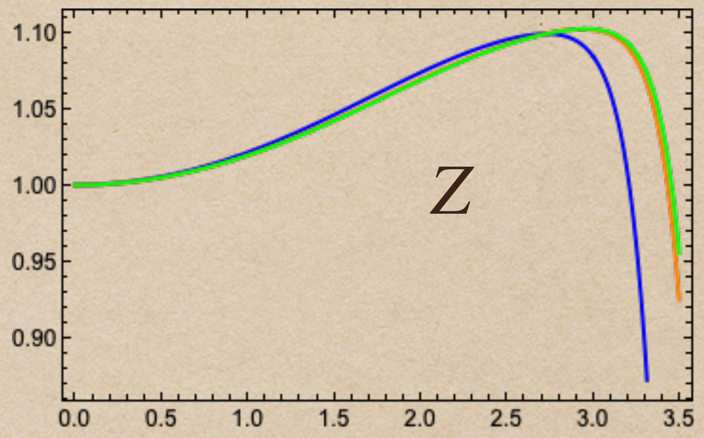
p	1	2	3
η	0.035096	0.035372	0.035381
ν	0.63017	0.63008	0.63009
ω	0.8251	0.8257	0.8259

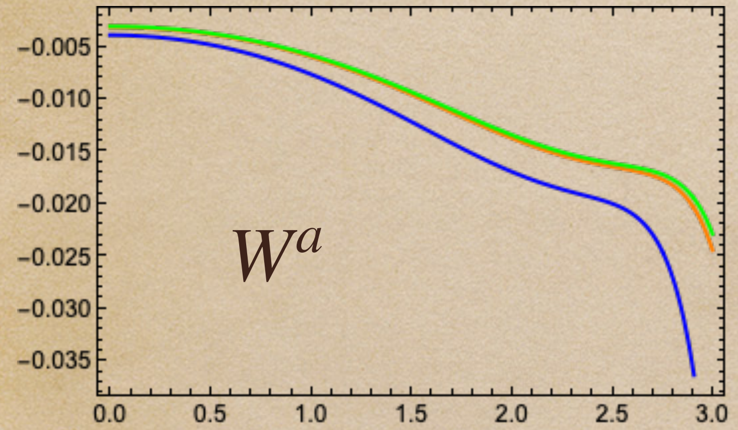
Moving from $p=1$ to $p=3$ the distance from the CB values of (η, ν, ω) drops of (24%, 40%, 18%).

