Flow equations of second order

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Motivation

- functional RG for non-perturbative quantum field theory
- real time dynamics
- analytic properties of correlation functions

Real time dynamics & analytic continuation

- information about real time dynamics at real frequencies
- analytic continuation of functional RG equations[1] [2]
- bound states
- transport properties

[1]S. Floerchinger, JHEP 05, 021 (2012)

[2]K. Kamikado, N. Strodthoff, L. von Smekal, J. Wambach, Eur.Phys.J. C74, 2806 (2014)

Unitarity

- Unitarity in Minkowski space
- Osterwalder-Schrader positivity in Euclidean space
- very important property of quantum field theories
- many non-perturbative properties follow from unitarity
- Example 1: spectral representations
- Example 2: analytic S-matrix programm
- unfortunately broken by many IR regularisation functions $R(p)$

The Callan-Symanzik regulator

$$R_k(p) = k^2$$

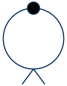
- does not break unitarity and makes analytic continuation straight-forward
- no additional unphysical poles in the complex plane
- physical interpretation: change in microscopic mass $m_\Lambda^2 \to m_\Lambda^2 + k^2$
- preserves locality
- preserves Lorentz-invariance and causality
- preserved most other symmetries of interest[3]

[3]see also: J. Braun, Y. Chen, W. Fu, A. Geißel, J. Horak, C. Huang, F. Ihssen, J. M. Pawlowski, M. Reichert, F. Rennecke, Y. Tan, S. Töpfel, J. Wessely, N. Wink, SciPost Phys. Core 6, 061 (2023)

Ultraviolet divergences

- one serious issue: ultraviolet divergences not fully tamed
- example: flow of squared mass m_k^2 in scalar theory with $R_k(p) = k^2$

$$\frac{\partial}{\partial k} m_k^2 = -\frac{1}{2}$$



$$= -\frac{\lambda}{2} \int \frac{d^4 p}{(2\pi)^4} \frac{\partial_k R_k(p)}{(p^2 + m_k^2 + R_k(p))^2} = -\frac{\lambda}{2} \int \frac{d^4 p}{(2\pi)^4} \frac{2k}{(p^2 + m_k^2 + k^2)^2}$$

- has logarithmic UV divergence
- perturbation theory: quardatic + subleading logarithmic divergence in Δm^2, scheme-dependent

Flow of effective potential

- flow of effective potential for $O(N)$ model in local potential approximation (with $\rho = \frac{1}{2}\sum_{n=1}^{N}\phi_n^2$)

$$\frac{1}{k}\frac{\partial}{\partial k}U_k(\rho) = \int \frac{d^4p}{(2\pi)^4}\left\{\frac{1}{p^2+U_k'(\rho)+2\rho U_k''(\rho)+k^2} + \frac{N-1}{p^2+U_k'(\rho)+k^2}\right\}$$

- flow of first derivative

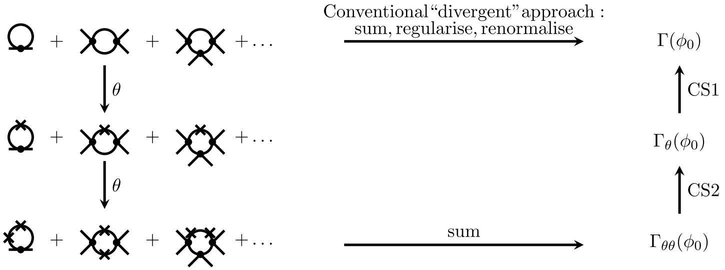
$$\frac{1}{k}\frac{\partial}{\partial k}U_k'(\rho) = \int \frac{d^4p}{(2\pi)^4}\left\{\frac{3U_k''(\rho)+2\rho U_k'''(\rho)}{[p^2+U_k'(\rho)+2\rho U_k''(\rho)+k^2]^2} + \frac{[N-1]U_k''(\rho)}{[p^2+U_k'(\rho)+k^2]^2}\right\}$$

- logarithmic UV divergence here and in all higher ρ derivatives

An explicit UV regulator?

