Linearity in the renormalization group equations

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Single operator model

The Callan-Symanzik Equation

$$\left(\mu^2\frac{d}{d\mu^2}+\beta(\alpha)\frac{d}{d\alpha}-\gamma(\alpha)\right)\Gamma\left(\frac{Q^2}{\mu^2},\alpha\right)=0$$

Running coupling

$$\frac{\partial}{\partial L}\bar{\alpha}[\alpha,L]=\beta(\bar{\alpha})=\beta_0\bar{\alpha}^2+\beta_1\bar{\alpha}^3+\beta_2\bar{\alpha}^4+\beta_3\bar{\alpha}^5+O(\bar{\alpha}^6)$$

$$L=\log(Q^2/\mu^2)$$

1-loop approximation

$$\frac{\partial}{\partial L}\alpha_1=\beta_0\alpha_1^2$$

2-loop approximation

$$\frac{\partial}{\partial L}\alpha_2=\beta_0\alpha_2^2+\beta_1\alpha_2^3$$

Initial conditions

$$\alpha_1[\alpha,0]=\alpha$$

$$\alpha_2[\alpha,0]=0$$

Solutions

$$\alpha_1=\frac{\alpha}{1-\beta_0\alpha L}$$

The solution contains a non-elementary Lambert function

$$\alpha_2=\frac{1}{1+W(\beta_0,\beta_1,L)}$$

Exact solutions for higher orders of the running coupling are not known

?

$$\alpha_3,\alpha_4\,..$$

Analysis of coefficients in perturbative expansion

Let us consider the n-loop approximation and the coefficients of the corresponding logarithms

$x = Q^2/\mu^2$

$$f_1(\beta_0)\log(x)^N + f_2(\beta_0, \beta_1)\log(x)^{N-1} + f_3(\beta_0, \beta_1, \beta_2)\log(x)^{N-2} + \dots$$

$$\frac{\partial}{\partial L}\alpha_1 = \beta_0\alpha_1^2 \qquad \frac{\partial}{\partial L}\alpha_2 = \beta_0\alpha_2^2 + \beta_1\alpha_2^3 \qquad \frac{\partial}{\partial L}\alpha_3 = \beta_0\alpha_3^2 + \beta_1\alpha_3^3 + \beta_2\alpha_3^4$$

However, it turns out that you can extract the third power of the logarithm from the equation in a subleading approximation

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However, it turns out that you can extract the third power of the logarithm from the equation in a subleading approximation

$$\alpha_2 = \alpha_2(\beta_0, \beta_1) + \xcancel{\alpha_2(\beta_0, \beta_1, \beta_2, \beta_3 \dots)}$$

Running coupling

Running coupling is the sum of logarithms in all orders of the PT

$$\bar{\alpha}[\alpha, L] = \alpha\left(1 + \sum_{k=1}^{\infty} A_k\, \alpha^k L^k + \sum_{k=2}^{\infty} B_k\, \alpha^k L^{k-1} + \sum_{k=3}^{\infty} C_k\, \alpha^k L^{k-2} + \ldots\right) \qquad L = \log(Q^2/\mu^2)$$

$$\bar{\alpha}[\alpha, L] = \sum_{i=1}^{\infty} \alpha_i = \alpha_1 + \alpha_2 + \alpha_3 \ldots$$

How can we extract the corresponding α_n ?

Kazakov, et al *Phys.Rev.D* 113 (2026) 12, 125021

Kazakov, et al *JHEP* 04 (2023) 128

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1) A general method that is applicable regardless of the renormalizability of the model

2) the resulting equation is always linear

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Let's make the substitution into the equation for the running coupling

$$\frac{\partial}{\partial L}\bar{\alpha} = \beta(\bar{\alpha}) = \beta_0\bar{\alpha}^2 + \beta_1\bar{\alpha}^3 + \beta_2\bar{\alpha}^4 + \beta_3\bar{\alpha}^5 + O(\bar{\alpha}^6)$$

We take only a linear part

And the same order for PT $[\alpha_n] = [\alpha_1^n]$

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$$\frac{d\alpha_1}{dL} = \beta_0\alpha_1^2$$

$$\frac{d\alpha_2}{dL} = 2\beta_0\alpha_1\alpha_2 + \beta_1\alpha_1^3$$

α_1 is excluded automatically

$$\frac{d\alpha_3}{dL} = 2\beta_0\alpha_1\alpha_3 + \beta_0\alpha_2^2 + 3\beta_1\alpha_1^2\alpha_2 + \beta_2\alpha_1^4,$$

