Renormalization Group Approach

to the Gravitational Two-Body Problem

Facundo Gutiérrez • Alessandro Codello • Kevin Falls

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1 The Two-Body Problem in GR

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1. The Two-Body Problem in GR
2. Exact Renormalization Group Approach

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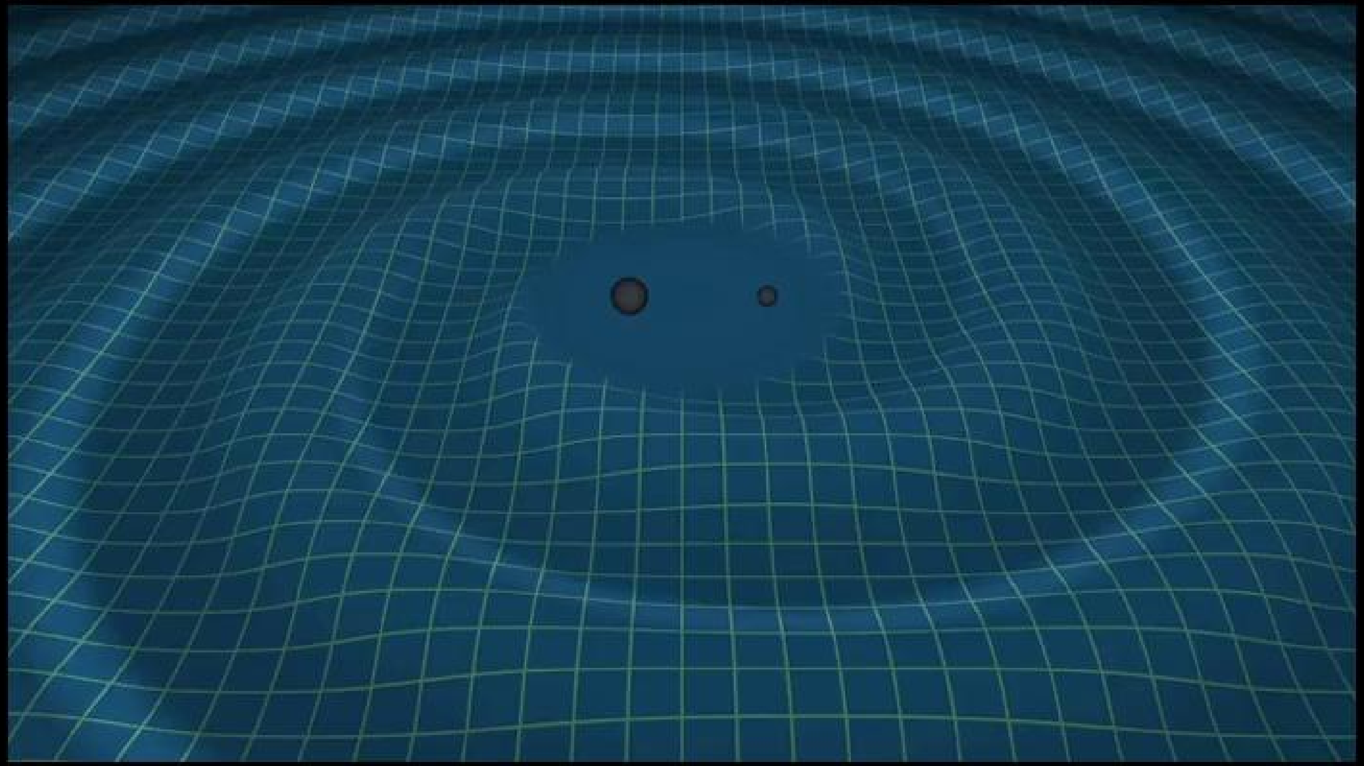
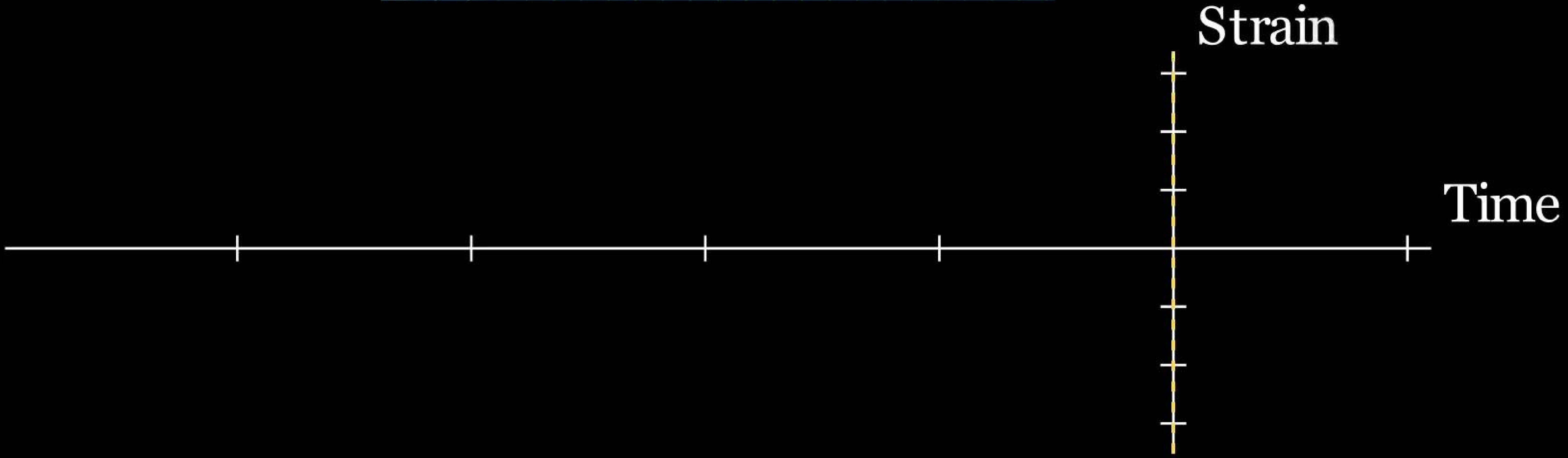
1. The Two-Body Problem in GR
2. Exact Renormalization Group Approach
3. Applications

Gravitational Waves





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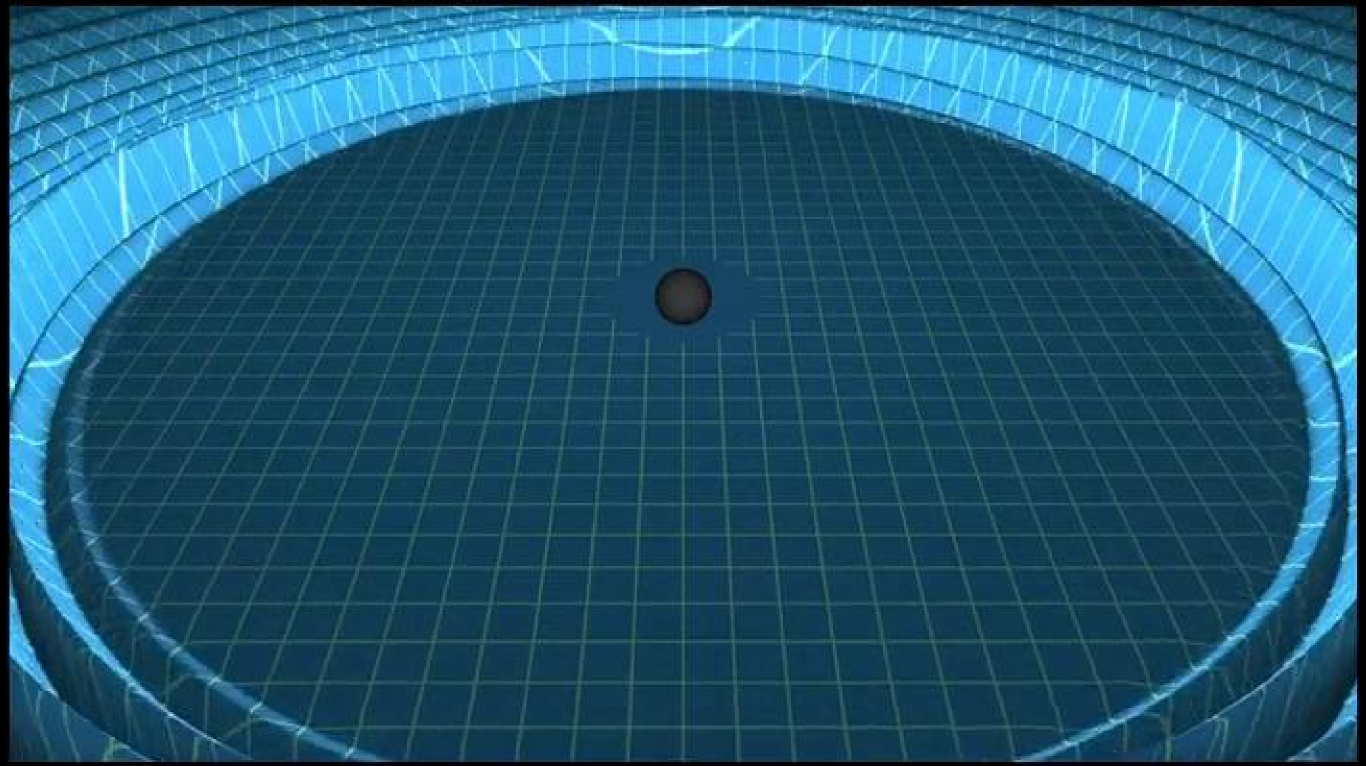

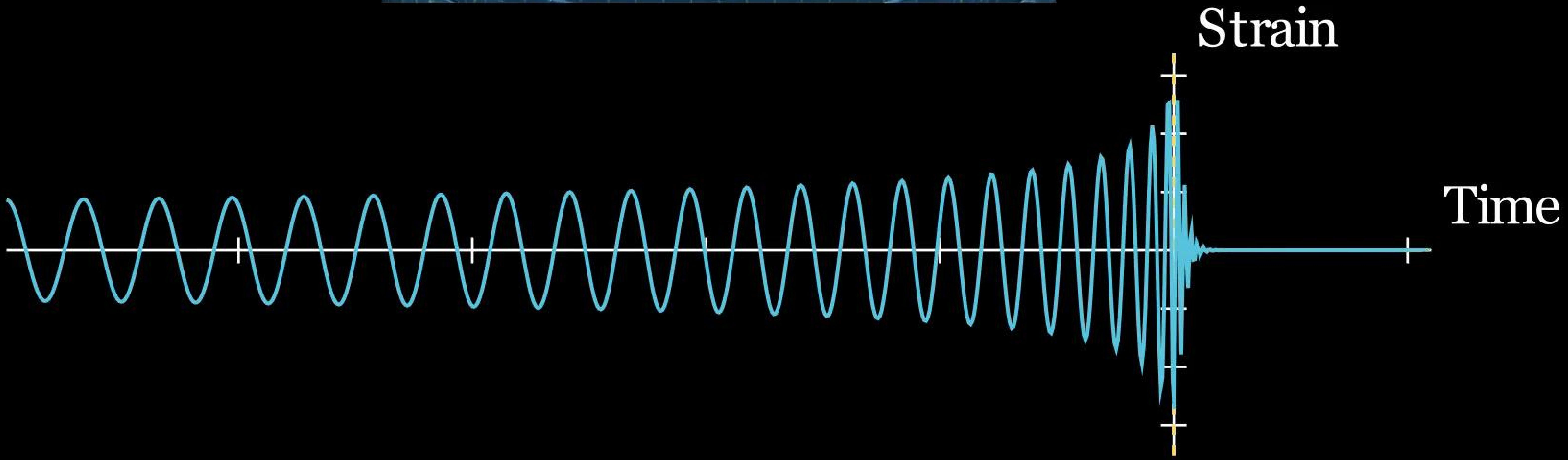

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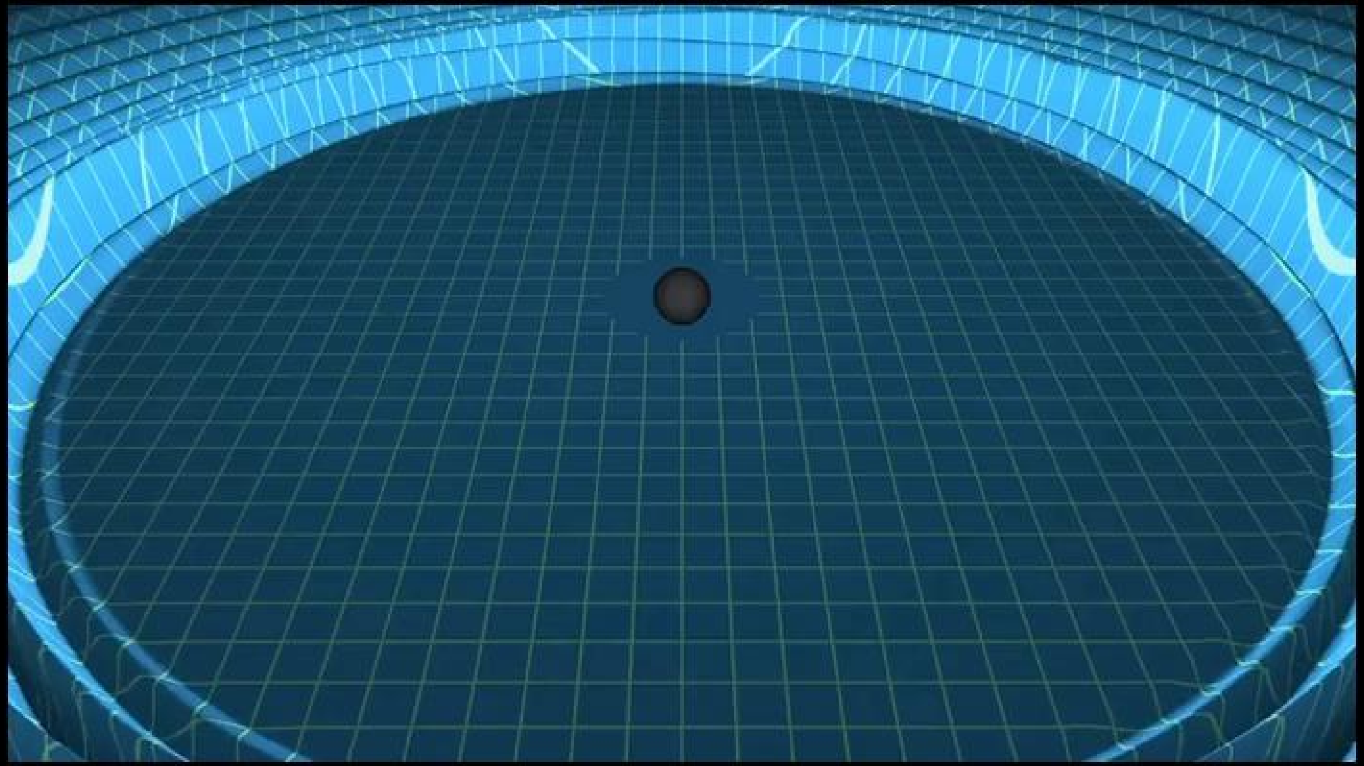