- most field theories are effective theories defined at some UV scale Λ
- why not derive flow equations in the presense of *some* UV regularisation?
- precise implementation of UV regularisation not known
- UV regularisations often break symmetries and / or unitarity
- can still be done, e.g. spacetime lattice, Schwinger proper time, point splitting, …
- resulting flow equations depend on the scheme!
- can one do something more systematic?

Callan's scheme[4][5]



- for bare vertex functions

$$\bar{\Gamma}^{(n)}_{\theta} = \frac{\partial}{\partial m_0^2}\bar{\Gamma}^{(n)}, \qquad \bar{\Gamma}^{(n)}_{\theta\theta} = \frac{\partial}{\partial m_0^2}\bar{\Gamma}^{(n)}_{\theta} = \frac{\partial^2}{\partial (m_0^2)^2}\bar{\Gamma}^{(n)},$$

[4]C. G. Callan, Jr., *Introduction to renormalization theory*, Methods in Field Theory 41-78 (1993)

[5]S. Mooij and M. Shaposhnikov, Nucl. Phys. B 990, 116117 (2023), Nucl. Phys. B 1006, 116642 (2024)

Callan-Symmanzik equations

- for renormalized vertex functions $\Gamma^n = Z^{n/2}\bar{\Gamma}^{(n)}, \quad \Gamma^n_\theta = Z_\theta Z^{n/2}\bar{\Gamma}^{(n)}_\theta, \quad \Gamma^n_{\theta\theta} = Z_\theta^2 Z^{n/2}\bar{\Gamma}^{(n)}_{\theta\theta}$
- first and second Callan-Symanzik equations

$$2m^2\left[1-\eta/2\right]\Gamma^{(n)}_\theta = \left[2m^2\frac{\partial}{\partial m^2} + \beta\frac{\partial}{\partial\lambda} - \frac{n}{2}\eta\right]\Gamma^{(n)}$$

$$2m^2\left[1-\eta/2\right]\Gamma^{(n)}_{\theta\theta} = \left[2m^2\frac{\partial}{\partial m^2} + \beta\frac{\partial}{\partial\lambda} - \frac{n}{2}\eta + \gamma_\theta\right]\Gamma^{(n)}_\theta$$

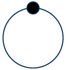
- imposes renormalization conditions

$$\Gamma^{(2)}(p) = m^2 + p^2 + \ldots, \qquad \Gamma^{(4)}(0) = \lambda, \qquad \Gamma^{(2)}_\theta(0) = 1.$$

First order flow equation

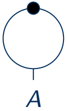
- flow equation for flowing action in abstract index notation

$$\frac{\partial}{\partial k}\Gamma_k[\Phi] = \frac{1}{2}\left(\frac{d}{dk}R_{kAB}\right)((\Gamma_k^{(2)}[\Phi] + R_k)^{-1})^{AB} = \frac{1}{2}$$



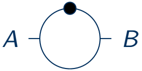
- for one point function

$$\frac{\partial}{\partial k}(\Gamma_k^{(1)}[\Phi])_A = \frac{\delta}{\delta\Phi^A}\frac{\partial}{\partial k}\Gamma_k[\Phi] = -\frac{1}{2}$$

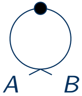


- for two-point function

$$\frac{\partial}{\partial k}(\Gamma_k^{(2)}[\Phi])_{AB} = \frac{\delta^2}{\delta\Phi^A\delta\Phi^B}\frac{\partial}{\partial k}\Gamma_k[\Phi] =$$



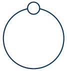
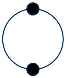
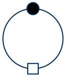
$$-\frac{1}{2}$$



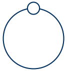
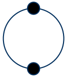
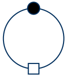
Second order flow equation

