Rethinking Dimensional Regularization in Critical Phenomena

Introducing *Functional Dimensional Regularization* (FDR)

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Critical Phenomena, the Renormalization Group and Universality
are major achievements of Statistical Field Theory

Dimensional Regularization (DR)
+ ε-expansion

- Standard approach for decades.
- Expand around a single upper critical dimension d_c: $\varepsilon = d_c - d$.
- **Pro:** Agile, minimal scheme dependence.
- **Con:** Reliable only near d_c (without resummations).

Functional RG (FRG)

- Based on an explicit mass cutoff (regulator).
- Works in any dimension, well below d_c.
- **Pro:** General, non-perturbative.
- **Con:** Introduces significant scheme dependence and complications from the regulator.

Question: Can we combine the agility of DR with the generality of FRG?

A Pattern of in the Beta Functions

When using DR for a scalar theory, we usually only subtract the pole corresponding to the critical dimension of interest (e.g., $d_c = 4$ for $\frac{\lambda_4}{4!}\phi^4$) to obtain the corresponding beta function $\beta_4^{\mathrm{DR}}|_{d_c=4} = \frac{3\lambda_4^2}{(4\pi)^2}$.

Key Observation: There are scheme-independent, one-loop DR beta functions at *all* critical dimensions $d_c = 0, 2, 4, 6, 8, \ldots$ and for *all* couplings:

	$d_c = 0$	$d_c = 2$	$d_c = 4$	$d_c = 6$	$d_c = 8$
β_2^{DR}	1	$-\lambda_4$	$\lambda_2\lambda_4$	$-\frac{1}{2}\lambda_2^2\lambda_4$	$\cdots$
β_4^{DR}	1	$-\lambda_6$	$3\lambda_4^2 + \lambda_2\lambda_6$	$-\frac{1}{2}\lambda_6\lambda_2^2 - 3\lambda_4^2\lambda_2$	$\cdots$
β_6^{DR}	1	$-\lambda_8$	$15\lambda_4\lambda_6 + \lambda_2\lambda_8$	$-15\lambda_4^3 - 15\lambda_2\lambda_6\lambda_4 - \frac{1}{2}\lambda_2^2\lambda_8$	$\cdots$

The twist: sum over all critical dimensions!

Summing along each row over $d_c = 0, 2, 4, 6, 8, \ldots$, with the extra-dimensionality factor μ^{d-d_c}, builds up an **exponential threshold function**:

$$
\begin{aligned}
\beta_2^{\mathrm{FDR}} &\equiv \mu^d + \mu^{d-2}(-\lambda_4) + \mu^{d-4}(\lambda_2\lambda_4) + \mu^{d-6}(-\tfrac{1}{2}\lambda_2^2\lambda_4) + \cdots = \mu^d(-\lambda_4)e^{-\lambda_2/\mu^2} \\
\beta_4^{\mathrm{FDR}} &\equiv \mu^d + \mu^{d-2}(-\lambda_6) + \mu^{d-4}(3\lambda_4^2 + \lambda_2\lambda_6) + \cdots = \mu^{d-4}(3\lambda_4^2 - \mu^2\lambda_6)e^{-\lambda_2/\mu^2}
\end{aligned}
$$

The Master Formula

The pattern generalizes. The FDR beta function for a coupling λ_i is the sum over all critical dimensions of its DR beta functions, each weighted by its extra-dimensionality factor:

Master Formula for Couplings

$$\beta_i^{\mathrm{FDR}}(d) = \sum_{d_c} \mu^{d-d_c}\, \beta_i^{\mathrm{DR}}(d_c)$$

- The sum over d_c includes both UV ($d_c > d$) and IR ($d_c < d$) dimensions.
- **Crucial point:** IR dimensions are essential. They allow d to be any positive value.
- Summing these contributions over all d_c reveals the threshold functions proper of DR.