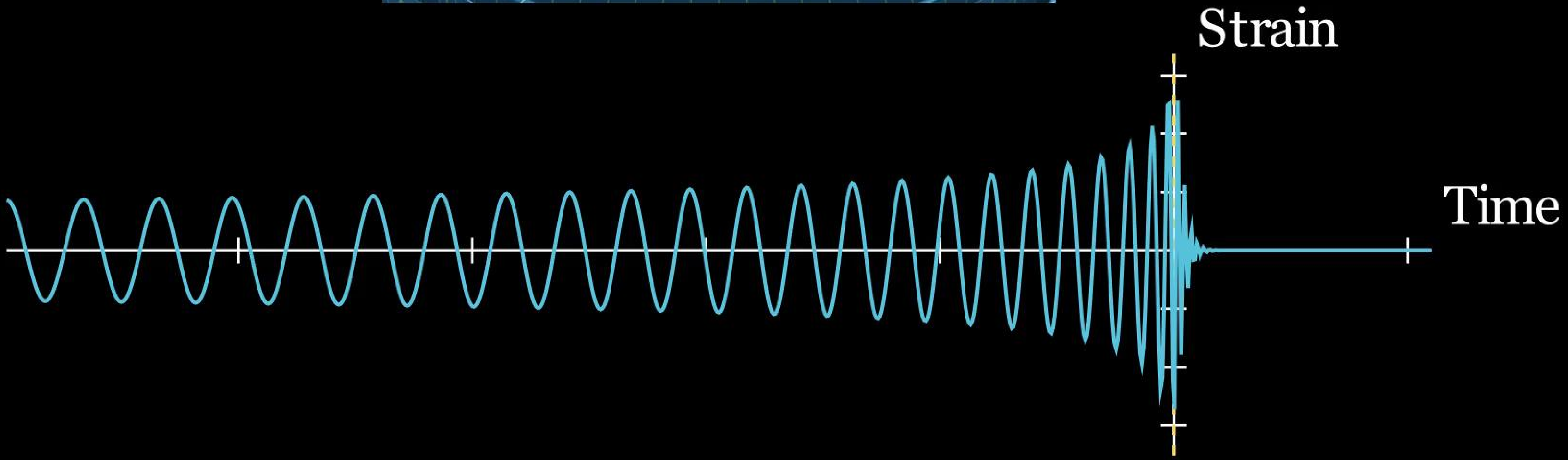

The Future of Gravitational Astronomy

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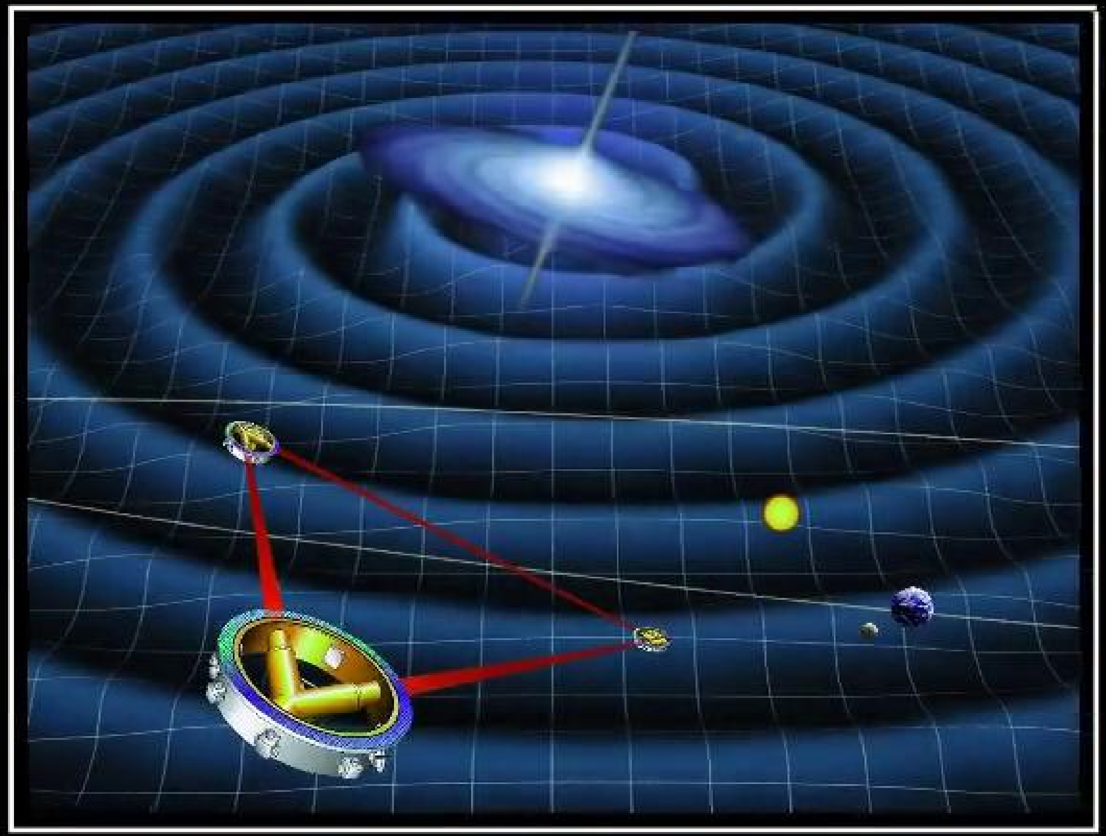

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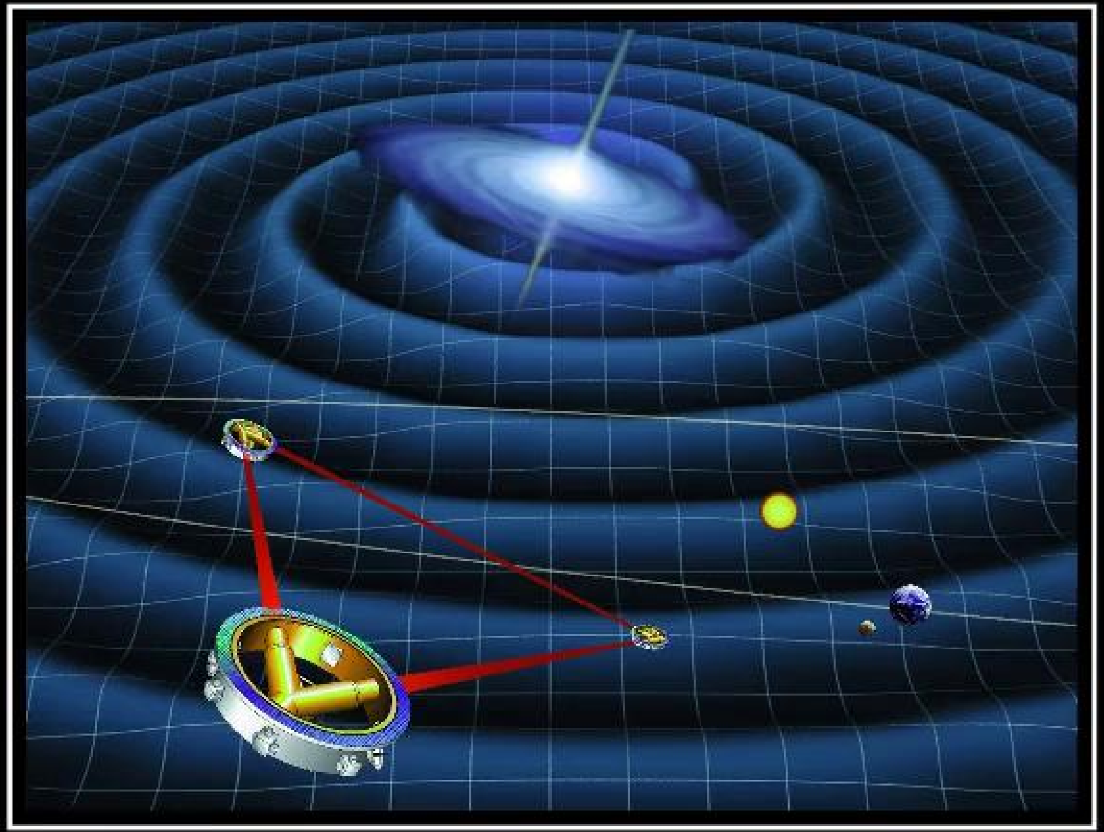