- idea to improve UV convergence: take further derivatives with respect to k
- one finds

$$
\begin{aligned}
\frac{\partial^2}{\partial k^2}\Gamma_k[\Phi] =& \frac{1}{2}\mathrm{Tr}\left\{\left(\frac{d^2}{dk^2}R_k\right)(\Gamma_k^{(2)}[\Phi]+R_k)^{-1}\right\} \\
&-\frac{1}{2}\mathrm{Tr}\left\{\left(\frac{d}{dk}R_k\right)((\Gamma_k^{(2)}[\Phi]+R_k)^{-1})\left(\frac{d}{dk}R_k\right)((\Gamma_k^{(2)}[\Phi]+R_k)^{-1})\right\} \\
&-\frac{1}{2}\mathrm{Tr}\left\{\left(\frac{d}{dk}R_k\right)((\Gamma_k^{(2)}[\Phi]+R_k)^{-1})\left(\frac{\partial}{\partial k}(\Gamma_k^{(2)}[\Phi])\right)((\Gamma_k^{(2)}[\Phi]+R_k)^{-1})\right\}
\end{aligned}
$$

$=\frac{1}{2}$  $-\frac{1}{2}$  $-\frac{1}{2}$ 

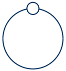
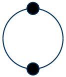
Second order flow equation

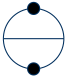
$$\frac{\partial^2}{\partial k^2}\Gamma_k[\Phi] = \frac{1}{2}$$  $$- \frac{1}{2}$$  $$- \frac{1}{2}$$ 

- second order flow equation
- first term on r. h. s. absent for flow with k^2
- UV convergence has improved, but logarithmic divergences are still present
- right hand side contains $\frac{\partial}{\partial k}\Gamma_k^{(2)}[\Phi]$
- two options:
  1. (what Callan did) extend parametrization / truncation to $\frac{\partial}{\partial k}\Gamma_k^{(2)}[\Phi]$
  2. plug in first order flow equation for $\frac{\partial}{\partial k}\Gamma_k^{(2)}[\Phi]$

Two-loop flow equation

- use first order flow equation for $\frac{\partial}{\partial k}\Gamma_k^{(2)}[\Phi]$ in second order flow equation[6]
- results in mix of one- and two-loop terms,

$$\frac{\partial^2}{\partial k^2}\Gamma_k[\Phi] = \frac{1}{2}$$

$$-\frac{1}{2}$$

$$+\frac{1}{4}$$

$$-\frac{1}{2}$$


- still conatins logarithmic UV divergences
- follows immediately from standard first order flow equation
- could be used to test truncations!

[6] see also: Lorenzo Rossi, diploma thesis, Uni. Bologna, 2018 (w. Gian-Paolo Vacca)