This leads directly to a new functional RG scheme: Functional Dimensional Regularization (FDR)

Towards a Functional Flow in Dimensional Regularization

We will present a simple computation that uses the Heat Kernel starting from the one-loop effective action

$$\boxed{\Gamma_1[\phi] = \tfrac{1}{2}\mathrm{Tr}\log S^{(2)}[\phi]}$$

We will work with a general potential (i.e. LPA) so the action is just

$$S[\phi] = \int \mathrm{d}^d x \left\{ \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + V(\phi) \right\}$$

From here we can compute the Hessian $S^{(2)} = -\Box + V''(\phi)$ which we can insert into Γ_1 and then perform some basic manipulations:

$$\Gamma_1 = \tfrac{1}{2}\mathrm{Tr}\log(-\Box + V'') = -\tfrac{1}{2}\int_0^\infty \frac{\mathrm{d}s}{s}\mathrm{Tr}\, e^{-s(-\Box+V'')} = -\Omega\tfrac{1}{2}\frac{1}{(4\pi)^{\frac{d}{2}}}\int_0^\infty \mathrm{d}s\, s^{-(\frac{d}{2}+1)}e^{-sV''}$$

where we assume ϕ constant and where we defined Ω as the volume. Now we do the proper-time integral which gives – diverging – Gamma functions ($\varepsilon = d_c - d$):

$$\boxed{\Gamma_1 = -\Omega\frac{\Gamma\left(-\frac{d}{2}\right)}{2(4\pi)^{\frac{d}{2}}}(V'')^{\frac{d}{2}}} \qquad \Gamma\left(-\tfrac{d}{2}\right) = \frac{1}{\varepsilon}(-1)^{\frac{d_c}{2}}\frac{2}{\left(\frac{d_c}{2}\right)!} + \ldots$$

The Central Insight: Subtracting all the Poles

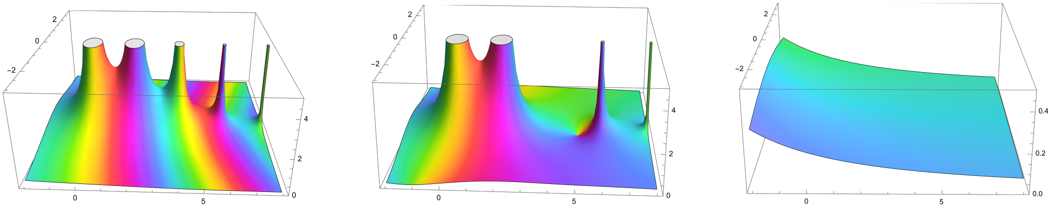

Figure: The function $\Gamma(-\frac{d}{2})$ in the complex d-plane with poles at $d_c = 0, 2, 4, 6, 8, \ldots$.

Our proposal: subtract ALL poles in the complex d-plane.

- Not just the one at the d_c we are expanding around!
- This naturally incorporates contributions from irrelevant operators.
- The result is finite over all the real-d line :)

From Couplings to a Functional Flow

The coefficients of the counter-term action ΔS_L define the DR beta functionals:

$$\Delta S_L = -\Gamma_\infty^{(L)} = \frac{1/L}{\varepsilon}\int \mathrm{d}^d x \left\{ \beta_V^{(L)}(\phi) + \beta_Z^{(L)}(\phi)\,\frac{1}{2}(\partial\phi)^2 + \cdots \right\}$$

From the $\frac{1}{\varepsilon}$-poles of Γ_1 we readily obtain:

$$\boxed{\beta_V^{\mathrm{DR}}(d_c) = \frac{1}{(\frac{d_c}{2})!}(-V'')^{\frac{d_c}{2}}}$$

We recover the known results $\beta_V^{\mathrm{DR}}(4) \propto (V'')^2$ and $\beta_V^{\mathrm{DR}}(6) \propto (V'')^3$, but have highlighted the general pattern.

The FDR master formula for couplings generalizes to functions (i.e. the potential in this case V):

The FDR flow of V

$$\beta_V^{\mathrm{FDR}}(d) = \sum_{d_c} \mu^{d-d_c}\beta_V^{\mathrm{DR}}(d_c) = \sum_{d_c} \mu^{d-d_c}\frac{1}{(\frac{d_c}{2})!}(-V'')^{\frac{d_c}{2}} = \mu^d e^{-V''/\mu^2}$$

The FDR-LPA is characterised by a simple exponential threshold function!