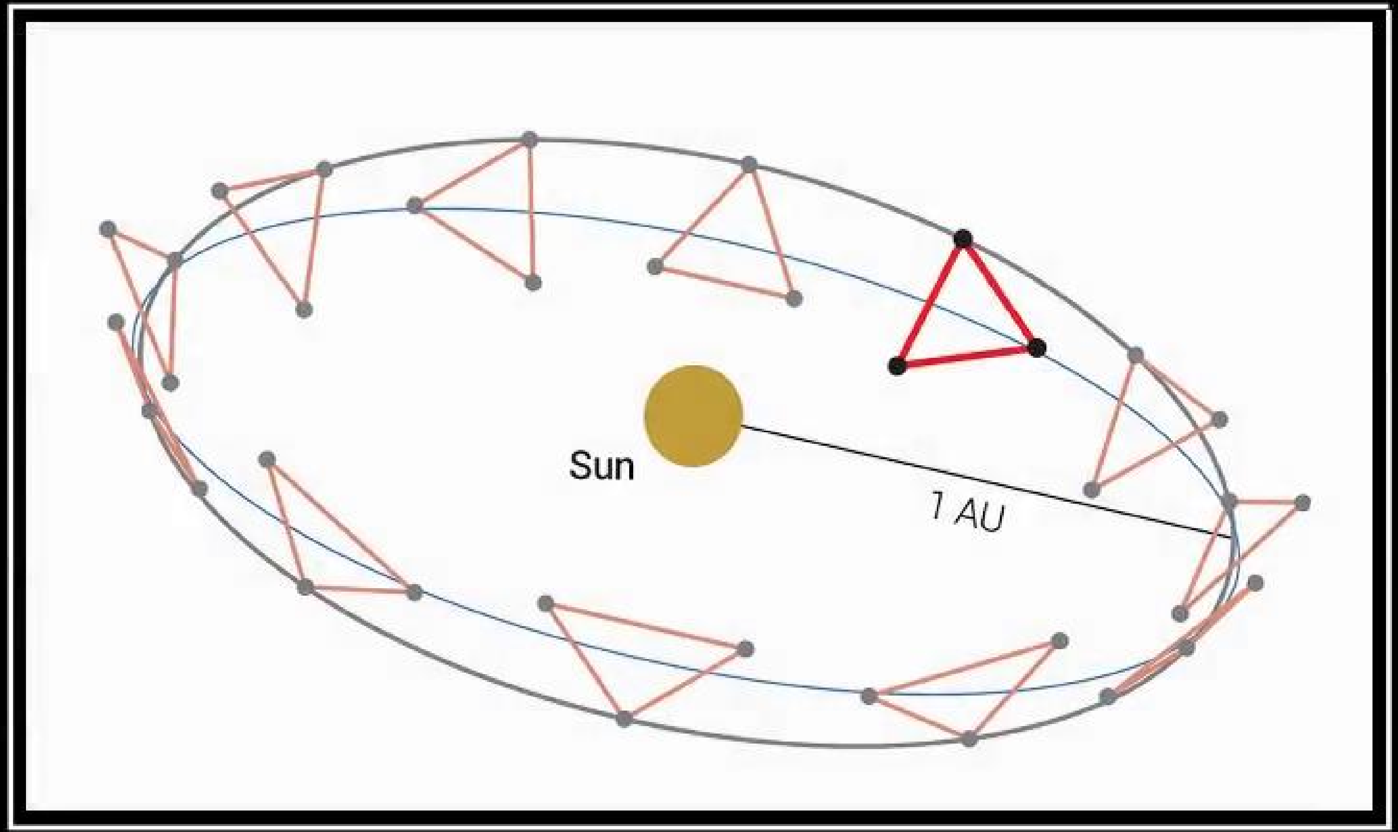

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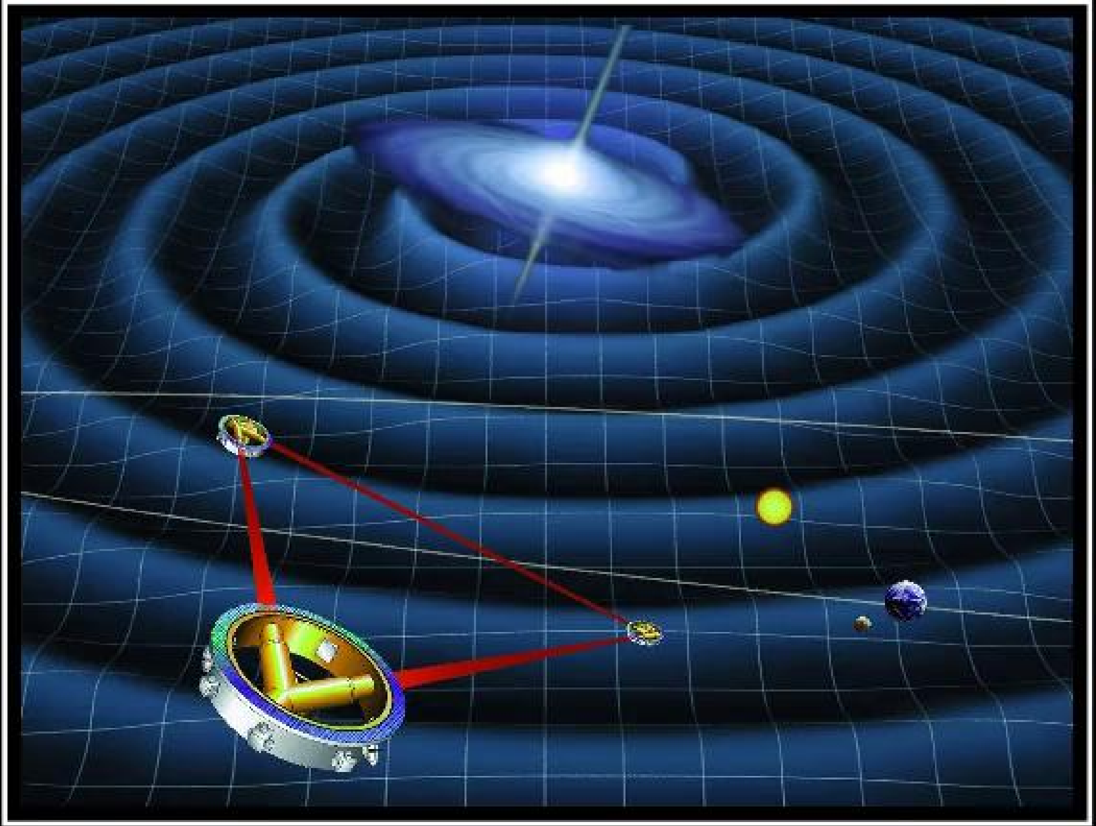
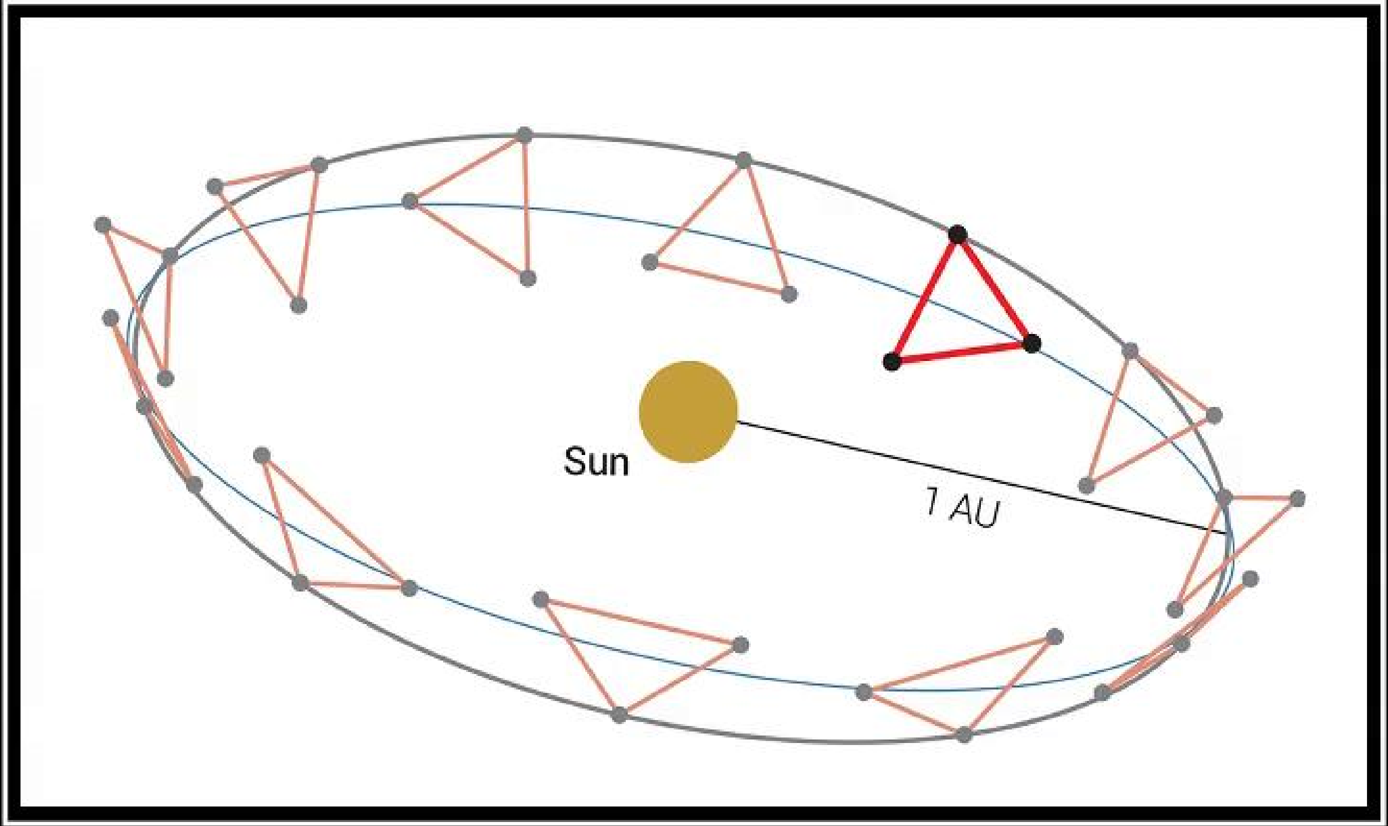
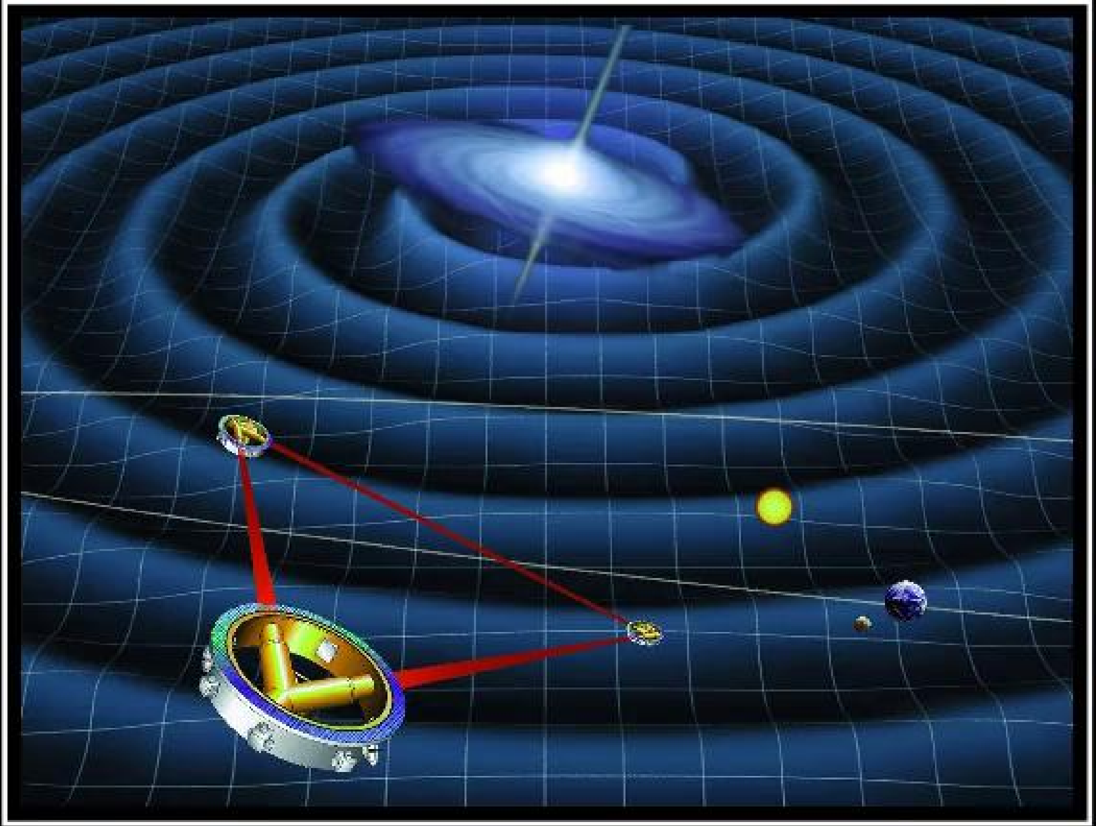
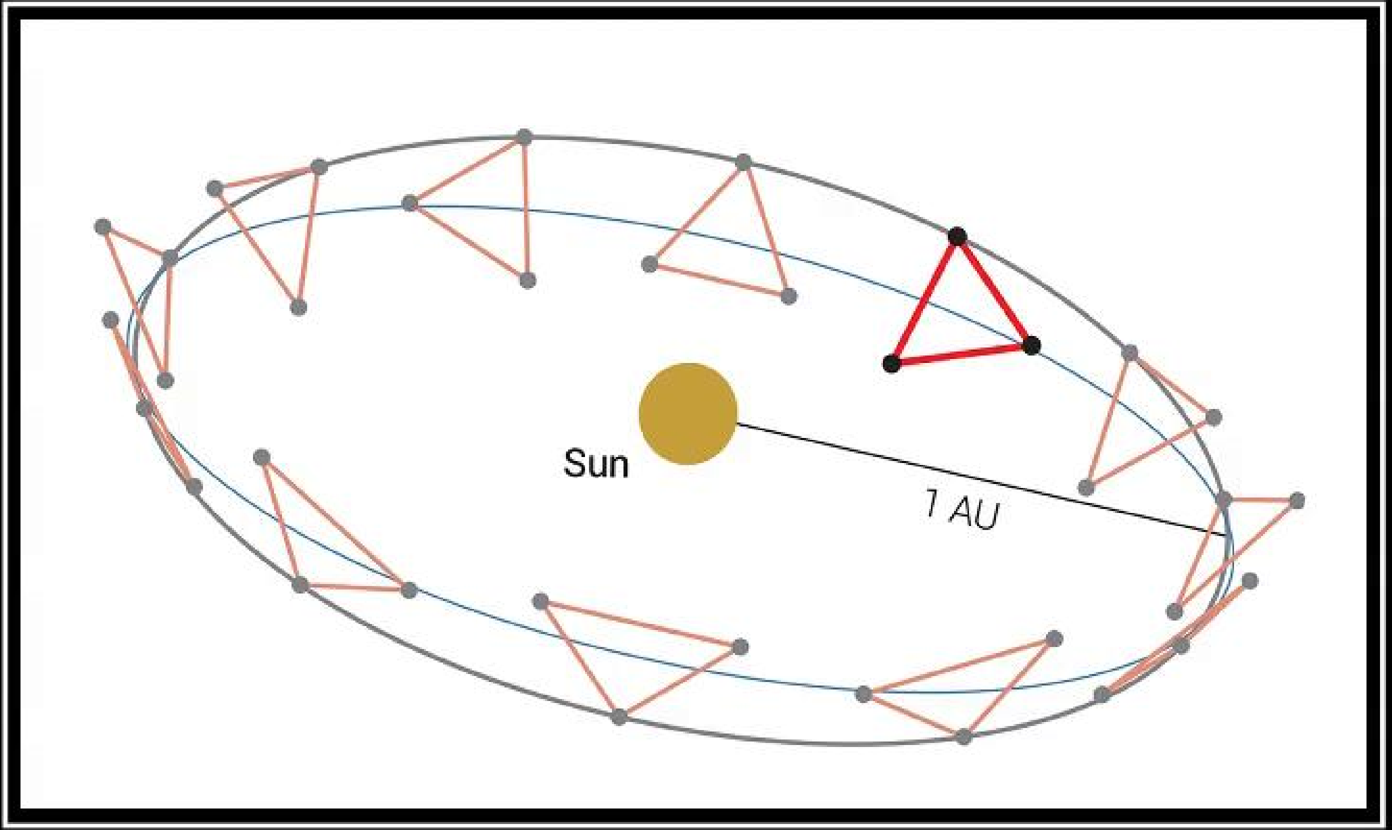




LISA Mission — 2035

The Future of Gravitational Astronomy





LISA Mission — 2035

Our starting point: the action

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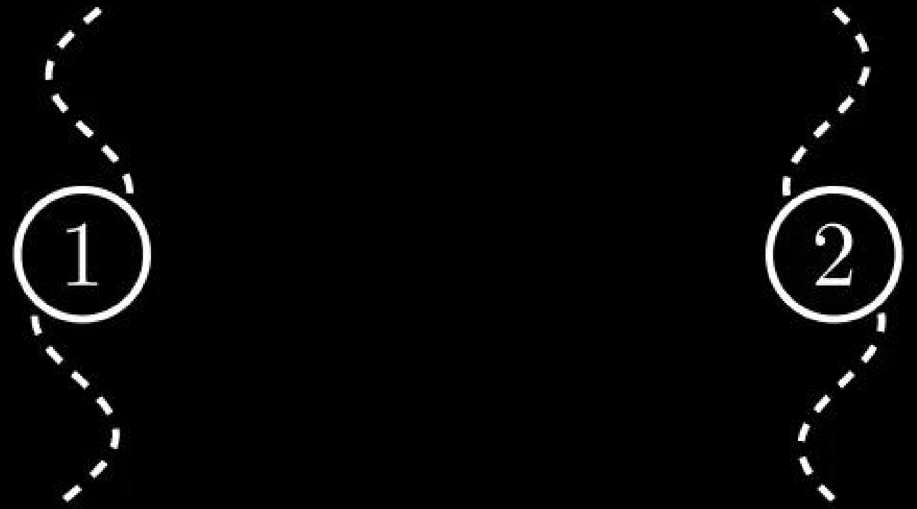
Trajectory 1 Trajectory 2



$x_1(t)$ $x_2(t)$

$$S[x_1, x_2]$$

Our starting point: the action

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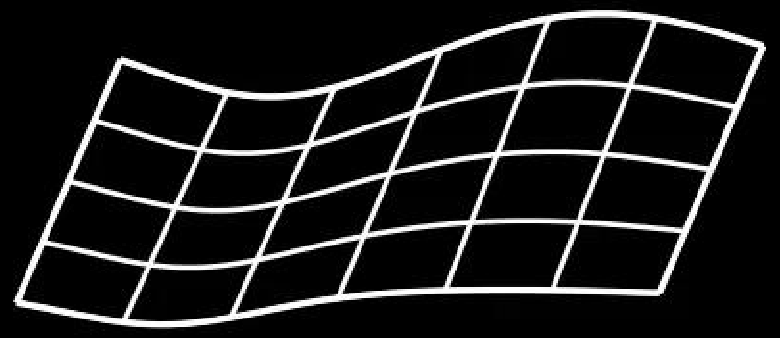
$x_1(t)$

Trajectory 2

$x_2(t)$

Spacetime



$g_{\mu\nu} \longrightarrow$ Metric

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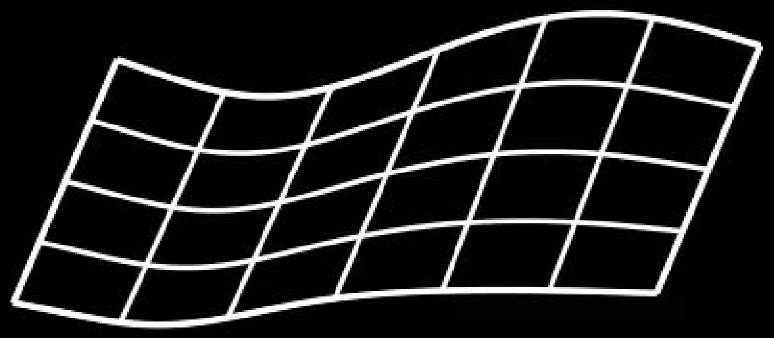

$g_{\mu\nu} \longrightarrow$ Metric

$$S_g[g] = \frac{c^4}{16\pi G}\int d^4x\,\sqrt{-g}\,R$$

$\downarrow$ Einstein-Hilbert action

$$S_{\rm pp}[g, x_1, x_2] = -\sum_{a=1,2} m_a c^2 \int dt\,\sqrt{-\frac{g_{\mu\nu}\dot{x}_a^\mu \dot{x}_a^\nu}{c^2}}$$

$\downarrow$ Point-particle action

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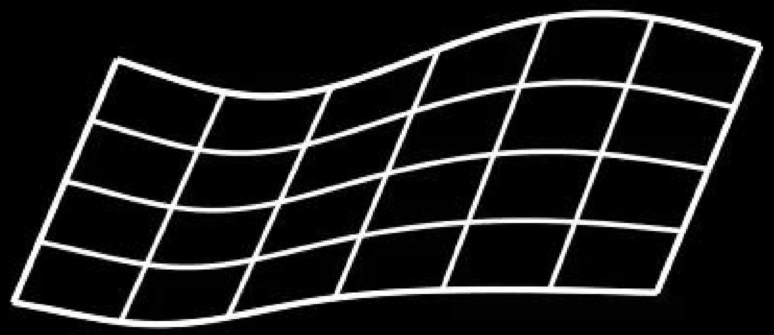

$g_{\mu\nu} \longrightarrow$ Metric

$$S[g, x_1, x_2] = \underbrace{\frac{c^4}{16\pi G}\int d^4x\,\sqrt{-g}\,R}_{\text{Einstein-Hilbert action}} - \underbrace{\sum_{a=1,2} m_a c^2 \int dt\,\sqrt{-\frac{g_{\mu\nu}\dot{x}_a^{\mu}\dot{x}_a^{\nu}}{c^2}}}_{\text{Point-particle action}}$$

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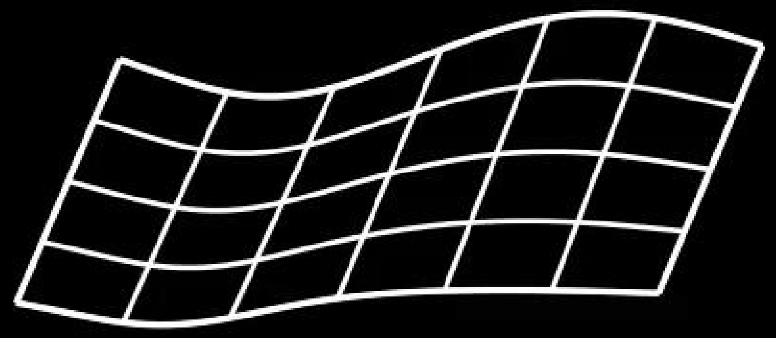

$g_{\mu\nu} \longrightarrow$ Metric

$$S[g, x_1, x_2] = \underbrace{\frac{1}{\kappa} S_g[g]}_{\text{Self-interaction of gravity}} + \underbrace{S_{\rm pp}[g, x_1, x_2]}_{\text{Interaction of particles with gravity}} \;, \qquad \kappa = 32\pi G$$

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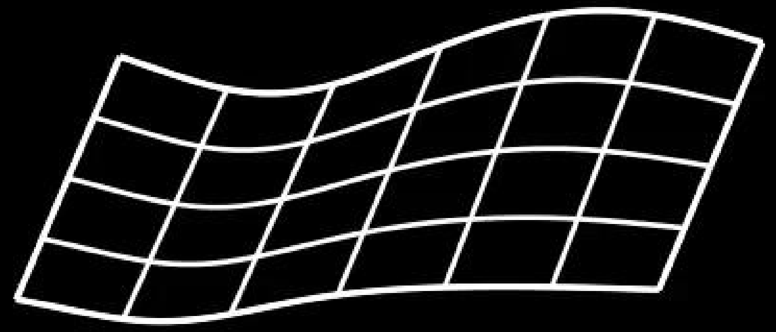

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Our goal: the effective action

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$$S_{\text{Newton}}[x_1, x_2] = \int \mathrm{d}t \, \left(\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 + \frac{G m_1 m_2}{r} \right)$$

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The metric does not appear

Our goal: the effective action

$$S_{\mathrm{tot}}[g, x_1, x_2] = \frac{c^4}{16\pi G}\int \mathrm{d}^4x\,\sqrt{-g}\,R - \sum_{a=1,2} m_a c^2 \int \mathrm{d}t\,\sqrt{-\frac{g_{\mu\nu}\dot{x}_a^\mu \dot{x}_a^\nu}{c^2}}$$

