New approach to OPE coefficients from the FRG

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13th International Conference on the Exact Renormalization Group,

University of Sussex, Sept. 2nd 2026

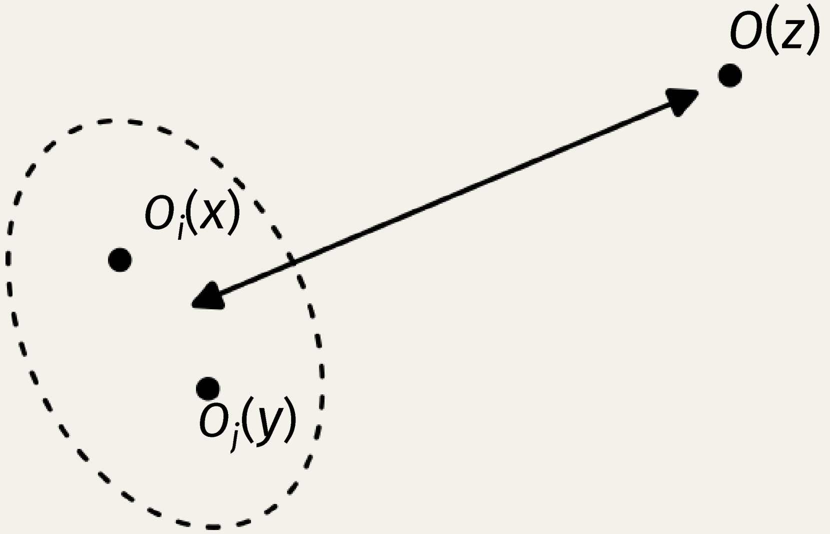
Introduction : the OPE

UV divergences → product of operators singular at short distance.

e.g., scalar field: $\lim_{y\to x}\langle\varphi(x)\varphi(y)\rangle = \infty \longrightarrow \varphi(x)\varphi(y)|_{y\to x} \neq \varphi(x)^2$.

Operator product expansion (OPE):

$$\text{for } y \to x, \quad \boxed{O_i(x)O_j(y) = \sum_k \underbrace{f^{ijk}(x-y)}_{\text{c number}} \overbrace{O_k(x)}^{\text{local operators}}}$$



- Sum over all local operators O_k;
- Singularities included in f_{ijk}: Wilson coefficients;
- Valid when inserted into correlation functions.

[Wilson '69, Kadanoff '70]

Applications

- Verified to all orders in perturbation and in conformal field theories.

  [Zimmerman, Ann Phys '73]

- Applied to

  - RG theory; perturbative RG formulation; QCD;

    [Novikov et al., PR '78]

  - Cold gases: thermodynamic relations for 3*d* interacting fermions.

Tan contact: $\dfrac{\partial\langle\hat{H}\rangle}{\partial(1/a)} \propto C.$ Related to singular HF tail of momentum distribution $\rho(\mathbf{k}) \overset{k\to\infty}{\sim} C/k^4$

$\to$ appears in the OPE expansion of $\hat{\psi}^\dagger(-\mathbf{r}/2)\hat{\psi}(\mathbf{r}/2)$.

[Braaten & Platter, PRL '08]

Conformal field theories

- Conformal field theories (CFT): OPE = basis for conformal bootstrap (CB).

[Poland et al., RMP '19]

- CFT: invariant under transforms that preserve angles.

(= scale transforms + special conformal transform)

$$\langle O_i(x) O_j(y)\rangle = \frac{\delta_{ij}}{|x-y|^{2\Delta_i}},$$
$$\langle O_i(x_1) O_j(x_2) O_k(x_3)\rangle = \frac{c_{ijk}}{x_{12}^{\Delta_i+\Delta_j-\Delta_k}\, x_{23}^{\Delta_j+\Delta_k-\Delta_i}\, x_{13}^{\Delta_i+\Delta_k-\Delta_j}}.$$

- Δ_i: scaling dimension.
- $x_{12} = |x_1 - x_2|$.
- c_{ijk}: OPE coefficient.