Dimensionless Flow and Field Expansion

Dimensionless flow. We study the flow in terms of dimensionless variables: field $\varphi = \phi\, \mu^{-(d-2+\eta)/2}$ and potential $v(\varphi) = \mu^{-d} V(\phi)$. The final dimensionless beta functional for the potential is:

$$\boxed{\beta_v = -d\, v + \tfrac{d-2+\eta}{2}\, \varphi\, v' + e^{-v''}}$$

Field expansion. For the `Ising` universality class use a $\mathbb{Z}_2$-symmetric potential

$$v(\varphi) = \lambda_0 + \tfrac{\lambda_2}{2}\varphi^2 + \tfrac{\lambda_4}{4!}\varphi^4 + \tfrac{\lambda_6}{6!}\varphi^6 + \dots$$

One beta functional generates all the beta functions:

$$\begin{aligned}
\beta_0 &= -d\lambda_0 + e^{-\lambda_2} \\
\beta_2 &= -(2-\eta)\lambda_2 - \lambda_4 e^{-\lambda_2} \\
\beta_4 &= (d-4+2\eta)\lambda_4 + 3\lambda_4^2 e^{-\lambda_2} - \lambda_6 e^{-\lambda_2} \\
\beta_6 &= (2d-6+3\eta)\lambda_6 - 15\lambda_4^3 e^{-\lambda_2} + 15\lambda_6\lambda_4 e^{-\lambda_2} - \lambda_8 e^{-\lambda_2} \\
\beta_8 &= (3d-8+4\eta)\lambda_8 + 105\lambda_4^4 e^{-\lambda_2} - 210\lambda_6\lambda_4^2 e^{-\lambda_2} + 28\lambda_8\lambda_4 e^{-\lambda_2} + 35\lambda_6^2 e^{-\lambda_2} - \lambda_{10} e^{-\lambda_2} \\
\beta_{10} &= \dots
\end{aligned}$$

Set $\beta_i = 0$ to look for the non-trivial fixed point in $d = 3$. Trick: truncate to N_{tr} couplings and solve iteratively $\lambda_0^*(\lambda_2)$, $\lambda_4^*(\lambda_2)$, $\lambda_6^*(\lambda_2)$, ... then solve numerically the last beta function in the truncation as a function of λ_2.

Wilson-Fisher Fixed Point (Ising)

Compute the RG spectrum form the stability matrix $M_{ij} = \partial\beta_i/\partial\lambda_j|_*$. Estimates converge (very) rapidly:

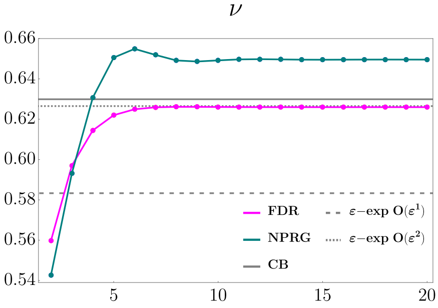

Figure: Critical exponent ν vs truncation order N_{tr}.

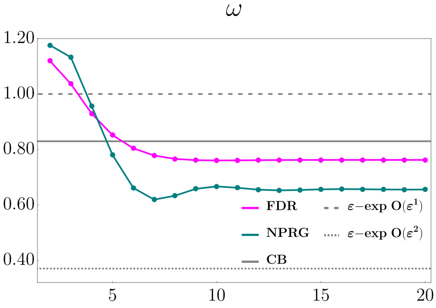

Figure: Critical exponent ω vs truncation order N_{tr}.

- FDR (magenta) shows excellent convergence compared to standard NPRG-LPA (teal).
- Estimates are remarkably close to Conformal Bootstrap (CB) values, given this is a simple one-loop calculation.