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We can switch to an action that accounts for curvature implicitly

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We can switch to an action that accounts for curvature implicitly

$$S_{\mathrm{eff}}[x_1, x_2] = \int \mathrm{d}t \, \left(\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 + \frac{G m_1 m_2}{r} \right) + \text{Corrections}$$

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$$S_{\mathrm{tot}}[g, x_1, x_2] = \frac{c^4}{16\pi G} \int \mathrm{d}^4x \, \sqrt{-g} \, R - \sum_{a=1,2} m_a c^2 \int \mathrm{d}t \, \sqrt{-\frac{g_{\mu\nu} \dot{x}_a^\mu \dot{x}_a^\nu}{c^2}}$$

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$$\downarrow$$

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$$\downarrow$$

Solution metric $\tilde{g}[x_1, x_2]$

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$$S_{\mathrm{eff}}[x_1, x_2] \equiv S[\tilde{g}, x_1, x_2]$$

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Post-Minkowskian expansion

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$$S[g, x_1, x_2] = \frac{1}{\kappa} S_g[g] + S_{\mathrm{pp}}[g, x_1, x_2]$$

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$$S_{\text{eff}} = \frac{1}{\kappa} S_{\text{eff}}^{-1} + S_{\text{eff}}^{0} + \kappa S_{\text{eff}}^{1} + \kappa^2 S_{\text{eff}}^{2} + \mathcal{O}(\kappa^3),$$

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$$\tilde{g} = \eta + \tilde{h} = \eta + \kappa \tilde{h}^1 + \kappa^2 \tilde{h}^2 + \kappa^3 \tilde{h}^3 + \cdots$$

Diagrammatic rules

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$$\left(\frac{\delta^2 S_g}{\delta g^a \delta g^b}[\eta]\right)^{-1} = a \text{ - - - - - - - - } b$$

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$$\Delta^{-1}_{ab} = a \text{ - - - - - - - - } b$$

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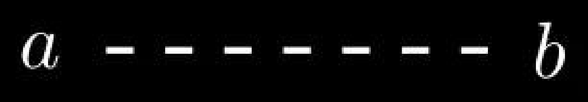
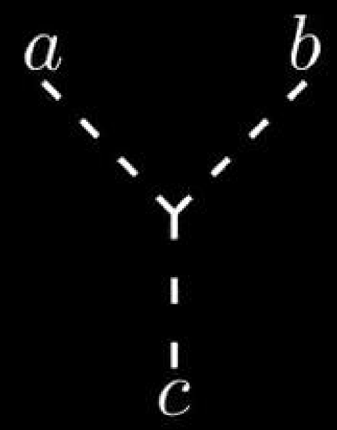
$\Delta_{ab}^{-1} =$ 

$S_g^{abc} =$ 

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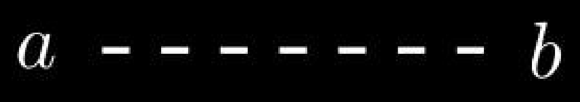
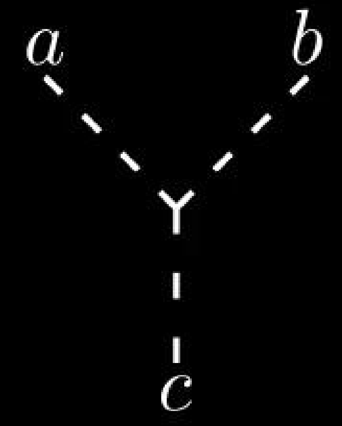
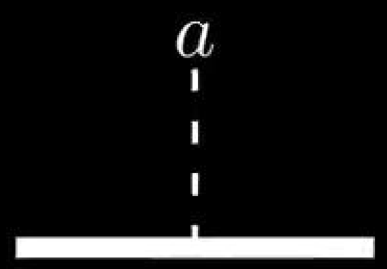
$\Delta_{ab}^{-1} =$ 

$S_g^{abc} =$ 

$S_{\mathrm{pp}}^{a} =$ 

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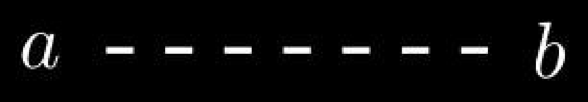
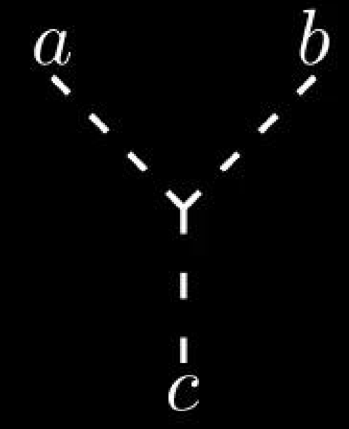
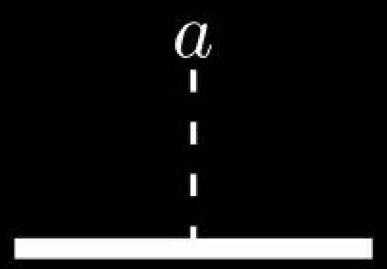
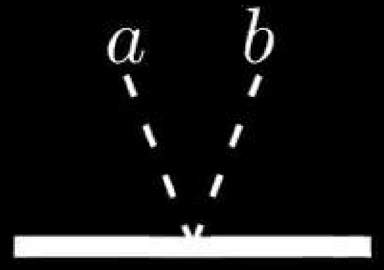
$\Delta_{ab}^{-1} =$ 

$S_g^{abc} =$ 

$S_{\mathrm{pp}}^{a} =$ 

$S_{\mathrm{pp}}^{ab} =$ 

Result at 2PM order

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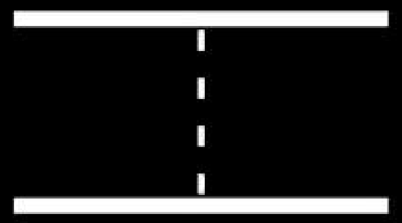
$$S_{\text{eff}} = S_{\text{pp}} - \frac{\kappa}{2}$$ 

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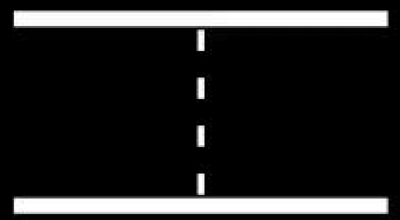
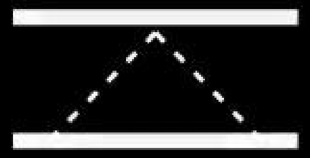
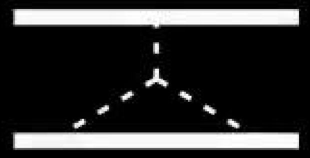
$S_{\text{eff}} = S_{\text{pp}} - \frac{\kappa}{2}$ 

$+\kappa^2 \Big(\frac{1}{2}$  $- \frac{1}{2}$  $\Big)$

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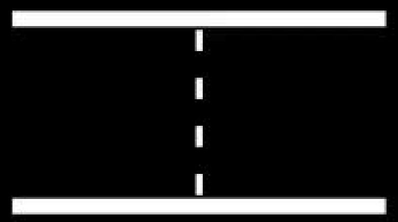
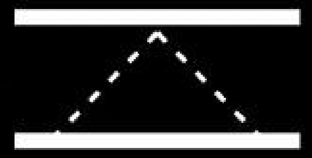
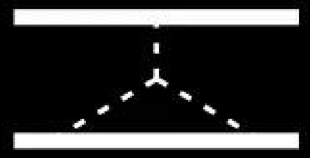
$S_{\text{eff}} = S_{\text{pp}} - \frac{\kappa}{2}$ 

$+\kappa^2 \Bigg(\frac{1}{2}$  $- \frac{1}{2}$  $\Bigg)$

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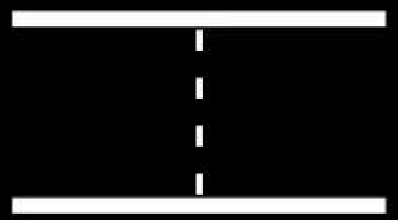
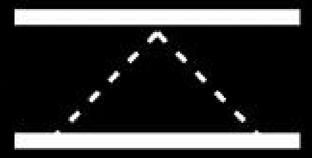
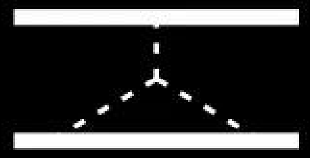
$S_{\text{eff}} = S_{\text{pp}} - \frac{\kappa}{2}$ 

$+\kappa^2 \Bigg(\frac{1}{2}$  $- \frac{1}{2}$  $\Bigg)$

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For a bound orbit, $\frac{v^2}{c^2} \sim \frac{GM}{c^2 R}$, so one must expand in both v and G.

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$$\boxed{\begin{aligned} S_{\text{eff}}^{\text{1PN}} = \int dt \Bigg\{ & \frac{1}{2}\left(m_1 \mathrm{v}_1^2 + m_2 \mathrm{v}_2^2\right) + \frac{G m_1 m_2}{R} \\ & + \frac{1}{c^2}\Bigg[\frac{1}{8}\left(m_1 \mathrm{v}_1^4 + m_2 \mathrm{v}_2^4\right) - \frac{1}{2}\,\frac{G^2 m_1 m_2 (m_1 + m_2)}{R^2} \\ & + \frac{G m_1 m_2}{R}\left(\frac{3}{2}\left(\mathrm{v}_1^2 + \mathrm{v}_2^2\right) - \frac{7}{2}\,\mathbf{v}_1 \cdot \mathbf{v}_2 - \frac{1}{2}\,(\mathbf{n}\cdot\mathbf{v}_1)(\mathbf{n}\cdot\mathbf{v}_2) \right)\Bigg]\Bigg\} + \mathcal{O}(c^{-4}) \end{aligned}}$$

Post-Newtonian expansion

For a bound orbit, $\frac{v^2}{c^2} \sim \frac{GM}{c^2 R}$, so one must expand in both v and G.

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$$S_{\text{eff}}^{\text{2PM}} \xrightarrow[\mathcal{O}(c^{-2})]{v/c \text{ expansion}} S_{\text{eff}}^{\text{1PN}}$$

$$\boxed{S_{\text{eff}} \supset \int dt \left(\frac{G m_1 m_2}{R} - \frac{1}{2} \frac{G^2 m_1 m_2 (m_1 + m_2)}{c^2 R^2} \right)}$$

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Extracting the complete 1PN action
from the 2PM result is technically challenging.

Classical RG

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CLASSICAL POLCHINSKI

$$\partial_k S_k[g, \mathrm{x}] = -\frac{\kappa}{2} \frac{\delta S_k[g, \mathrm{x}]}{\delta g} \partial_k \left(\frac{1}{\frac{\delta^2 S_g[\eta]}{\delta g^2} + R_k} \right) \frac{\delta S_k[g, \mathrm{x}]}{\delta g}$$

Classical RG

INITIAL CONDITION

$$S_{k\to\infty}[g,\mathrm{x}] = S_{\mathrm{tot}}[g,\mathrm{x}] - \frac{1}{2\kappa}\, h\, \frac{\delta^2 S_g[\eta]}{\delta g^2}\, h$$

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- It is sufficient to evaluate directly at $g = \eta$: $$\partial_k S_k[g] = -\frac{\kappa}{2}\frac{\delta S_k[g]}{\delta g}\partial_k\left(\frac{1}{\frac{\delta^2 S_g}{\delta g^2}[g] + R_k}\right)\frac{\delta S_k[g]}{\delta g}$$

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Left-hand side

$$\begin{aligned}
S_k = \int \mathrm{d}t \Bigg\{ & \frac{m_1\mathbf{v}_1^2 + m_2\mathbf{v}_2^2}{2} + \frac{m_1\mathbf{v}_1^4 + m_2\mathbf{v}_2^4}{8c^2} \\
& + N_k\frac{Gm_1m_2}{r} + H_k\frac{G^2m_1m_2\,(m_1+m_2)}{2c^2r^2} \\
& + \frac{Gm_1m_2}{c^2r}\Big[C_k(\mathbf{n}\cdot\mathbf{v}_1)(\mathbf{n}\cdot\mathbf{v}_2) + A_k(\mathbf{v}_1^2+\mathbf{v}_2^2) + B_k(\mathbf{v}_1\cdot\mathbf{v}_2) \\
& + D_k\left[(\mathbf{n}\cdot\mathbf{v}_1)^2 + (\mathbf{n}\cdot\mathbf{v}_2)^2\right] + F_k(\mathbf{n}\cdot\mathbf{a}_1 - \mathbf{n}\cdot\mathbf{a}_2)\,r\Big]\Bigg\} + \mathcal{O}(1/c^4)
\end{aligned}$$

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Left-hand side

$$\partial_k S_k = \int \mathrm{d}t \Bigg\{ \partial_k N_k \frac{G m_1 m_2}{r} + \partial_k H_k \frac{G^2 m_1 m_2 (m_1 + m_2)}{2c^2 r^2} + \frac{G m_1 m_2}{c^2 r}\Big[\partial_k C_k (\mathbf{n}\cdot\mathbf{v}_1)(\mathbf{n}\cdot\mathbf{v}_2) + \partial_k A_k (\mathbf{v}_1^2 + \mathbf{v}_2^2) + \partial_k B_k (\mathbf{v}_1\cdot\mathbf{v}_2) + \partial_k D_k\left[(\mathbf{n}\cdot\mathbf{v}_1)^2 + (\mathbf{n}\cdot\mathbf{v}_2)^2\right] + \partial_k F_k (\mathbf{n}\cdot\mathbf{a}_1 - \mathbf{n}\cdot\mathbf{a}_2)\, r\Big]\Bigg\}$$

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$$\partial_k S_k = \int dt\, \left(\partial_k N_k \frac{G m_1 m_2}{R} + \partial_k H_k \frac{G^2 m_1 m_2 (m_1 + m_2)}{c^2 R^2}\right) + \cdots$$

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Energy-momentum tensor $\Rightarrow$ $$T^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g_{\mu\nu}}$$

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We can find $T_k^{\mu\nu}$ from the conserved quantities

Recovering the 1PN

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$$T_k^{00} = \sum_{a=1,2}\left[m_a c^2 + \frac{1}{2} m_a v_a^2 - \frac{N_k}{2}\frac{G m_1 m_2}{r}\right]\delta^{(3)}(\mathbf{x} - \mathbf{x}_a)$$

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Recovering the 1PN

$$\partial_k S_k = \frac{\kappa}{8c^4} \int_{x,y} T_k^{\mu\nu}(x)\, P_{\mu\nu\alpha\beta}\, T_k^{\alpha\beta}(y) \int_q \frac{e^{-iq\cdot(x-y)}\, \partial_k R_k(\mathbf{q}^2)}{\left(q^2 - R_k(\mathbf{q}^2)\right)^2}$$

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*D. F. Litim, Phys. Rev. D **64**, 105007 (2001)*

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- Using $T_k^{\mu\nu}$ and R_k, we find the flows of $(N_k,\, A_k,\, B_k,\, C_k,\, D_k,\, F_k,\, H_k)$