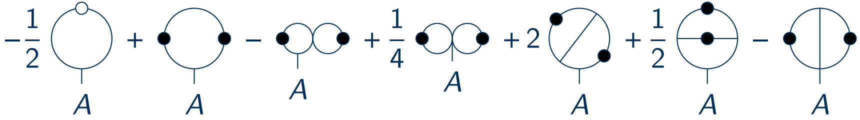
Two-loop flow equation

$$
\begin{aligned}
\frac{\partial^2}{\partial k^2}\Gamma_k[\Phi] =& \frac{1}{2}\left(\frac{d^2}{dk^2}R_{kAB}\right)((\Gamma_k^{(2)}[\Phi]+R_k)^{-1})^{AB} \\
&- \frac{1}{2}\left(\frac{d}{dk}R_{kCF}\right)\left(\frac{d}{dk}R_{kDE}\right)((\Gamma_k^{(2)}[\Phi]+R_k)^{-1})^{CD}((\Gamma_k^{(2)}[\Phi]+R_k)^{-1})^{EF} \\
&+ \frac{1}{4}\left(\frac{d}{dk}R_{kIJ}\right)\left(\frac{d}{dk}R_{kCF}\right)((\Gamma_k^{(2)}[\Phi]+R_k)^{-1})^{IA}((\Gamma_k^{(2)}[\Phi]+R_k)^{-1})^{CD} \\
&\qquad \times (\Gamma_k^{(4)}[\Phi])_{ABDE}((\Gamma_k^{(2)}[\Phi]+R_k)^{-1})^{EF}((\Gamma_k^{(2)}[\Phi]+R_k)^{-1})^{BJ} \\
&- \frac{1}{2}\left(\frac{d}{dk}R_{kIJ}\right)\left(\frac{d}{dk}R_{kCH}\right)((\Gamma_k^{(2)}[\Phi]+R_k)^{-1})^{IA}((\Gamma_k^{(2)}[\Phi]+R_k)^{-1})^{CD} \\
&\qquad \times (\Gamma_k^{(3)}[\Phi])_{ADE}((\Gamma_k^{(2)}[\Phi]+R_k)^{-1})^{EF}(\Gamma_k^{(3)}[\Phi])_{BFG} \\
&\qquad \times ((\Gamma_k^{(2)}[\Phi]+R_k)^{-1})^{GH}((\Gamma_k^{(2)}[\Phi]+R_k)^{-1})^{BJ}
\end{aligned}
$$

Second order flow of correlation functions

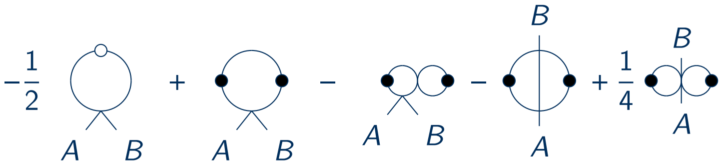
- second order flow of one-point function

$$\frac{\partial^2}{\partial k^2}(\Gamma_k^{(1)}[\Phi])_A =$$



- second order flow of two-point function (with Z_2 symmetry)

$$\frac{\partial^2}{\partial k^2}(\Gamma_k^{(2)}[\Phi])_{AB} =$$



Seconary flow

- another idea: generalize the right hand die of the flow equation
- introduce

$$\mathsf{B}_{k,l}[\Phi] = \frac{1}{2}\left(\frac{d}{dk}R_{kAB}\right)((\Gamma_k^{(2)}[\Phi] + R_k + Q_l)^{-1})^{AB}.$$

- so that

$$\frac{\partial}{\partial k}\Gamma_k[\Phi] = \mathsf{B}_{k,0}[\Phi].$$

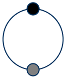
- take $R_k = k^2$ and new additional regulator $Q_l = l^2$
- consider $B_{k,l}[\Phi]$ as functional of combination $\Gamma_k^{(2)}[\Phi] + R_k + Q_l$

Secondary flow

- new flow eaution

$$\frac{\partial}{\partial l}\mathsf{B}_{k,l}[\Phi] = -\frac{1}{2}\left(\frac{d}{dk}R_{kAB}\right)\left(\frac{d}{dl}Q_{lCD}\right)((\Gamma_k^{(2)}[\Phi]+R_k+Q_l)^{-1})^{AC}((\Gamma_k^{(2)}[\Phi]+R_k+Q_l)^{-1})^{DB}$$

- diagrammatically

$$\frac{\partial}{\partial l}\mathsf{B}_{k,l}[\Phi] = -\frac{1}{2}$$ 

- further derivatives with respect to l can be easily taken
- improved UV convergence
- solutions need boundary conditions

Secondary flow of effective potential

- $O(N)$ model in local potential approximation
- flow equation for effective potential

$$\frac{1}{k}\frac{\partial}{\partial k}U_k(\rho) = b_{k,0}(\rho),$$

- with

$$b_{k,l}(\rho) = \frac{1}{(4\pi)^{d/2}} f^{(d)}\left(U_k'(\rho) + 2\rho U_k''(\rho) + k^2 + l^2\right) + \frac{N-1}{(4\pi)^{d/2}} f^{(d)}\left(U_k'(\rho) + k^2 + l^2\right).$$