Beyond LPA: Anomalous Dimension η

To go further, we need the flow of the wave-function renormalization $Z(\phi)$

$$S = \int \mathrm{d}^d x \left\{ \frac{1}{2}(\partial\phi)^2 + V(\phi) + \frac{1}{2} Z(\phi)(\partial\phi)^2 \right\}$$

which gives the anomalous dimension:

$$\boxed{\eta = -\mu^{\eta} \beta_Z \Big|_{\varphi \to 0}}$$

A calculation similar to the above gives:

The FDR flow of Z

$$\beta_Z(d) = -\mu^{d-6} \tfrac{1}{6} (V''')^2 e^{-V''/\mu^2} + \underbrace{\left\{ -\mu^{d-2} Z'' + \mu^{d-4} V''' Z' + \mu^{d-6} \tfrac{1}{3} V'' V''' Z' \right\} e^{-V''/\mu^2}}_{\text{from including the } Z \text{ operator in the action}}$$

For an **even** theory like `Ising`, the leading contribution vanishes. The dominant term comes from the expansion of $z(\varphi) = \mu^{\eta} Z(\phi)$ itself:

$$\boxed{\eta = z_2 \, e^{-\lambda_2}}$$

where $z(\varphi) = \frac{1}{2} z_2 \varphi^2 + \dots$.

Anomalous Dimension η in $d = 3$ for the `Ising` universality class

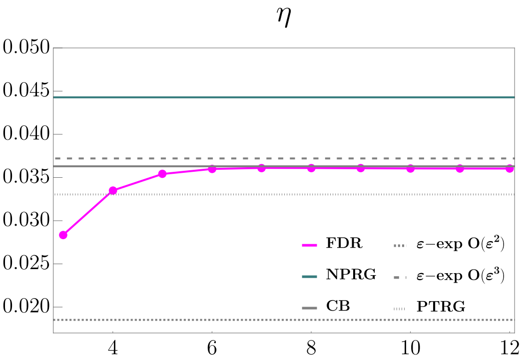

Figure: Convergence of η with truncation order N_{tr}.

- **FDR result:** $\eta \approx 0.0360$
- **State-of-the-art:**
  - CB: $\eta = 0.0362978(20)$ from 10^9 CPU hours!
  - NPRG (DE ∂^2): $\eta \approx 0.0443$ is 30% off...
  - NPRG (DE ∂^6): $\eta \approx 0.0365$ form equations +100 pages long!
  - PTRG (DE ∂^2): $\eta \approx 0.033$ (the limt $m \to \infty$ is related to FDR)
  - ε-exp. $O(\varepsilon^2)$: $\eta = 0.0185$ at 2-loop
  - ε-exp. $O(\varepsilon^3)$: $\eta = 0.0370$ at 3-loop
- Remarkable precision with very few couplings ($N_{\mathrm{tr}} \sim 12$) and minimal computational cost (few minutes on laptop).

A Surprising Result: Two-Loop from One-Loop

Let's test FDR at $d_c = 4$ by expanding the exponential in the dimensionless flow of the marginal coupling:

Expanding the β_4 flow

$$\beta_4 = 2\eta\lambda_4 + (3\lambda_4^2 - \lambda_6)e^{-\lambda_2} = \underbrace{3\lambda_4^2}_{\text{1-loop}} + \underbrace{2z_2\lambda_4 - 3\lambda_2\lambda_4^2 - \lambda_6}_{\text{2-loop}} + O(\lambda_4^4)$$

By solving the fixed-point equations $\beta_2 = 0$, $\beta_6 = 0$ and $\beta_{z_2} = 0$ for $\lambda_2^*(\lambda_4), \lambda_6^*(\lambda_4), z_2^*(\lambda_4)$ and substituting back in β_4, one recovers **exactly** the standard two-loop DR beta function:

$$\beta_4 = 3\lambda_4^2 - \frac{17}{3}\lambda_4^3 + O(\lambda_4^4) \qquad\qquad \eta = \frac{1}{6}\lambda_4^2 + O(\lambda_4^3)$$

This equivalence extends to the full RG spectrum in $d = 4 - \varepsilon$. From the one-loop FDR flow one computes:

FDR (1-loop) = DR (2-loop)

- $\nu = \frac{1}{2} + \frac{\varepsilon}{12} + \frac{7}{162}\varepsilon^2 + O(\varepsilon^3)$
- $\omega = \varepsilon - \frac{17}{27}\varepsilon^2 + O(\varepsilon^3)$
- $\eta = \frac{\varepsilon^2}{54} + O(\varepsilon^3)$

Implications:

- FDR is ε-correct!
- FDR at one-loop order is intrinsically *one-loop better* than standard DR.
- It suggests the FDR functional flow already resums an infinite set of perturbative contributions.