D. F. Litim, Phys. Rev. D ***64****, 105007 (2001)*

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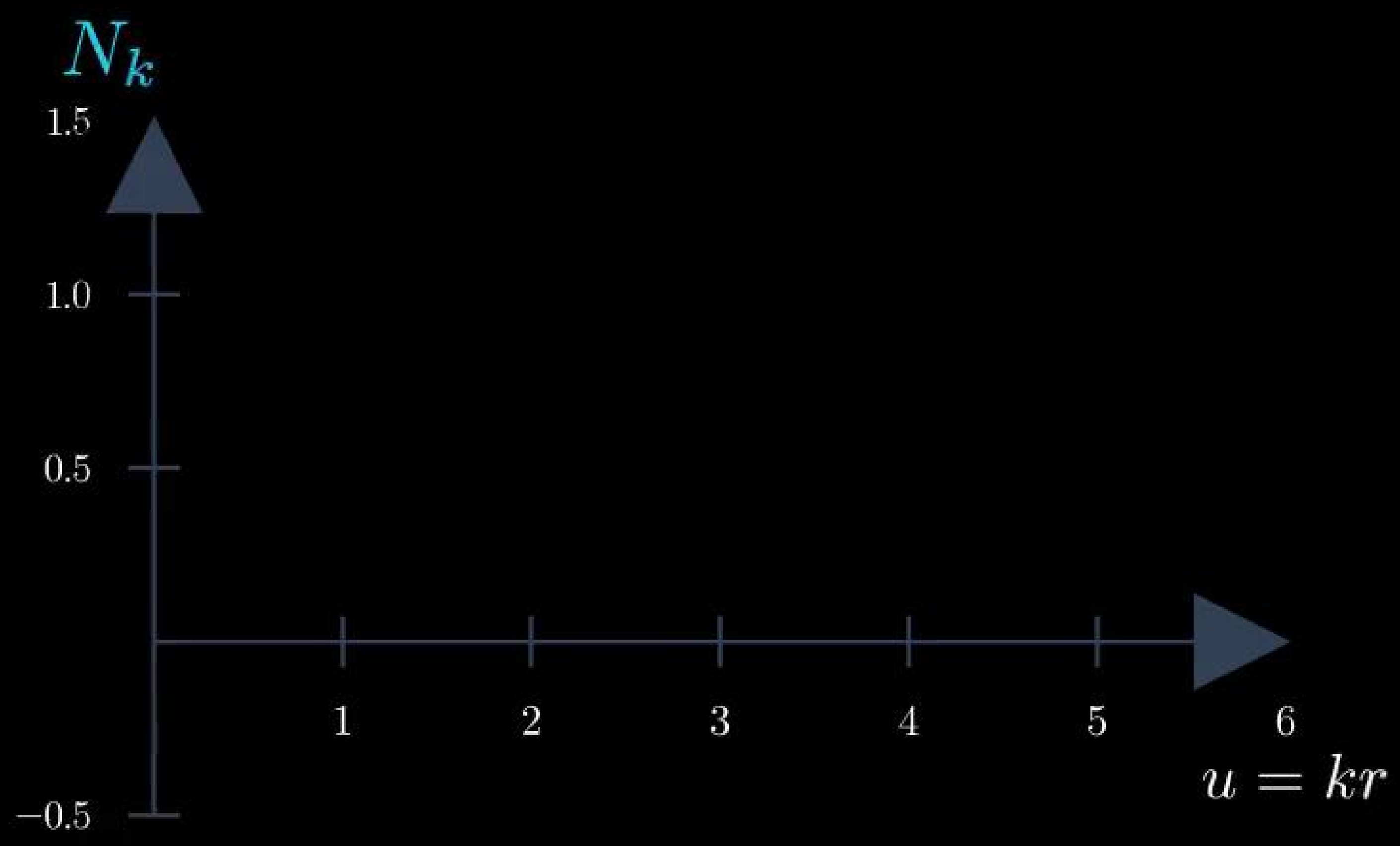

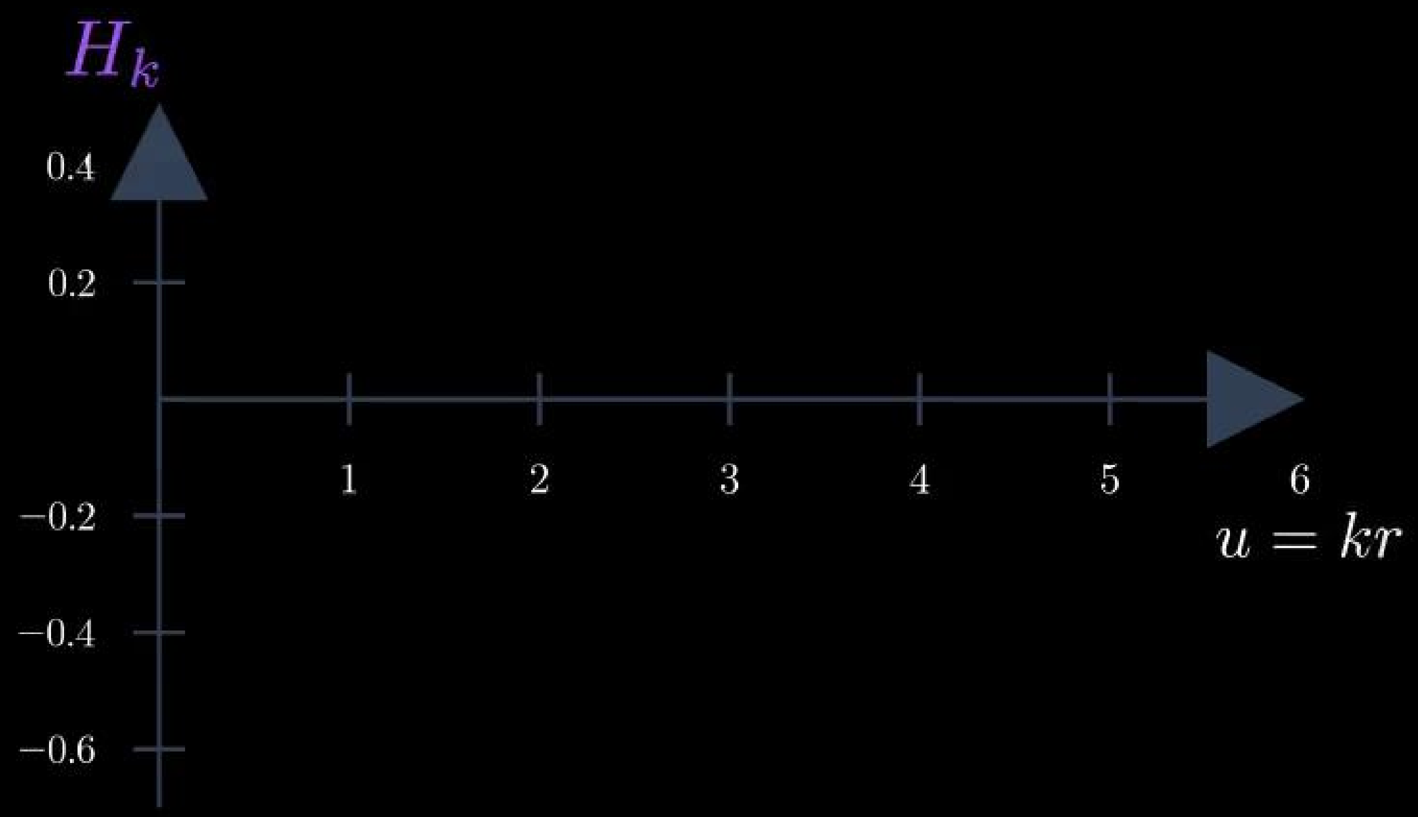

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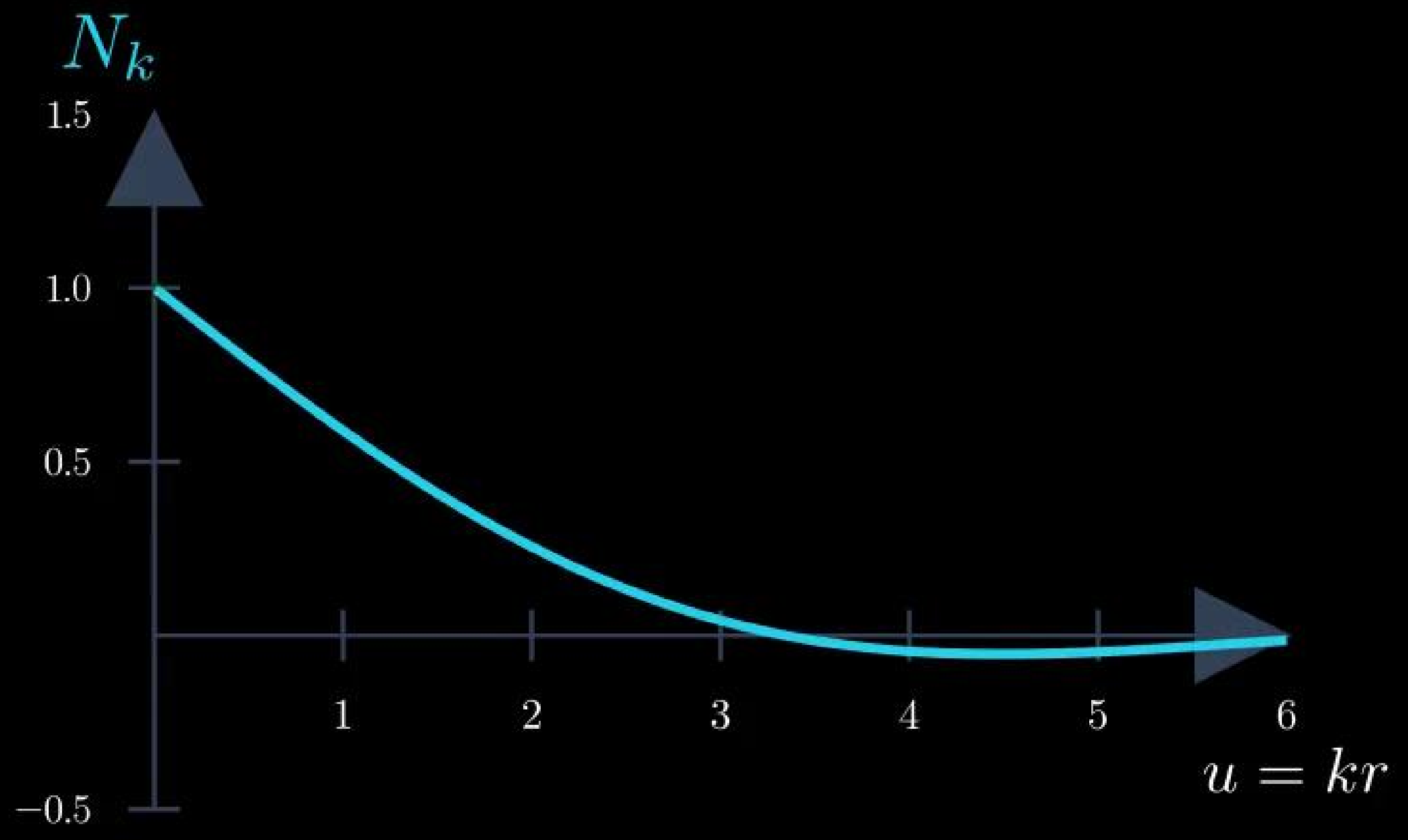