- we want to determine $f^{(d)}(z)$

Characterizing $f^{(d)}(z)$ by derivatives

- taking enough derivatives leads to finite result

$$\frac{\partial^n}{\partial z^n} f_\Lambda^{(d)}(z) = (-1)^n (4\pi)^{d/2} \int_\Lambda \frac{d^d p}{(2\pi)^d} \frac{n!}{(p^2+z)^{n+1}} = (-1)^n \frac{\Gamma(1+n-d/2)}{z^{1+n-d/2}}$$

- from here one can integrate again
- example $d = 2$

$$\frac{\partial}{\partial z} f^{(2)}(z) = -\frac{1}{z}, \qquad f^{(2)}(z) = \ln\left(\frac{L_1^2}{z}\right)$$

with integration constant L_1 of dimension mass

Flow of the effective potential

- resulting flow equation in $d = 4$ dimensions after two integrations

$$
\begin{aligned}
\frac{1}{k}\frac{\partial}{\partial k}U_k(\rho) =& N\frac{L_2^2}{16\pi^2} - \frac{U_k'(\rho) + 2\rho U_k''(\rho) + k^2}{16\pi^2}\ln\left(\frac{L_1^2}{U_k'(\rho) + 2\rho U_k''(\rho) + k^2}\right) \\
&- (N-1)\frac{U_k'(\rho) + k^2}{16\pi^2}\ln\left(\frac{L_1^2}{U_k'(\rho) + k^2}\right).
\end{aligned} \tag{1}
$$

- integration constants L_1 (relevant) and L_2 (irrelevant)
- what determines the integration constants?

Taylor expansion

- use ansatz

$$U_k(\rho) = u_k + m_k^2 \rho + \frac{\lambda_k}{2}\rho^2 + \frac{\gamma_k}{3!}\rho^3 + \frac{\zeta_k}{4!}\rho^4 + \ldots,$$

- obtain flow equations

$$k\frac{d}{dk}m_k^2 = -\frac{k^2[N+2]\lambda_k}{16\pi^2}\ln\left(\frac{L_1^2}{m_k^2+k^2}\right),$$

$$k\frac{d}{dk}\lambda_k = \frac{k^2[N+8]\lambda_k^2}{16\pi^2[m_k^2+k^2]} - \frac{[N+4]\gamma_k}{32\pi^2}\ln\left(\frac{L_1^2}{m_k^2+k^2}\right)$$

Choice of integration constants

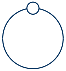
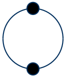
- interesting choice $L_1^2 = m_0^2$
- implies $\frac{d}{dk} m_k = 0$ at $k = 0$
- implies $m_k^2 + k^2 = m_0^2 + k^2$ to lowest order in k
- flow parameter is change in renormalized mass (Callan's scheme)

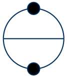
Are the integration constants physical?

- integration constants L_1 and L_2 are UV dominated
- are they similar to relevant or marginal coupling constants?
- can one fix them by an experiment?
- yes, when response of correlation functions to change in microscopic mass-squared can be measured
- otherwise: residual scheme-dependence similar to choice of regulator

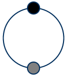
Conclusions

- exact flow equation of second order can help to test approximations/truncations

$$\frac{\partial^2}{\partial k^2}\Gamma_k[\Phi] = \frac{1}{2}$$

$$- \frac{1}{2}$$

$$+ \frac{1}{4}$$

$$- \frac{1}{2}$$


- secondary flow allows to work with $R_k = k^2$ supplemented with $Q_l = l^2$

$$\frac{\partial}{\partial k}\Gamma_k[\Phi] = \mathrm{B}_{k,0}[\Phi] \qquad \frac{\partial}{\partial l}\mathrm{B}_{k,l}[\Phi] = -\frac{1}{2}$$


- two approaches could be combined
- non-perturbative generalisation of the Callan-Symanzik scheme