OPE in CFTs

$$O_i(x) O_j(y) = \sum_k \frac{c_{ijk}}{|x-y|^{\Delta_i+\Delta_j-\Delta_k}} O_k(x) + \text{spinful fields}.$$

True for all d.

[Di Francesco, Mathieu, Sénéchal, CFT, Springer]

Motivation

Near continuous phase transitions: OPE coefficients = universal numbers.

Functional RG (FRG):
→ universal quantities.

- Critical exponents;
  [De Polsi et al., PRE ’20 & many more]
- Amplitude ratios;
  [Berges et al., PR ’02; Rançon et al., PRE ’13; De Polsi et al., PRE ’21]
- Scaling functions.
  [e.g. Rose et al., PRB ’15; Rose and Dupuis PRB ’17]

Interest from field theory community:

- CB
  [Kos et al. JHEP ‘16; Cappeli et al. JHEP ‘19]
- Fuzzy sphere
  [Zhu et al. PRX ’23; Hu et al. PRL ‘23]
- Monte-Carlo
  [Caselle et al. PRD ’15; Hasenbusch, PRB ’20]
- $4-\varepsilon$ expansion
  [Dey et al., JHEP ’17; Carmi et al., SciPost ’21]

Our goal: determine OPE coefficients in critical O(N) theories.

The O(N) model

Similar to φ^4 theory. φ: N-component real field.

$$S[\boldsymbol{\varphi}] = \int \mathrm{d}^d x \left\{ \frac{1}{2} \left(\partial_\mu \boldsymbol{\varphi}\right)^2 + r_0 \boldsymbol{\varphi}^2 + u_0 (\boldsymbol{\varphi}^2)^2 \right\}$$

Phase transition controlled by the Wilson-Fisher fixed point for $d<4$.

- N=1, 2, 3: universality classes of physical systems (Ising, XY, Heisenberg).
- $N=\infty$: exact results.

At the phase transition: emergent conformal invariance!

Most relevant operators: $O_1 \sim \varphi$, $O_2 \sim \varphi^2$, $O_4 \sim \varphi^4$. $\Delta_1=(d-2+\eta)/2$, $\Delta_2=d-1/\nu$, $\Delta_4=d+\omega$.

FRG strategy

c_{abc} can be deduced from correlations [Pagani & Sonoda, PRD '20]

$$\text{for } |p_1| \gg |p_2|,\ \langle O_a(p_1) O_b(p_2) O_c(-p_1 - p_2)\rangle = \frac{c_{abc}\times\text{const.}}{|p_1|^{d-\Delta_b}|p_2|^{d-2\Delta_c}}$$

- **Strategy**: composite operators → add sources K_a, coupling $\int_x K_a O_a$

Effective action $\Gamma[\phi, K_a]$

E.g. for c_{11n}, n=2, 4, 6, …

OPE of ϕ_i, ϕ_i, and ϕ^n.

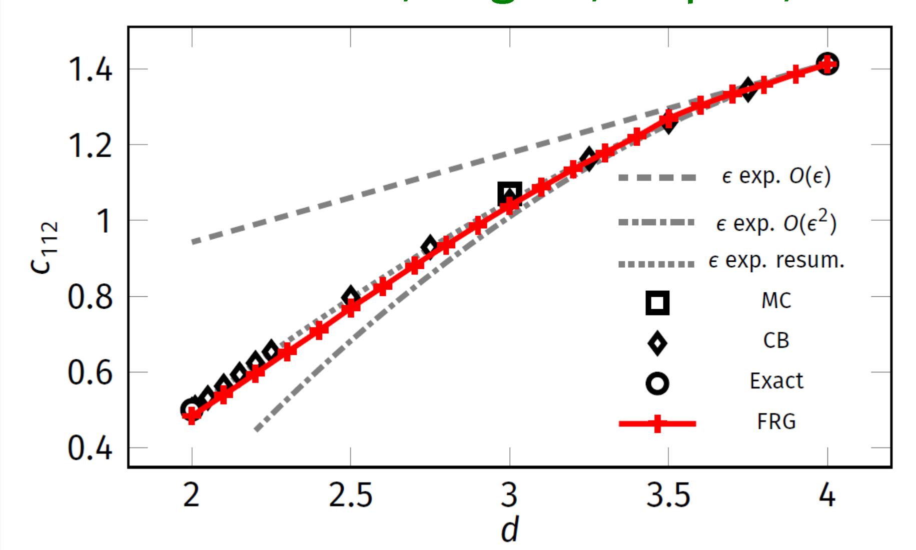
$$\boxed{c_{11n} = \text{const.} \times \lim_{p\to 0} \frac{\Gamma^{(2,1)}_{iin}(p,0)}{|p|^{\Delta_n - 2\Delta_1}}}$$