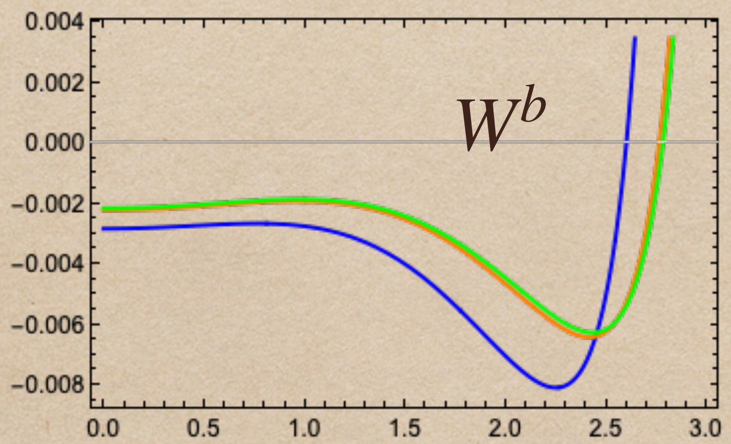
P-dependence in truncated functions

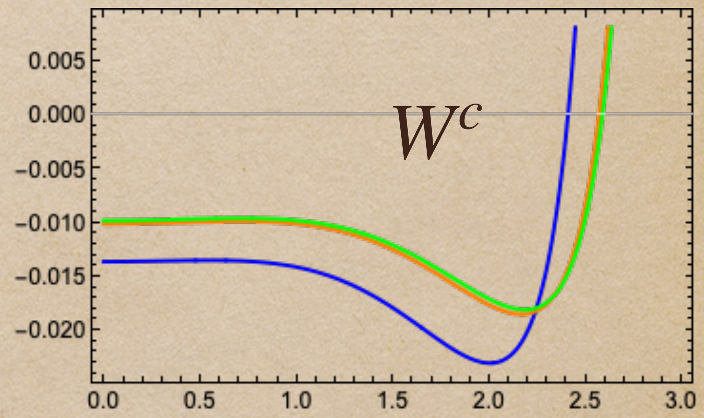
DE4 case: $t=10$, $p = 1,2,3$







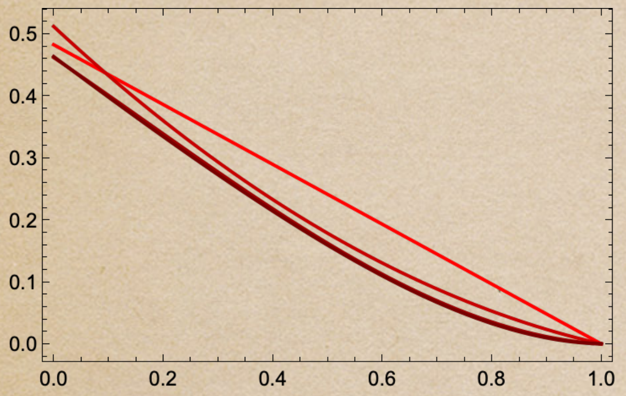




Moving to higher p, apart from deforming the shapes, could slightly increase the region of “convergence”.

Comparing optimized p-regulators (DE2)

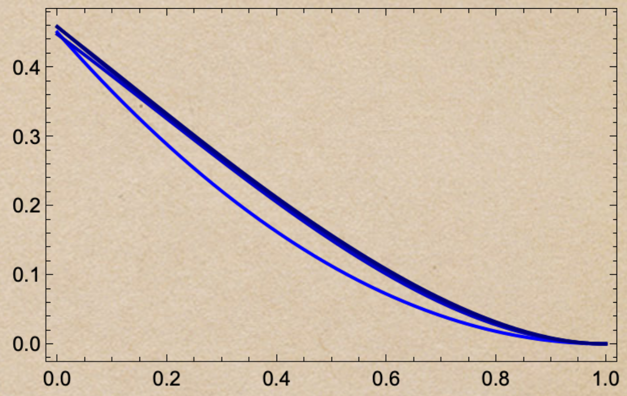
$\Theta_{k,p}$

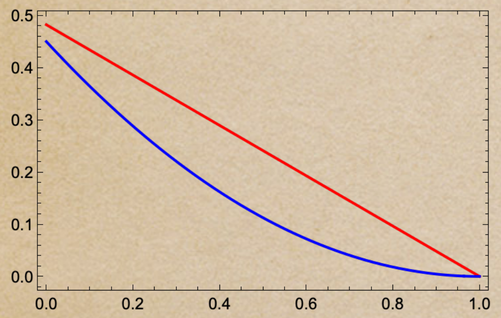


Fixed t , $p = 1,2,3,4$

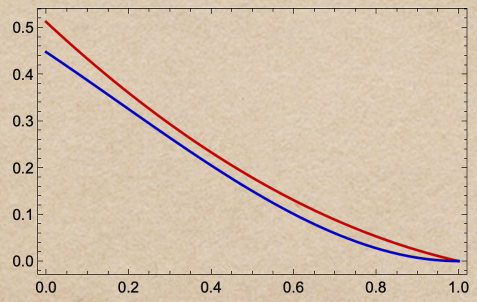
Grater p with darker color

$\Theta^2_{k,p}$

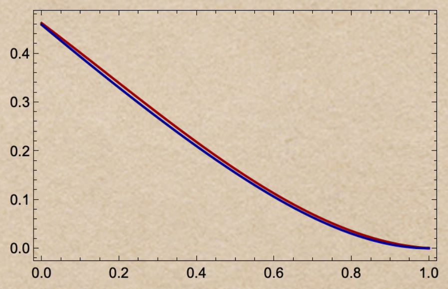




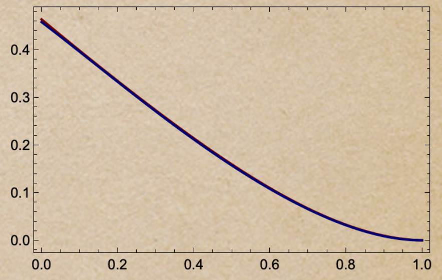
$p = 1$



$p = 2$



$p = 3$

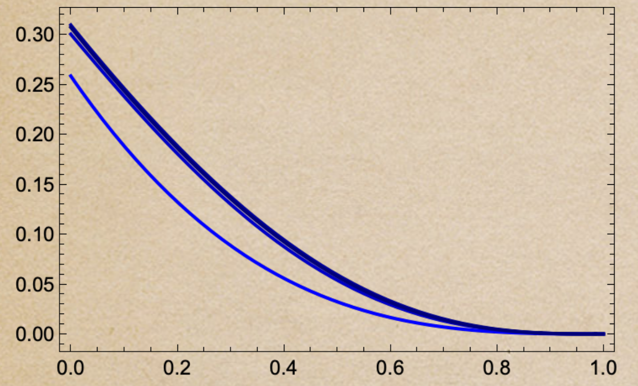


$p = 4$

Different families of regulators get closer and closer with increasing p
Critical exponents from the two regulators becomes almost indistinguishable with increasing p