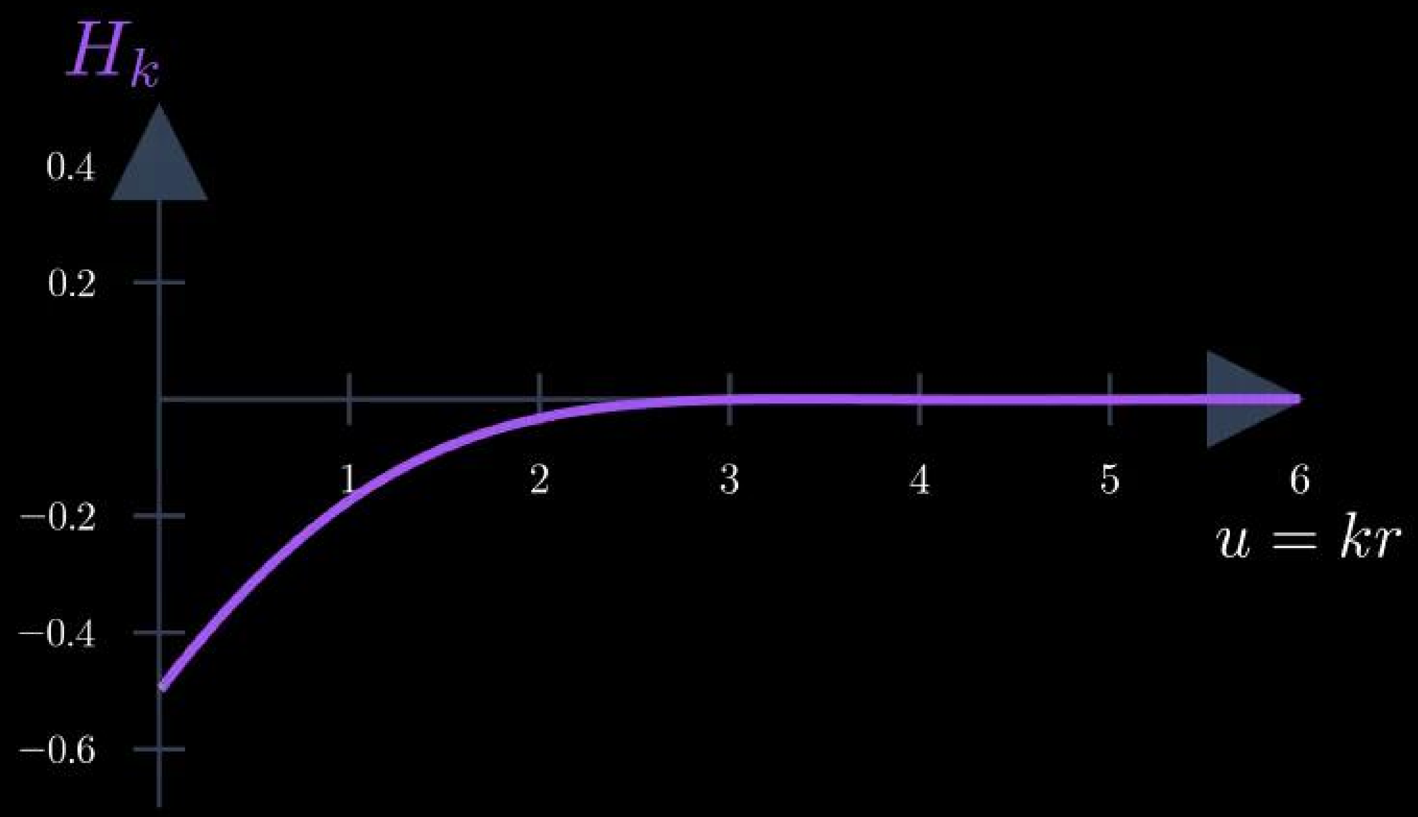

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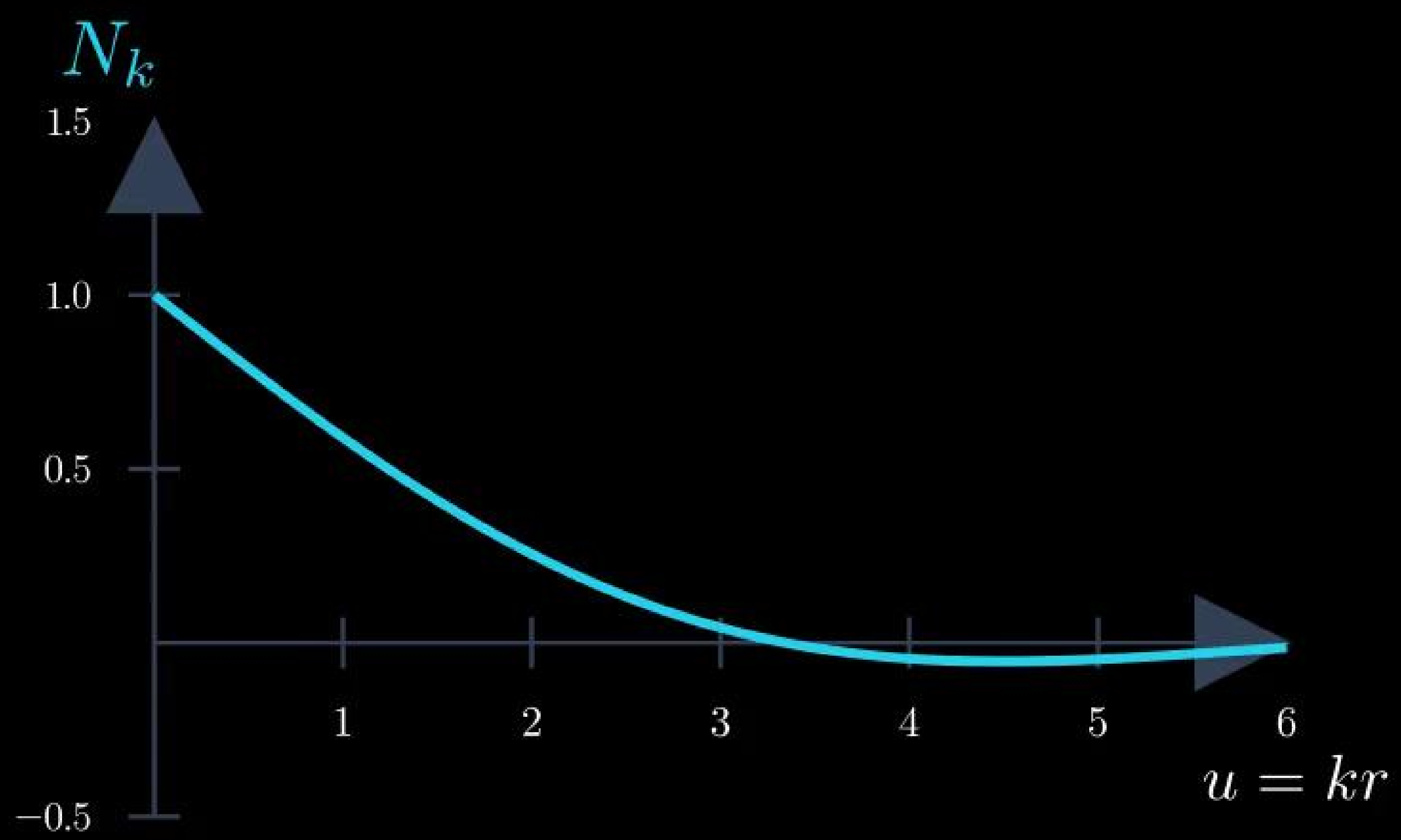
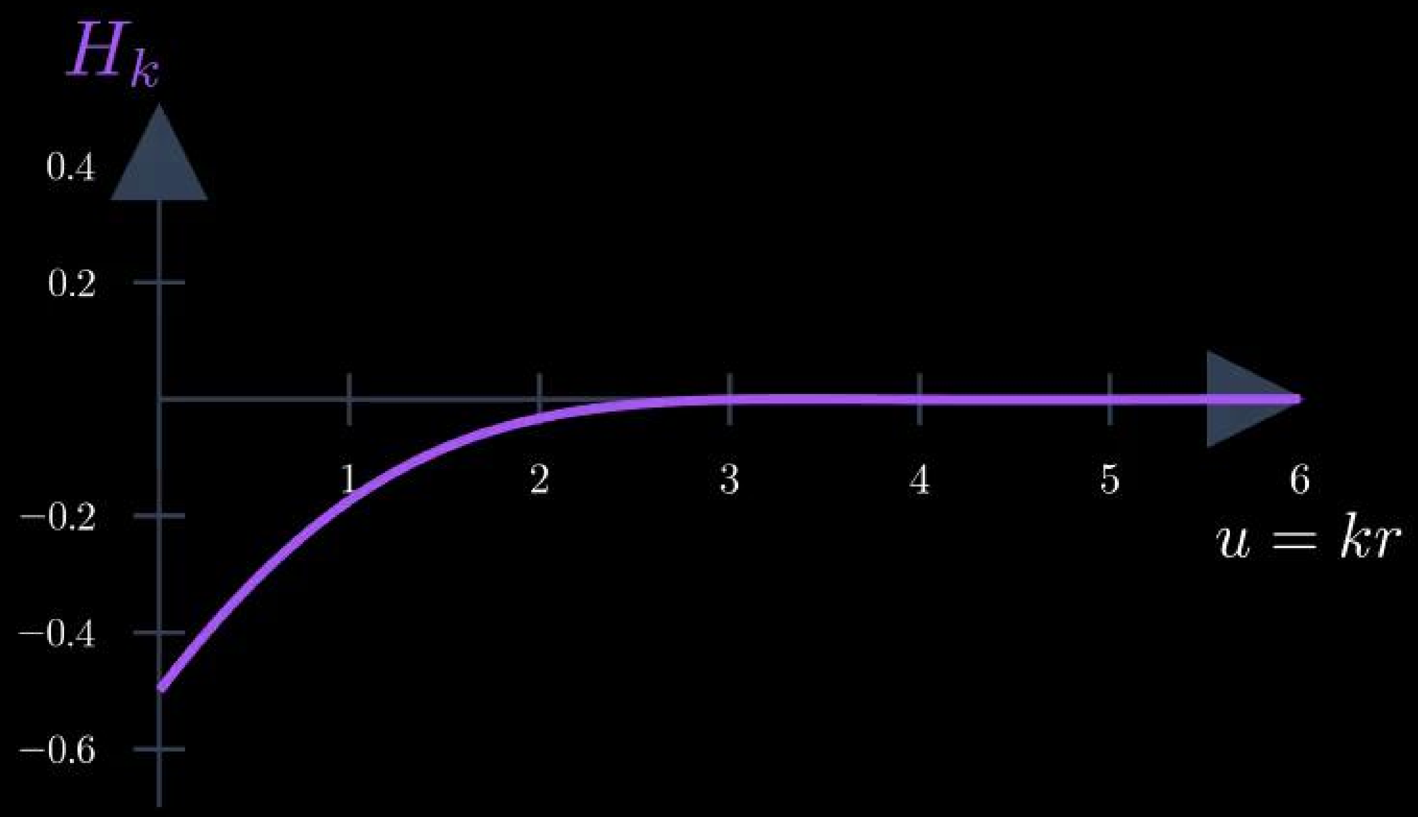
$$\Longrightarrow \boxed{N_{k=0} = 1, \qquad H_{k=0} = -\frac{1}{2}}$$





Why this is useful?

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Flow equations

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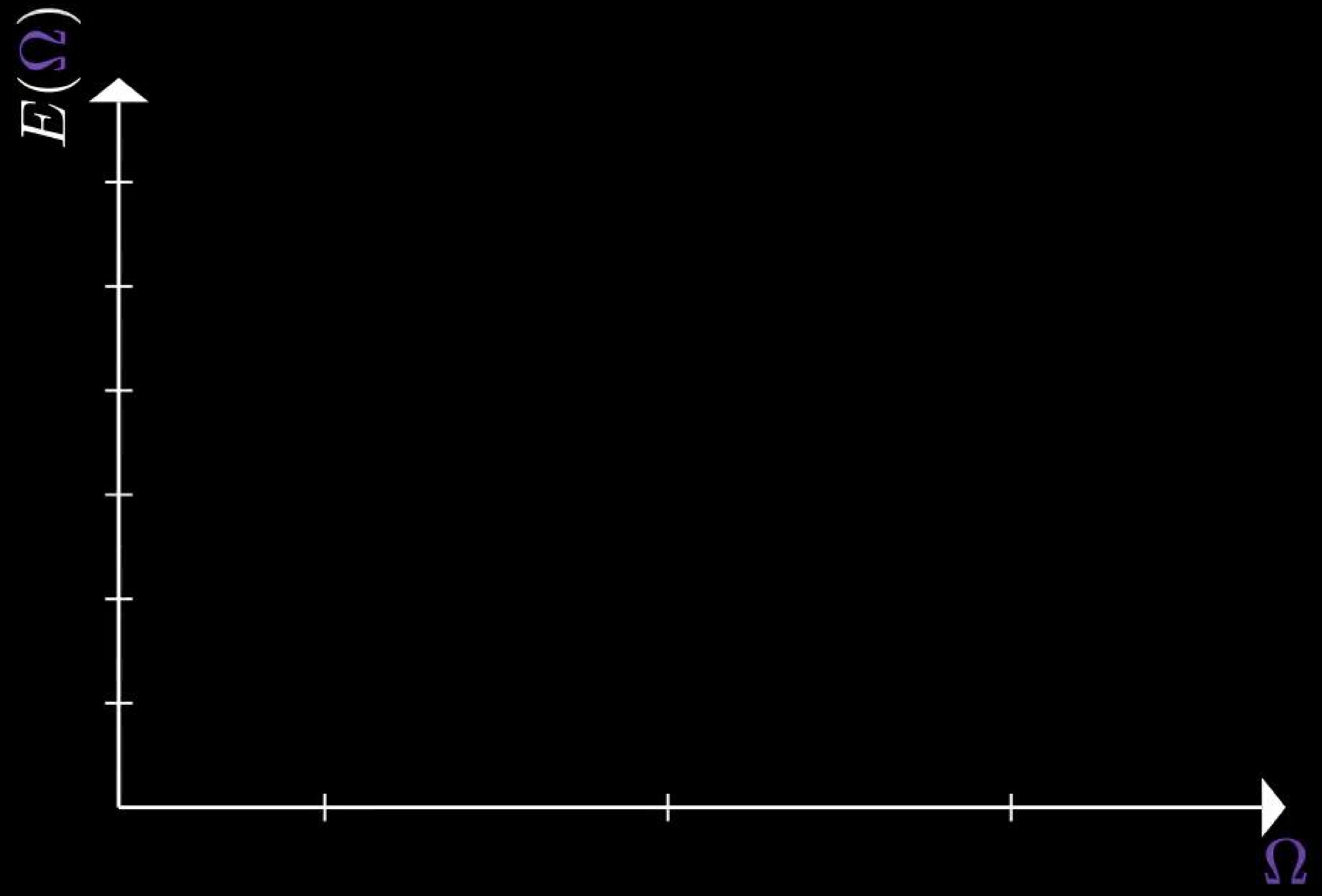
Something to compare with

Energy of an equal-mass binary on a circular orbit (CM frame).



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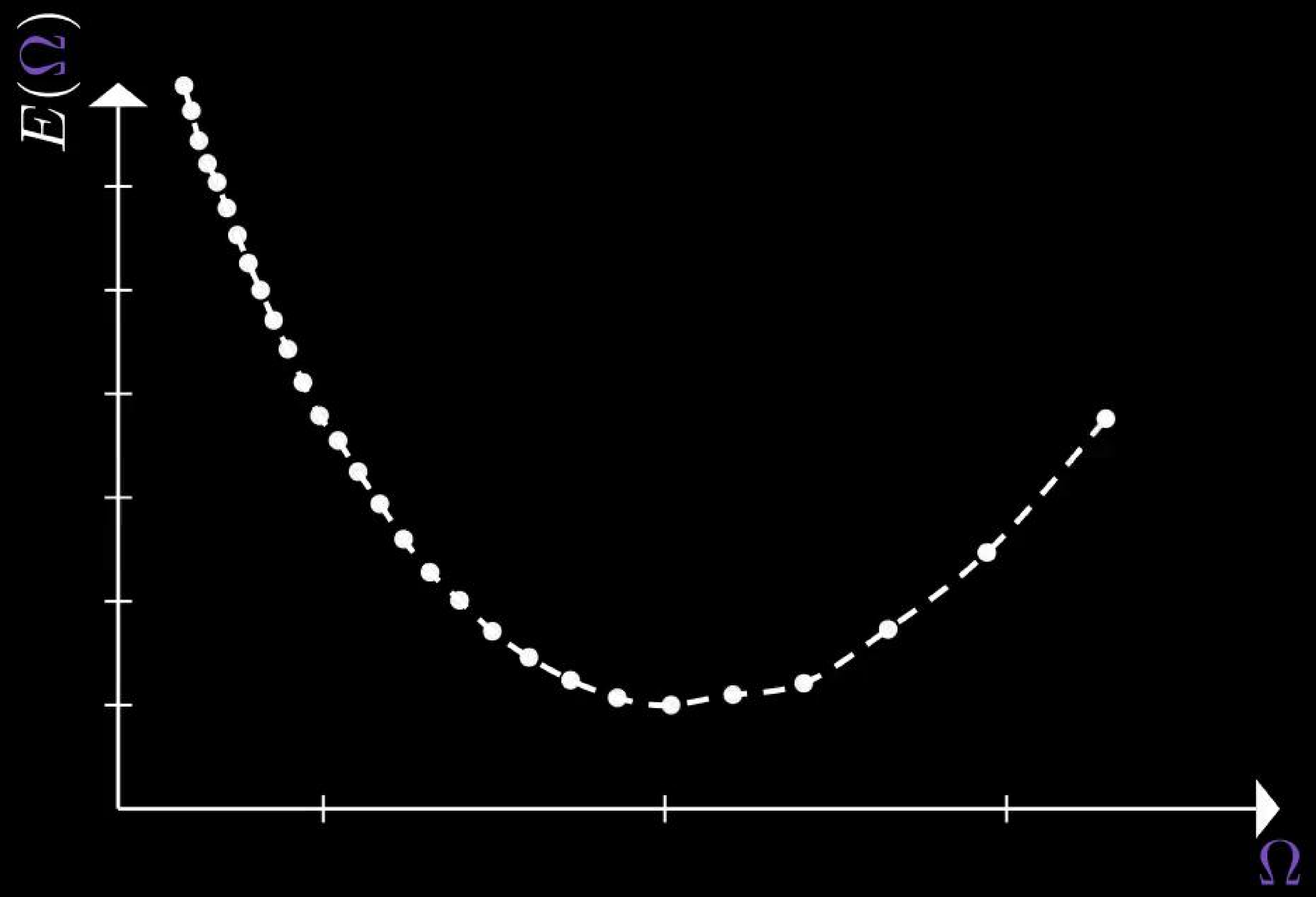

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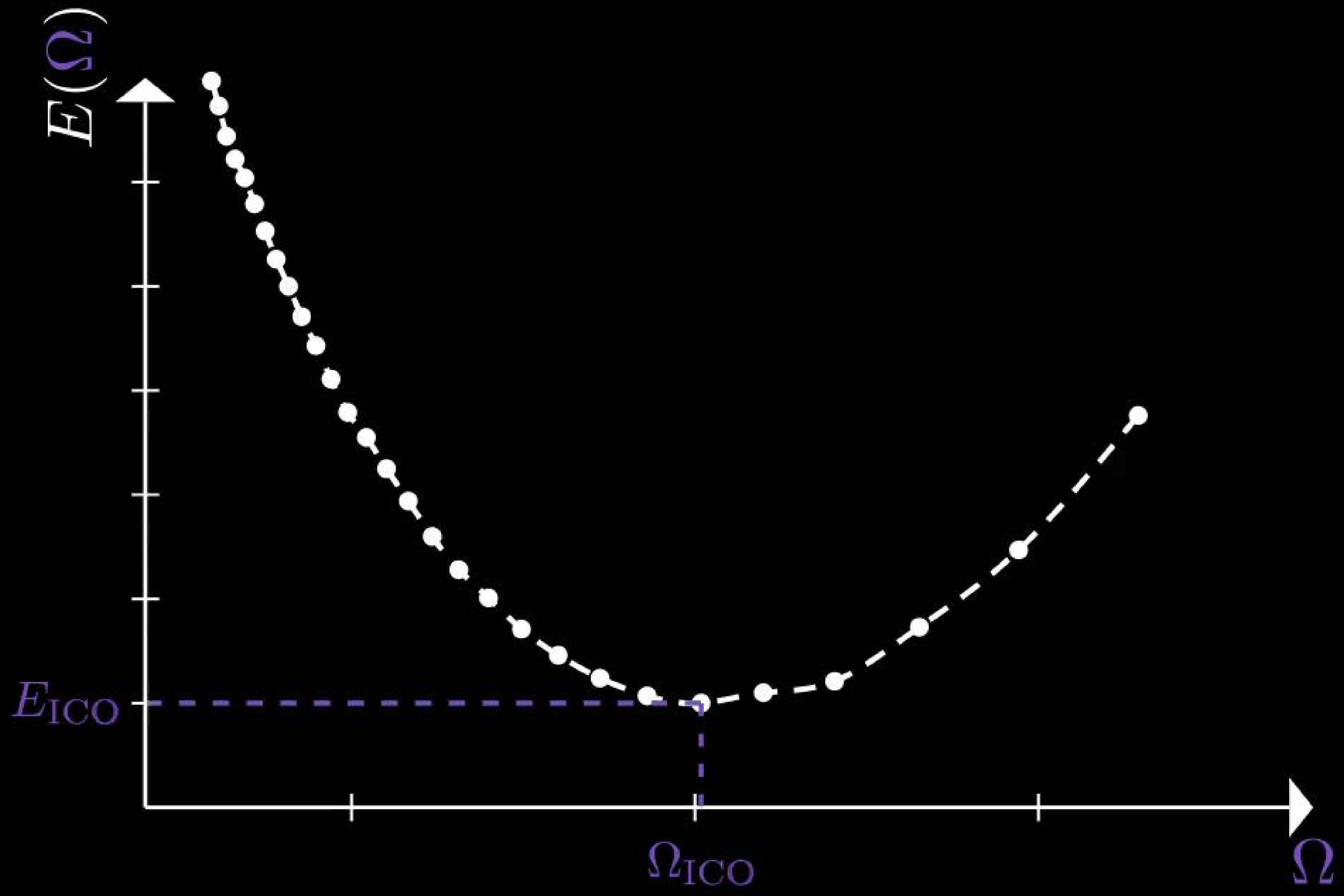