Momentum dependency
⇨ BMW scheme

Vertex: $\Gamma^{(2,1)}_{iin} = \delta^3\Gamma/\delta\phi_i\delta\phi_i\delta K_n$

Success and shortcomings

[Rose, Pagani, Dupuis, PRD '22]



- c_{112} determined for Ising ($2 \le d \le 4$) and 3*d* O(*N*) for all *N*.
- Limited success! Cannot be applied to more complicated operators...

[Hu, He, Zu, PRL '23]

TABLE I. List of OPE coefficients of primary operators obtained on the fuzzy sphere. The primary operators in consideration have scaling dimensions $\Delta_\sigma \approx 0.51815$, $\Delta_\epsilon \approx 1.4126$, $\Delta_{\epsilon'} \approx 3.8297$, $\Delta_{\sigma'} \approx 5.2906$, $\Delta_{T_{\mu\nu}} = 3$, and $\Delta_{\sigma_{\mu\nu}} \approx 4.1803$. The conformal bootstrap (CB) data is from Ref. [34], where some of unavailable data is labeled by 'NA'. We note that our convention for $f_{T_{\mu\nu}\epsilon T_{\rho\eta}}$ is $f_{T_{\mu\nu}\epsilon T_{\rho\eta}} = \frac{1}{\sqrt{4\pi}} \int d\mathbf{\Omega}\, \langle T_{\mu\nu}, m = 0|\hat{\epsilon}(\mathbf{\Omega})|T_{\rho\eta}, m = 0\rangle$ [30].

Operators	Spin	Z_2	$f_{\alpha\beta\gamma}$ (Fuzzy Sphere)	$f_{\alpha\beta\gamma}$ (CB)
σ	0	$-$	$f_{\sigma\sigma\epsilon} \approx 1.0539(18)$	$f_{\sigma\sigma\epsilon} \approx 1.0519$
ϵ	0	$+$	$f_{\epsilon\epsilon\epsilon} \approx 1.5441(23)$	$f_{\epsilon\epsilon\epsilon} \approx 1.5324$
ϵ'	0	$+$	$f_{\sigma\sigma\epsilon'} \approx 0.0529(16)$	$f_{\sigma\sigma\epsilon'} \approx 0.0530$
			$f_{\epsilon\epsilon\epsilon'} \approx 1.566(68)$	$f_{\epsilon\epsilon\epsilon'} \approx 1.5360$
σ'	0	$-$	$f_{\sigma'\sigma\epsilon} \approx 0.0515(42)$	$f_{\sigma'\sigma\epsilon} \approx 0.0572$
			$f_{\sigma'\sigma\epsilon'} \approx 1.294(51)$	NA
			$f_{\sigma'\epsilon\sigma'} \approx 2.98(13)$	NA
$T_{\mu\nu}$	2	$+$	$f_{\sigma\sigma T} \approx 0.3248(35)$	$f_{\sigma\sigma T} \approx 0.3261$
			$f_{\sigma'\sigma T} \approx -0.00007(96)$	$f_{\sigma'\sigma T} = 0$
			$f_{\epsilon\epsilon T} \approx 0.8951(35)$	$f_{\epsilon\epsilon T} \approx 0.8892$
			$f_{T\epsilon T} \approx 0.8658(69)$	NA
$\sigma_{\mu\nu}$	2	$-$	$f_{\sigma\epsilon\sigma_{\mu\nu}} \approx 0.400(33)$	$f_{\sigma\epsilon\sigma_{\mu\nu}} \approx 0.3892$
			$f_{\sigma\epsilon'\sigma_{\mu\nu}} \approx 0.18256(69)$	NA

- ...contrary to Fuzzy sphere!