Comparing optimized p-regulators (DE4)

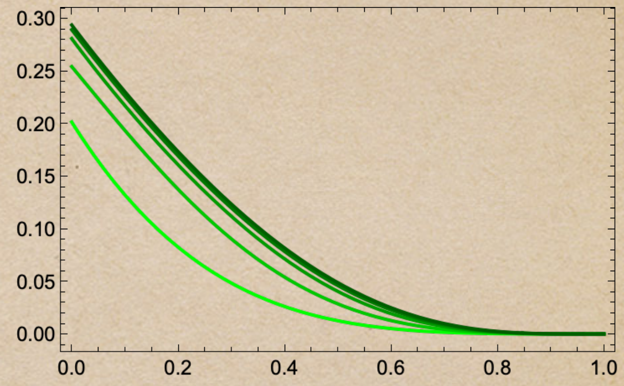
$\Theta^3_{k,p}$

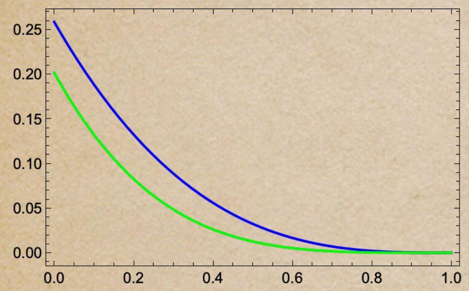


Fixed t , $p = 1,2,3,4,5$

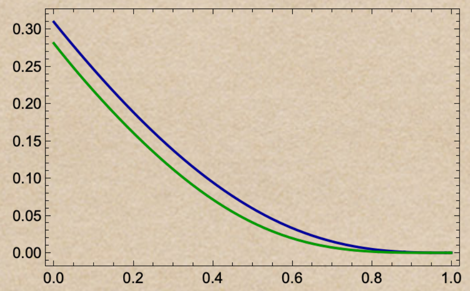
Grater p with darker color

$\Theta^4_{k,p}$

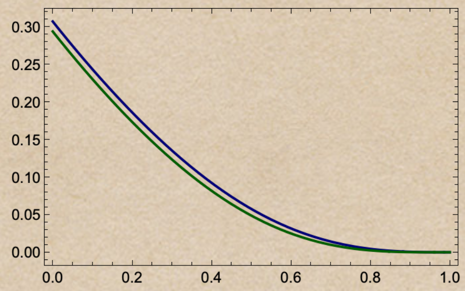




$p = 1$



$p = 3$



$p = 5$

$\Theta^3_{k,p}$, $\Theta^4_{k,p}$ regulators get closer with increasing p

$\Theta^2_{k,p}$ has a special behavior!

Optimization $\Rightarrow$ $\Theta^2_{k,2} = \Theta^3_{k,1}$ $\Rightarrow$ $\Theta^2_{k,p+1} = \Theta^3_{k,p}$

In DE4 the “delta-like” terms multiply a coefficient whose optimal value is zero within the precision used.

Conclusions

- Focus on minimizing the regulator dependence in FRG fixed-point solutions by applying in a multi parameter scenario the principle of minimal sensitivity to a critical exponent. In particular to η associated to the field anomalous dimension.

- Considered here families of compact regulators in a polynomial basis.

- Tested on the 3D Ising UC, allowing #(parameters)>1 in DE truncations:

  DE2 $\longrightarrow$ we see little improvement
  small changes in regulator shape

  DE4 $\longrightarrow$ visibly alters the regulator shape and shifts η by ~1% towards CB

- In DE4 studied with $\Theta^3_{k,p}$ 2-3 parameters seem enough to provide most of the improvement. Clearly this is a figure which depends on the precision required.

- In general the functional shape of regulators in different function spaces tend to almost overlap after multi parameter optimization.

Paper to appear soon in the arXiv.

Outlook

- Applications to the study of different critical QFT and Universality classes
- Families of multi parameter non compact regulators can be also studied.
- A better quantitative control of PMS, as shown, should improve the analysis of its connection with conformal invariance.

  On the other hand, what about critical theories not conformal?
  It would be natural to think that the PMS on physical observable
  could still be a guiding idea… and requires specific less trivial testings
- Go to the next step in the derivative expansion, but at DE6 it looks hard!

Thank you