$$\frac{d\alpha_4}{dL} = 2\beta_0\alpha_1\alpha_4 + 2\beta_0\alpha_2\alpha_3 + 3\beta_1\alpha_1^2\alpha_3 + 3\beta_1\alpha_1\alpha_2^2 + 4\beta_2\alpha_1^3\alpha_2 + \beta_3\alpha_1^5$$

$$\frac{d\alpha_5}{dL} = 2\beta_0\alpha_1\alpha_5 + 2\beta_0\alpha_2\alpha_4 + \beta_0\alpha_3^2 + \beta_1\alpha_2^3 + 3\beta_1\alpha_1^2\alpha_4 + 6\beta_1\alpha_1\alpha_2\alpha_3 +$$

$$+4\beta_2\alpha_1^3\alpha_3 + 6\beta_2\alpha_1^2\alpha_2^2 + 5\beta_3\alpha_1^4\alpha_2 + \beta_4\alpha_1^6$$

Solving equations

$$\alpha_1[\alpha, L] = \frac{\alpha}{1 - \beta_0 \alpha L}$$

$$\bar{\beta}_N = \beta_N / \beta_0$$

It is important that the solutions to these equations are easy to find and expressed in elementary functions.

$$\alpha_2[\alpha, L] = \bar{\beta}_1 \alpha_1^2 \log\left(\frac{\alpha_1}{\alpha}\right)$$

$$\alpha_3[\alpha, L] = \bar{\beta}_1^2 \alpha_1^3 \left(\log^2\left(\frac{\alpha_1}{\alpha}\right) + \log\left(\frac{\alpha_1}{\alpha}\right)\right) + \alpha_1^2(\alpha_1 - \alpha)(\bar{\beta}_2 - \bar{\beta}_1^2)$$

$$\alpha_4[\alpha, L] = \bar{\beta}_1^3 \alpha_1^4 \left(\log^3\left(\frac{\alpha_1}{\alpha}\right) + \frac{5}{2}\log^2\left(\frac{\alpha_1}{\alpha}\right) - \bar{\beta}_2(3\alpha_1 - 2\alpha))\log\left(\frac{\alpha_1}{\alpha}\right)\right) + \text{et al}$$

$$\alpha_5[\alpha, L] = \bar{\beta}_1^4 \alpha_1^5 \left(\log^4\left(\frac{\alpha_1}{\alpha}\right) + \frac{13}{3}\log^3\left(\frac{\alpha_1}{\alpha}\right) - \frac{3}{2}\log^2\left(\frac{\alpha_1}{\alpha}\right) - 4\log\left(\frac{\alpha_1}{\alpha}\right)\right) +$$

$$+ (6\alpha_1^5 \bar{\beta}_2 \bar{\beta}_1^2 + 3\alpha_1^4 \alpha \bar{\beta}_1^4 - 3\alpha_1^4 \alpha \bar{\beta}_2 \bar{\beta}_1^2)\log^2\left(\frac{\alpha_1}{\alpha}\right) + \text{et al}$$

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$$\boxed{\alpha_1[\text{column sum}] = \frac{\alpha}{1 - \beta_0 \alpha \left[\bar{\beta}_1 \alpha_1 \log\left(\frac{\alpha_1}{\alpha}\right)\right]}}$$

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$$\alpha_1[\text{column sum}] = \alpha_1\left[\alpha, \bar{\beta}_1\alpha_1 \log\left(\frac{\alpha_1}{\alpha}\right)\right]$$

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New approximation

$$\alpha_N[\text{new approximation for N order of PT}] = \sum_{i=1}^{N} \alpha_i(L) + \sum_{i=1}^{N} \alpha_1^i \alpha_i \left[\bar{\beta}_1 \alpha_1 \log\left(\frac{\alpha_1}{\alpha}\right)\right]$$

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An object with anomalous dimensions

$$\left(\mu^2 \frac{d}{d\mu^2} + \beta(\alpha)\frac{d}{d\alpha} - \gamma(\alpha)\right)\Gamma\left(\frac{Q^2}{\mu^2}, \alpha\right) = 0$$

$$\frac{d\log\Gamma}{d\log\mu^2} = \gamma(\alpha)$$

Formal solution

$$\log\Gamma = \int_{\alpha}^{\bar{\alpha}} \frac{\gamma(x)}{\beta(x)} dx$$

$$\gamma(\alpha) = \gamma_0\alpha + \gamma_1\alpha^2 + \gamma_2\alpha^3 + \ldots$$

Linearisation

$$\log\Gamma = \log\Gamma_1 + \log\Gamma_2 + \log\Gamma_3 + \ldots$$

Summary

✓A method has been found for obtaining analytical solutions to the renormalisation group equations in the single-coupling model

✓A new approximation formula for taking into account quantum corrections in a given order of perturbation theory is proposed

Thank you for attention!