Issue with higher-order coefficients

Naively $O_2 \sim \varphi^2$, $O_4 \sim \varphi^4$.

Actual scaling operators = eigenperturbations at the fixed point.

Initial condition can be decomposed onto scaling operators:

$$\phi^4 \sim a_2 O_2 + a_4 O_4 + \ldots$$

$$\Gamma^{(2,1)}_{iin}(p,0) = \# c_{112} |p|^{\Delta_2 - 2\Delta_1} + \# c_{114} |p|^{\Delta_4 - 2\Delta_1} + \ldots$$

c_{114} always subleading !

Solution

[Delamotte, De Polsi, Tissier, Wschebor, PRE '24]
[Cabera, De Polsi, Wschebor, PRE '25]

Connection between

- Eigenperturbation near the fixed point $\Gamma^*[\phi, K_a=0]$

$$\gamma_k[\tilde{\phi}_a] = \Gamma_k\left[\tilde{\phi}_a, K=0\right] - \Gamma_*\left[\tilde{\phi}_a, K=0\right],$$

$$\gamma_k[\vec{\phi}] = \exp(\lambda t)\hat{\gamma}[\vec{\phi}],$$

- Fixed point equation for $\Gamma^{(0,1)}{}_a[\phi, K_a=0]$

$$\hat{\Gamma}[\vec{\phi}] = \int_x \Gamma_k^{(0,1)}\Big|_{K=0} = -\Big\langle \int_x \mathcal{O}(x)\Big\rangle_{J,K=0}.$$

Implementation

Two different fixed points:

- First, find RG fixed point (FP) $\Gamma^*[\phi, K_a=0]$
- Then, find 2nd FP for $\Gamma^{(0,1)}{}_a[\phi]$

(pyramidal structure of equations)
$\partial_t\Gamma^{(n,0)} \Rightarrow \partial_t\Gamma^{(n,1)} \Rightarrow \partial_t\Gamma^{(n,2)} \Rightarrow \ldots$

$$(D_{\mathcal{O}} - d)\hat{\Gamma}^*[\vec{\phi}] = -\int_x \hat{\Gamma}_a^{*(1)}[\vec{\phi};x]\big(D_\varphi + x_\mu\partial_\mu\big)\phi_a + \frac{1}{2}\mathrm{Tr}\big[\dot{R}_k G_k \cdot \hat{\Gamma}_k^{*(2)}[\vec{\phi}] \cdot G_k\big].$$

2nd FP only reached if sources couples to scaling operator!

Strategy

Once O_n is found (BMW), 2 and pseudo-3-point vertices determined.

$$\langle O_n(p) O_n(-p) \rangle \qquad \langle \phi_i(p) \phi_i(0) O_n(-p) \rangle$$

Correct FP?

FP root-finding algorithm ⇨ dynamics in theory space.

Start with good initial guess!

- DE: diagonalize stability matrix
- BMW: start from d=3,99... and lower d

Outline and perspectives

- Preliminary work: extension to O(N), odd and spinful operators, ...

- Long term: connection between conformal constraints and BMW?

Thank you for your attention!

Solution

[Delamotte, De Polsi, Tissier, Wschebor, PRE ‘24]
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Connection between

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$$\gamma_k[\vec{\phi}] = \exp(\lambda t)\hat{\gamma}[\vec{\phi}],$$

$$\partial_t \gamma_k[\vec{\phi}] + \int_x \gamma_{k,a}^{(1)}[\vec{\phi};x]\left(D_\varphi + x_\mu \partial_\mu\right)\phi_a(x) = \frac{1}{2}\mathrm{Tr}\left[\dot{R}_k G_k \cdot \gamma_k^{(2)}[\vec{\phi}] \cdot G_k\right],$$

- Fixed point equation for $\Gamma^{(0,1)}{}_a[\phi]$

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