A Stronger Test: Unitary Multi-critical Models

Can we resolve the series of unitary multi-critical models in $d = 2$?

Yes! The *Landscape of Field Theories* maps FDR-DE2 scaling solutions:

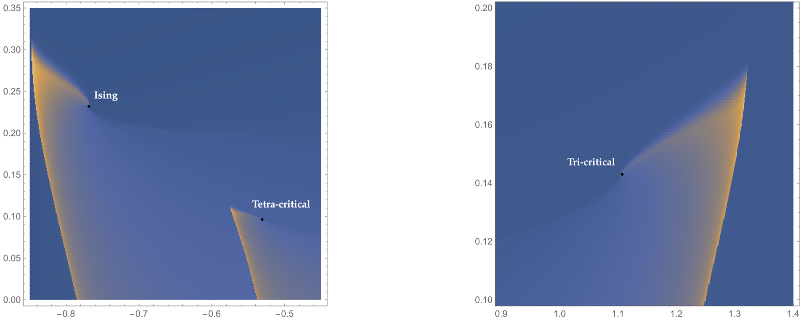

Figure: Density plot of the landscape around the `Ising` ($\sigma \simeq -0.769$ and $\eta = 0.232$ [$\eta_{\rm CFT} = 0.25$]) and `Tetra-critical` ($\sigma \simeq -0.531$ and $\eta = 0.096$ [$\eta_{\rm CFT} = 0.1$]) fixed points (left) and around the `Tri-critical` ($\sigma \simeq 0.107$ and $\eta = 0.143$ [$\eta_{\rm CFT} = 0.15$]) fixed point (right).

A one-loop (functional) DR computations sees all unitary CFTs in two dimensions!

Outlook: A Multi-Loop Master Formula

How do we go beyond the one-loop order? A natural generalization is:

Generalized Master Formula

$$\beta^{\mathrm{FDR}}(d) = \sum_{L} \sum_{d_c \in L} \mu^{L(d-d_c)} \beta_L^{\mathrm{DR}}(d_c)$$

Here the sum is over loop order L, where at each L we sum over a set of critical dimensions d_c.

Possible choices for the extension of the FDR scheme:

(a) Keep only **leading order** dimensions for full universality.

(b) Include **scheme-dependent** contributions (starting at NLO).

(c) Use **RG improvement**: keep one-loop structure but include all operators.

Work in progress...

Summary: A New Functional RG is Born

We have constructed a new functional RG scheme, FDR, by rethinking dimensional regularization.

The Core Principle

Subtract **all** one-loop poles in the complex d-plane, not just the one at the critical dimension of interest. This naturally produces a functional flow without an explicit mass cutoff.

Key Achievements

- **Combines the best of both worlds:** The agility of DR with the generality of FRG.
- **Minimizes scheme dependence:** No arbitrary regulator function.
- **Excellent performance:** Critical exponents in $d = 3$ rivaling state-of-the-art methods, even at simple truncations.
- **Intrinsic resummation:** Reproduces standard 2-loop results from a 1-loop calculation.

Future Directions and Impact

Immediate Next Steps:

- Generalization to `O(N)` (see `2604.26207`), `O(N)`$\times$`O(2)` and `Dipolar` universality classes.
- Apply to non-unitary theories (e.g., `Lee-Yang` universality class).
- Unitary `multi-critical` models in $2 \leq d \leq 4$.
- Explore contributions beyond one-loop, higher order derivative expansion and essential RG.

Broader Impact:

- Towards a *classification of universality classes* for N-components field theories.
- DR is central to Quantum Field Theory. This framework can have applications far beyond critical phenomena.
- Potential impact on High-Energy Physics, Gauge Theories, and Quantum Gravity, where a functional, minimally scheme-dependent RG is highly desirable.

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Thank You!

Questions?