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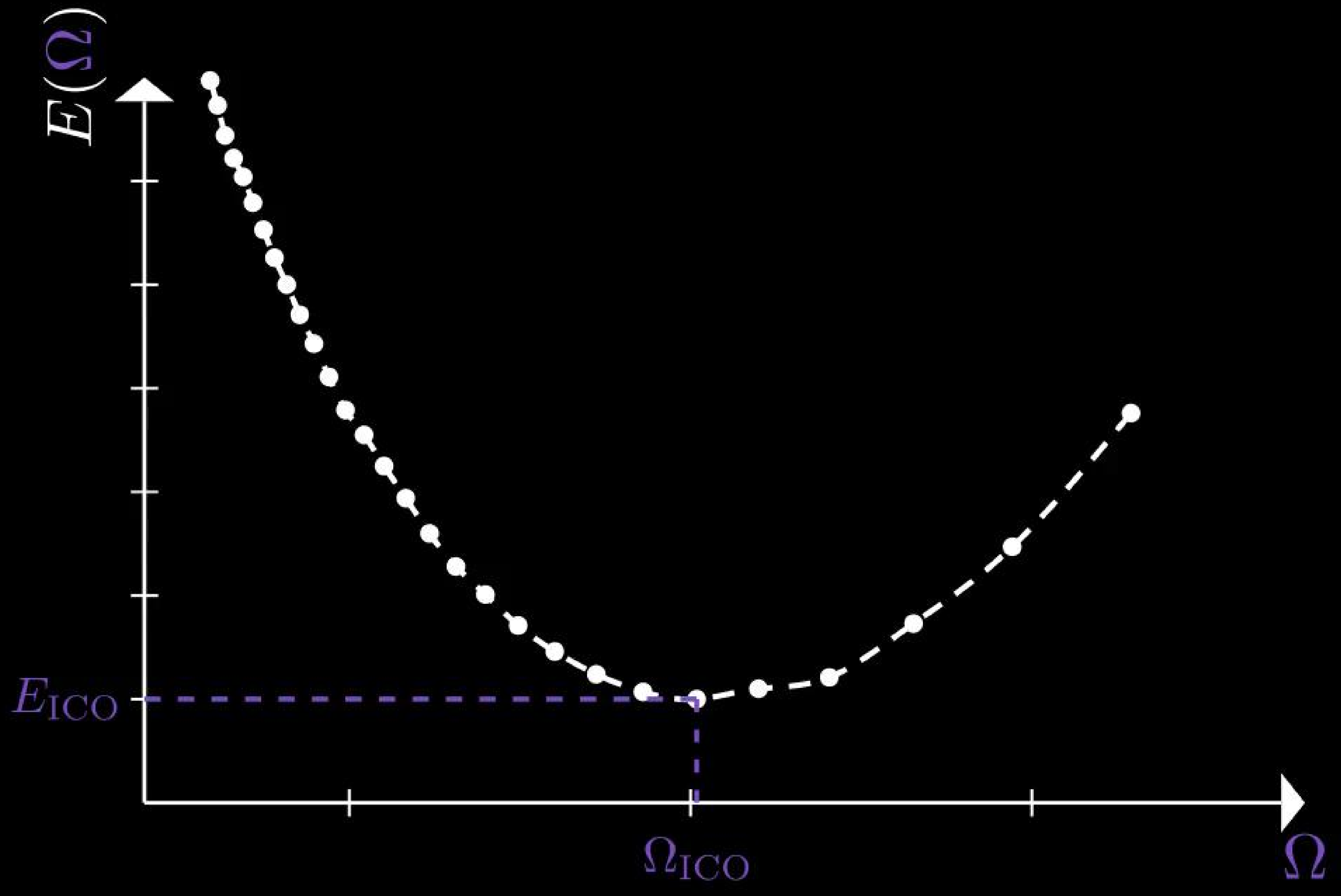

Non-perturbative analysis

Non-perturbative analysis

$$S_k = S_{\rm pp} + \int \mathrm{d}t \Bigg\{ N_k \frac{G m_1 m_2}{r} + H_k \frac{G^2 m_1 m_2 \left(m_1 + m_2\right)}{2c^2 r^2}$$

$$+ \frac{G m_1 m_2}{c^2 r}\Big[C_k (\mathbf{n}\cdot\mathbf{v}_1)(\mathbf{n}\cdot\mathbf{v}_2) + A_k (\mathbf{v}_1^2 + \mathbf{v}_2^2) + B_k (\mathbf{v}_1\cdot\mathbf{v}_2)$$

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Non-perturbative information is encoded in an arbitrary function of distance.

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Comparison of E_{ICO}

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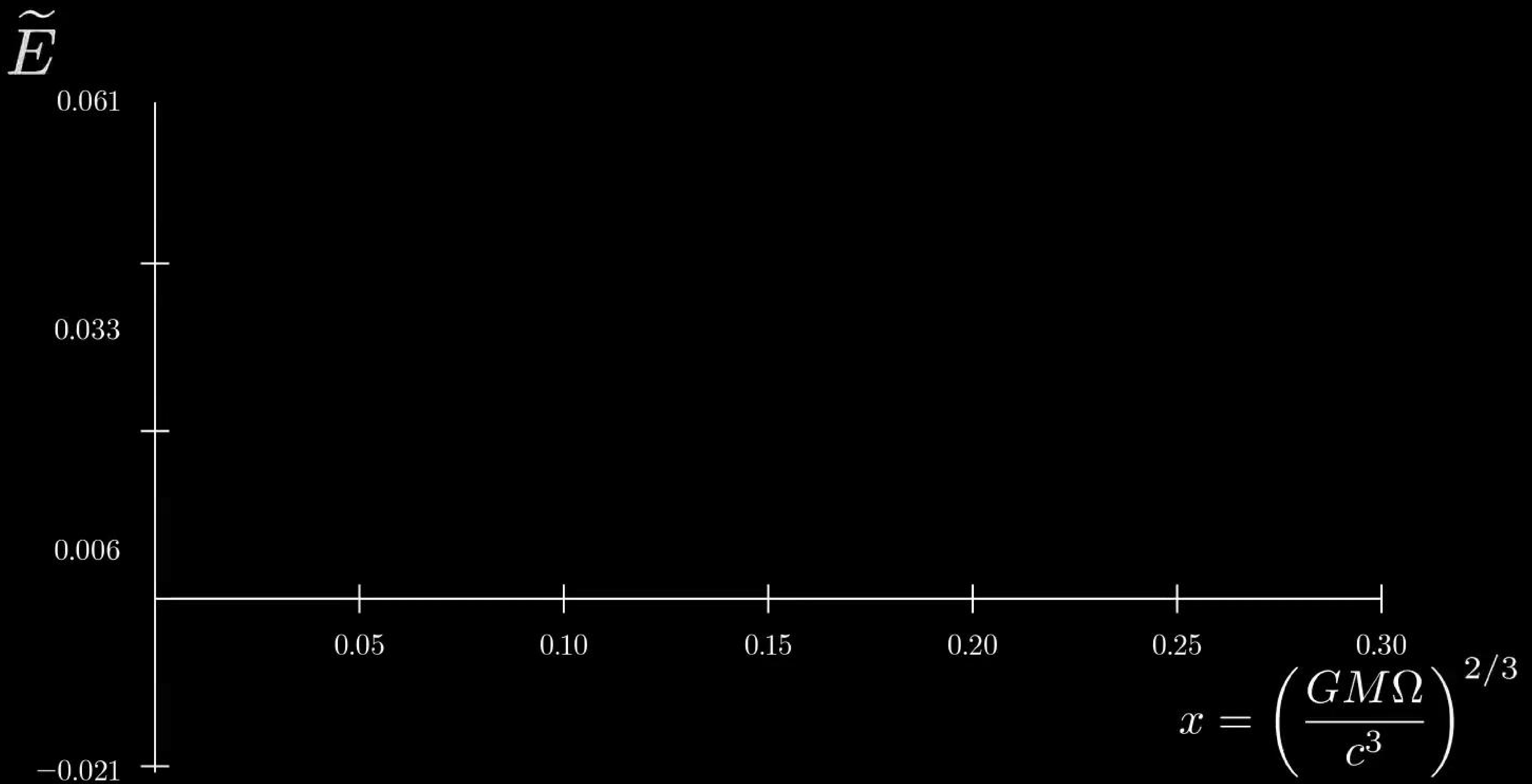

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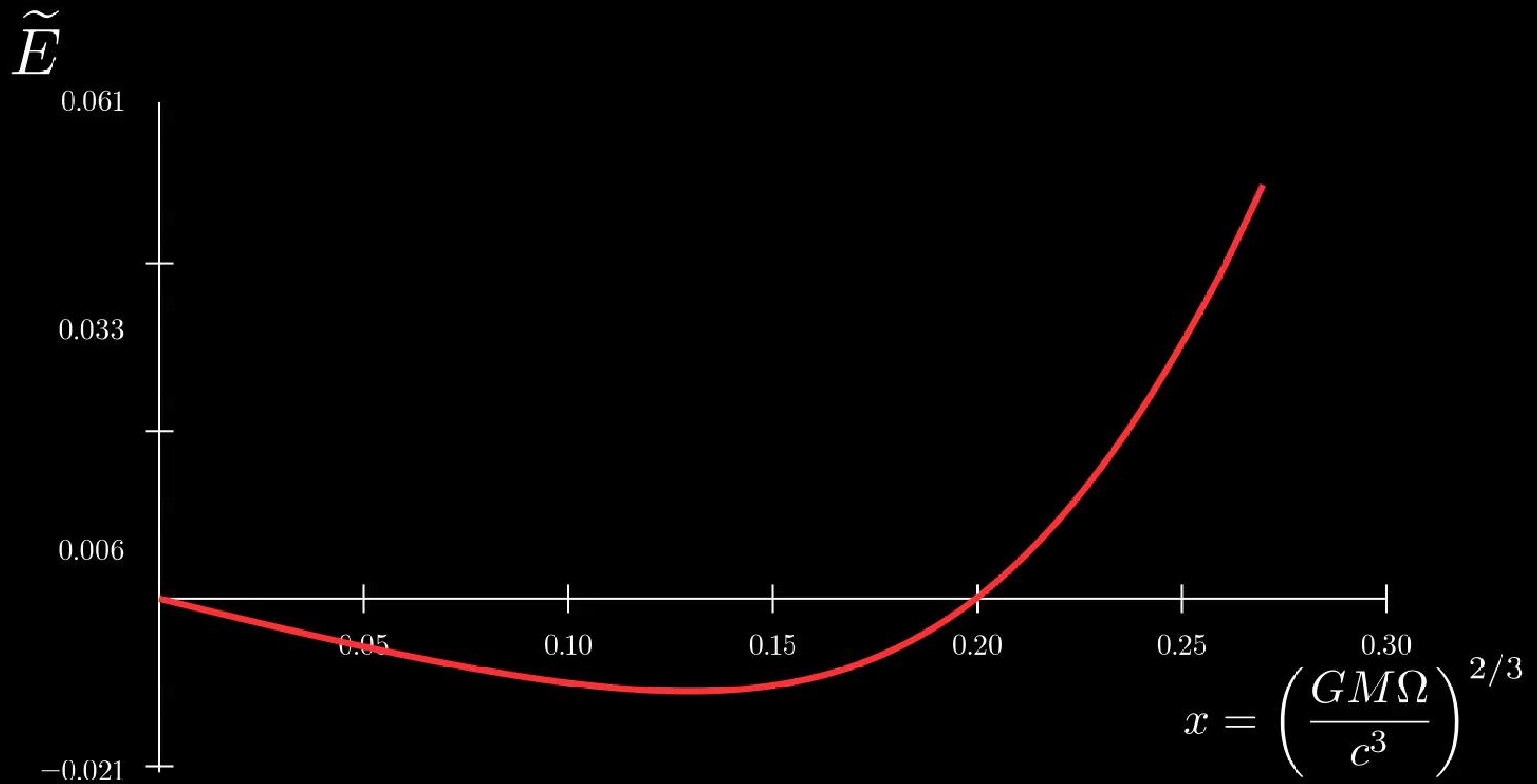

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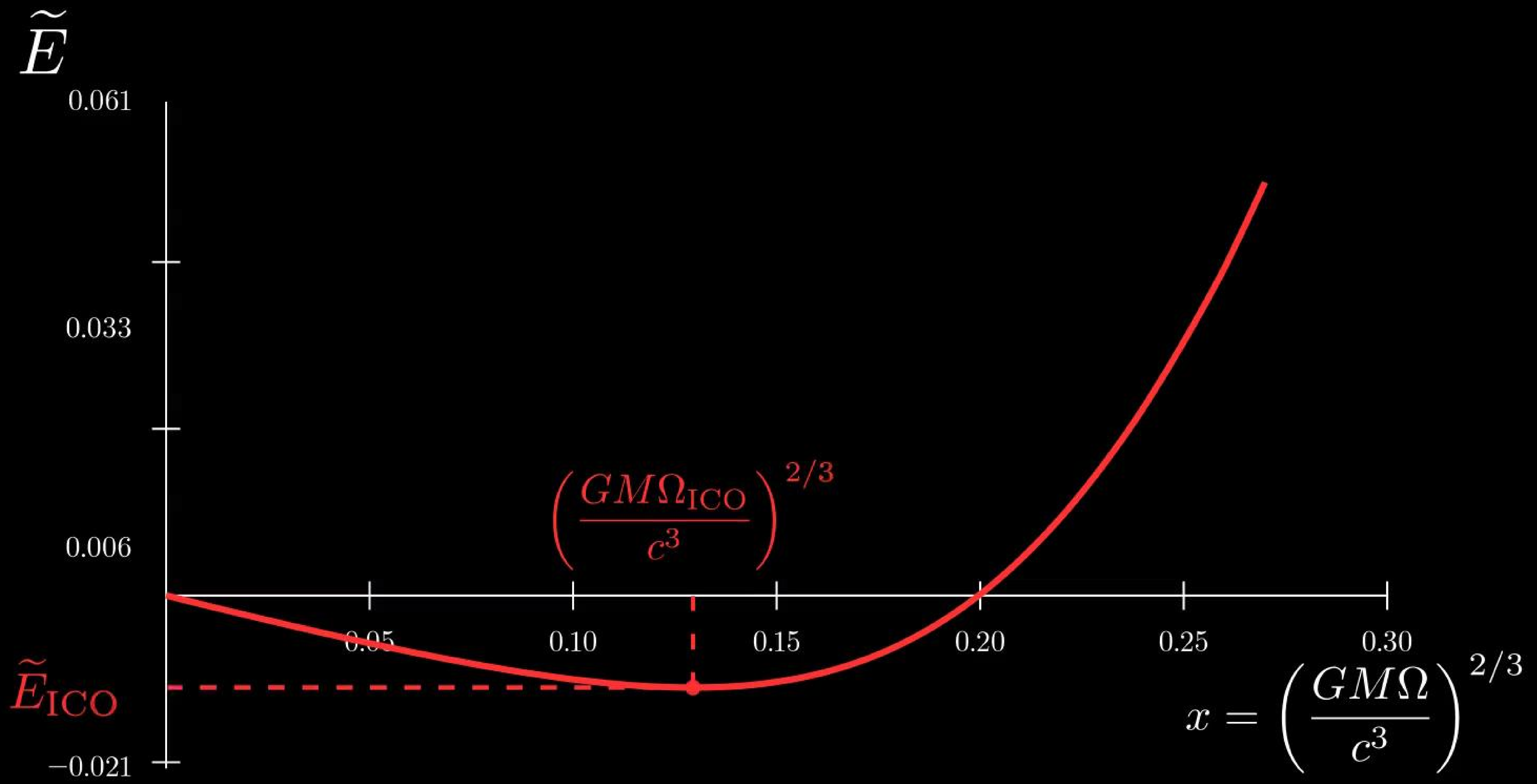

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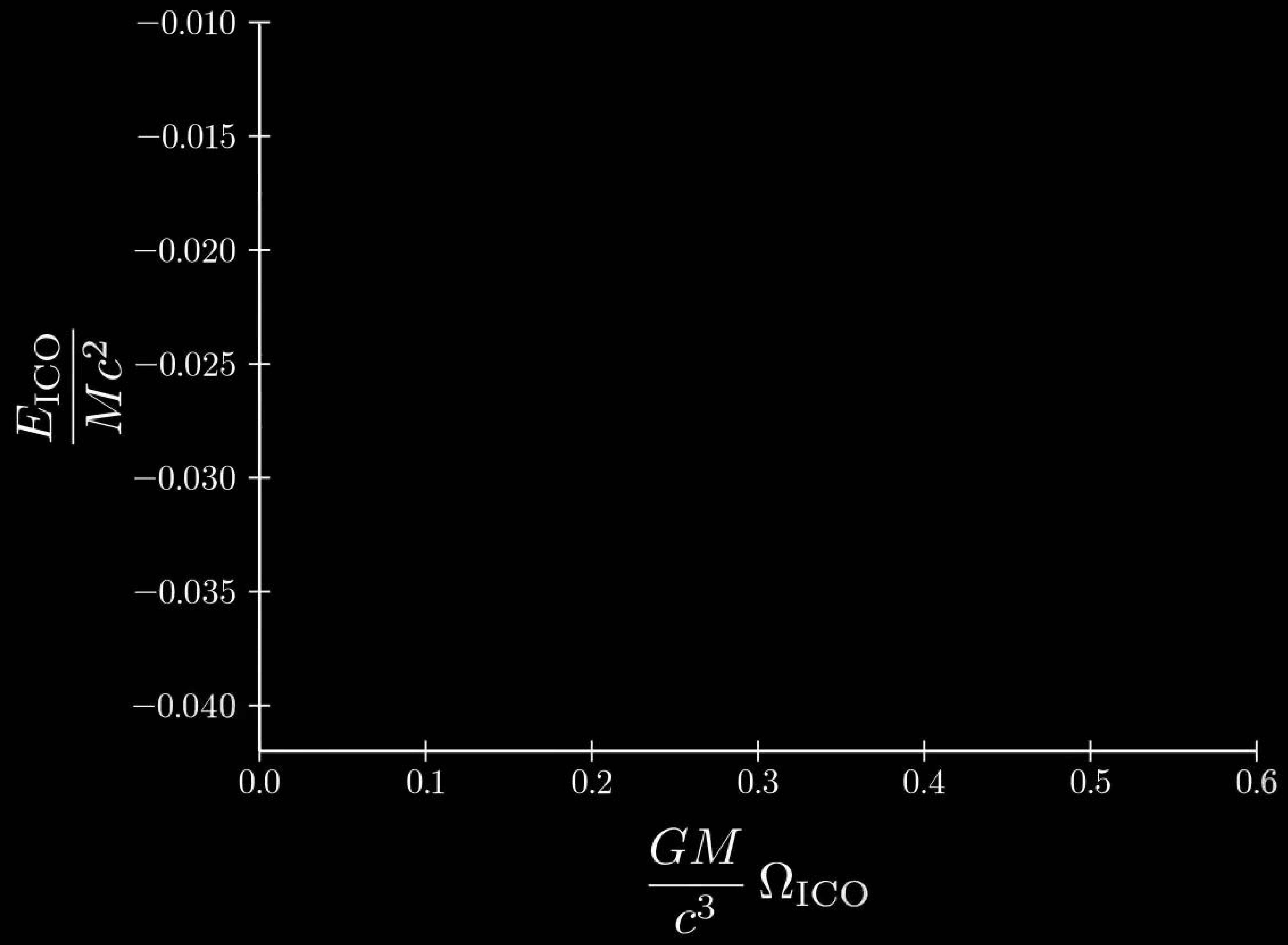

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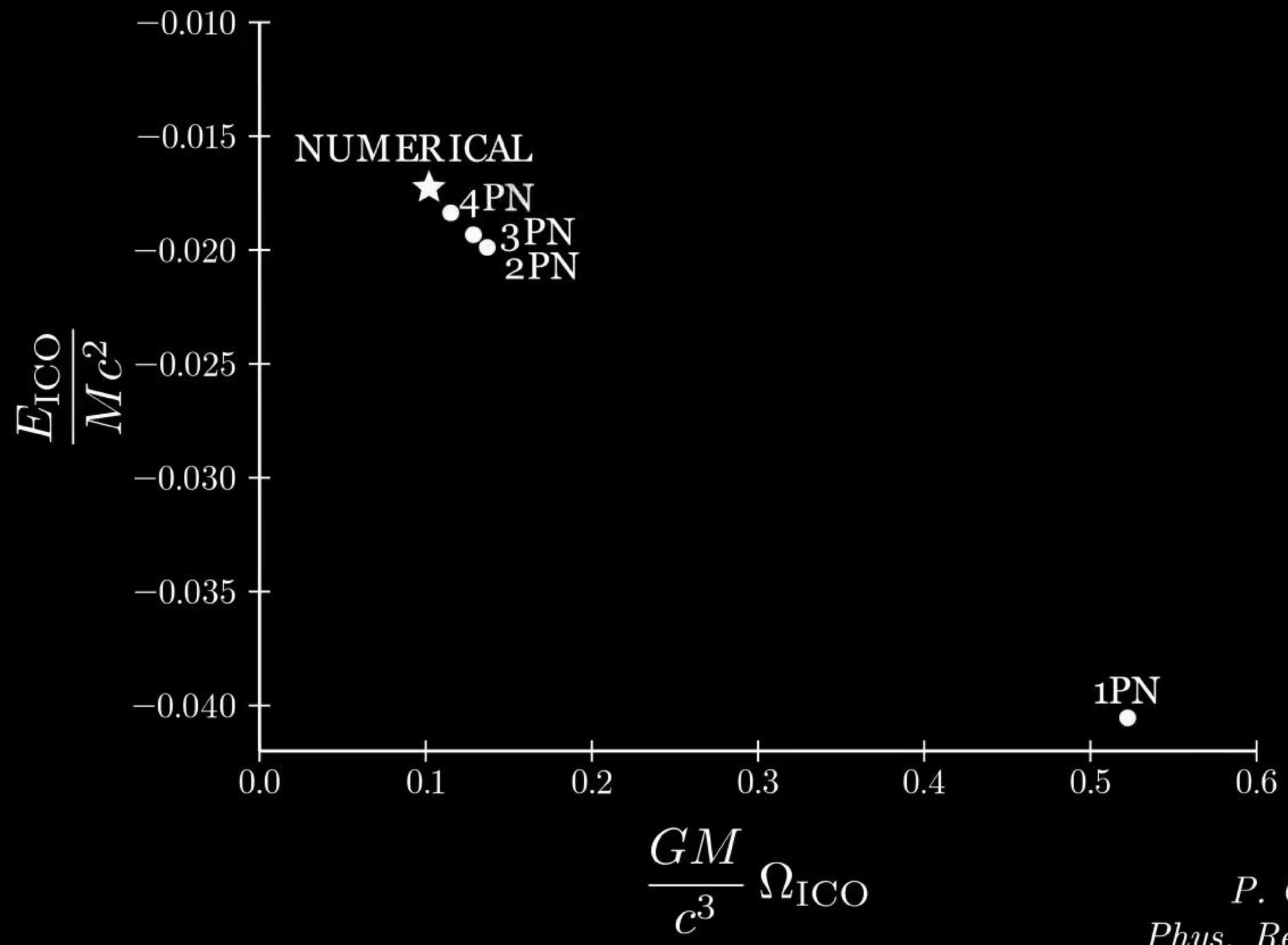

P. Grandclément et al.
Phys. Rev. D ***65****, 044021 (2002)*

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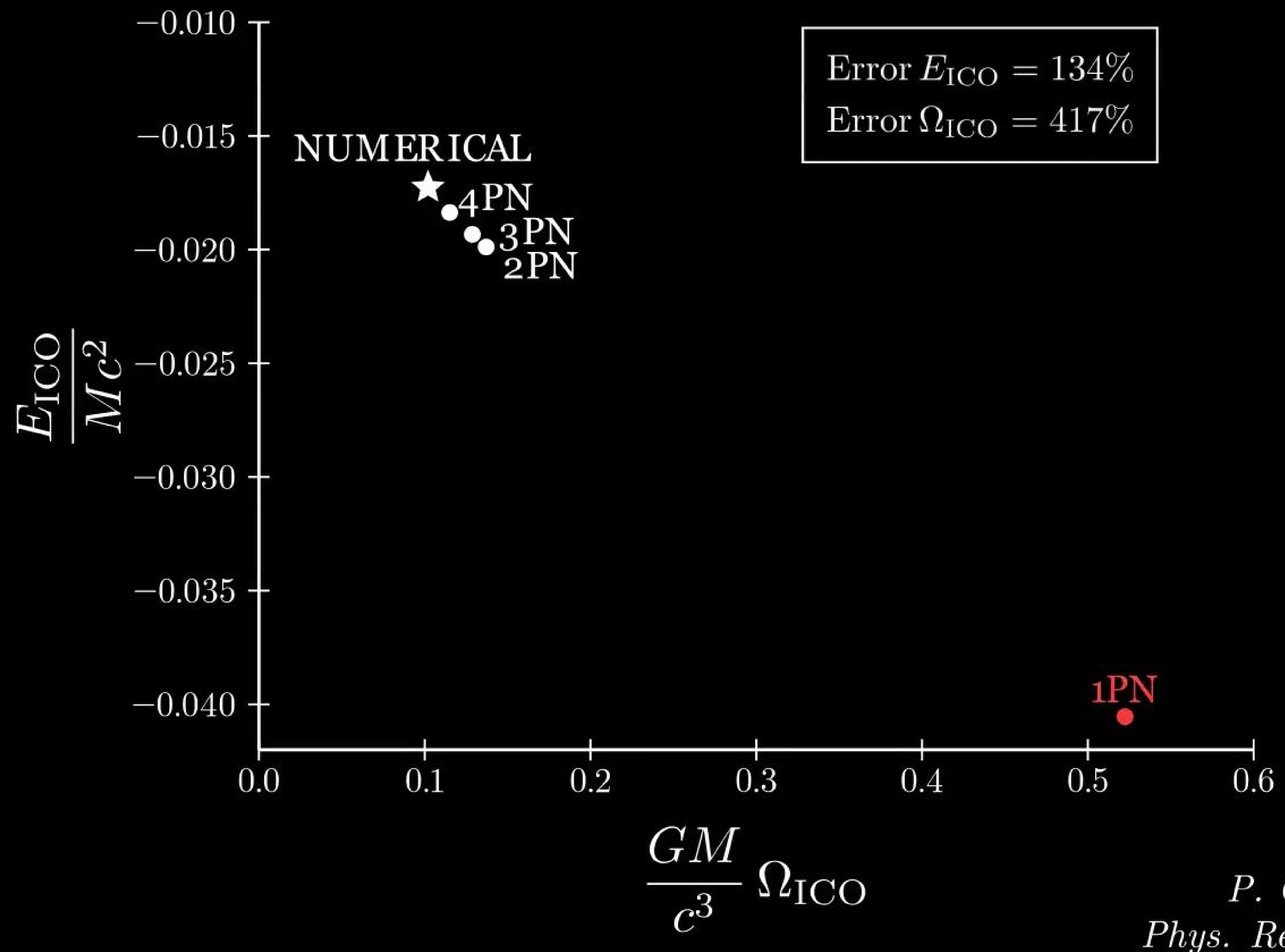

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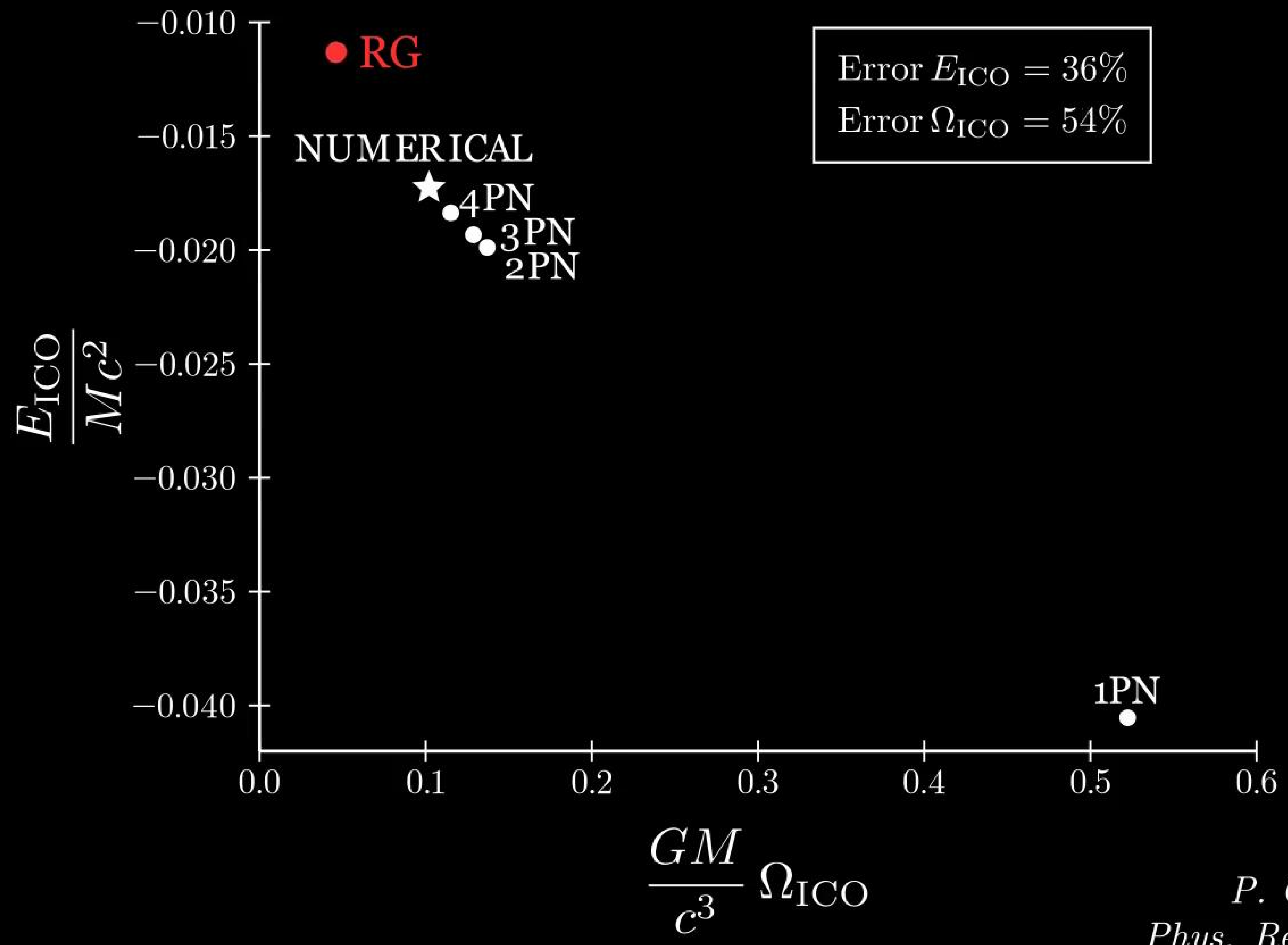

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Future Directions

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- Improve the nonperturbative analysis by including higher-order terms in the derivative expansion.

- Compare with other observables, such as the angular momentum and the scattering angle.

- Propose a covariant ansatz for the effective action, from which the $T_k^{\mu\nu}$ can be computed directly via a functional derivative.

- Derive an RG equation for the radiative sector, with the aim of computing the waveform.

Thank you