Summation of logarithmic quantum corrections based on the locality of QFT

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⋆ Callan-Symanzik RG equations [Callan'1970, Symanzik'1970]

$$\left(\mu\frac{\partial}{\partial\mu}+\beta_g\frac{\partial}{\partial g}-\gamma_\phi\right)\Gamma_{n,R}=0$$

⋆ Exact RG [J. Polchinski'1984, C. Wetterich'1993]

$$k\partial_k\Gamma_k=\frac{1}{2}STr\left((\Gamma_k^{(2)}+R_k)^{-1}k\partial_k R_k\right)$$

⋆ BPHZ R-operation [Bogoliubov, Parasiuk'1957, Hepp'1966, Zimmerman'1967]

$$\mathcal{L}\to\mathcal{L}+\Delta\mathcal{L}$$

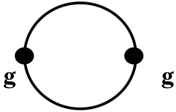
Main ideas

- ⋆ $\mathcal{R}$-operation works for (non)renormalizable theories and leads to local counter terms
- ⋆ *Bogoliubov-Parasiuk theorem*: In any local quantum field theory after subtracting the UV divergences in subgraphs the resulting counterterms are always local in coordinate space or at most are polynomials of external momenta in momentum space in each order of perturbation theory
- ⋆ Bogoliubov-Parasiuk theorem is valid for all types of interactions
- ⋆ Reccurence relations for Green functions, amplitudes, potential, etc

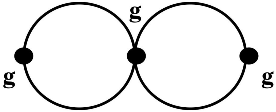
$\mathcal{R}'$-operation [Bogoliubov, Parasiuk'1957, Hepp'1966, Zimmerman'1967]

$\mathcal{R}$-operation is a procedure of substraction of all UV-devergencies in the Feynman graph G

$$\mathcal{R}G = \sum_{\gamma}(1 - K_{\gamma})G,$$
$$\mathcal{R}' = 1 - \sum_{\gamma} K\mathcal{R}'\gamma - \sum_{\gamma,\gamma'} K\mathcal{R}'\gamma K\mathcal{R}'\gamma - \dots$$



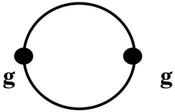
$$\sim \frac{g^2}{\epsilon} + g^2 \log\frac{\mu^2}{Q^2}$$



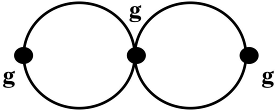
$$\sim \frac{g^3}{\epsilon^2} + g^3\log^2\frac{\mu^2}{Q^2} + 2g^3\frac{\log\frac{\mu^2}{Q^2}}{\epsilon}$$

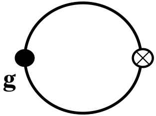
$R'G_n$ is local, i.e. terms like $\log^k \mu^2/\epsilon^m$ should cancel for any k and m

$\mathcal{R}'$-operation [Bogoliubov, Parasiuk'1957, Hepp'1966, Zimmerman'1967]



$$\sim \frac{g^2}{\epsilon} + g^2 \log \frac{\mu^2}{Q^2}$$



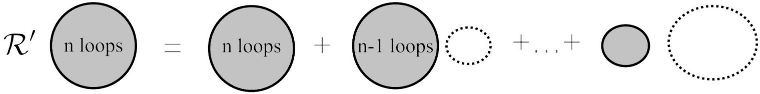


$$\sim g^3 \left(\frac{1}{\epsilon^2} + \log^2 \frac{\mu^2}{Q^2} + 2 \frac{\log \frac{\mu^2}{Q^2}}{\epsilon} \right)$$

$$\sim -2g \frac{g^2}{\epsilon} \left(\frac{1}{\epsilon} + \log \frac{\mu^2}{Q^2} \right)$$

$R'G_n$ is local, i.e. terms like $\log^k \mu^2/\epsilon^m$ should cancel for any k and m

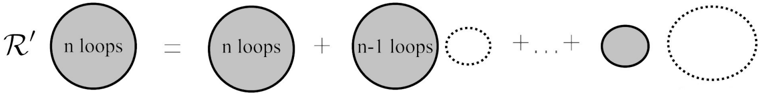
$\mathcal{R}'$-operation



$$
\begin{aligned}
\mathcal{R}' G_n = & \frac{\mathcal{A}_n^{(n)}}{\epsilon^n} (\mu^2)^{n\epsilon} + \frac{\mathcal{A}_{n-1}^{(n)}}{\epsilon^n} (\mu^2)^{(n-1)\epsilon} + \ldots \frac{\mathcal{A}_1^{(n)}}{\epsilon^n} \mu^\epsilon + \\
& + \frac{\mathcal{B}_n^{(n)}}{\epsilon^{n-1}} (\mu^2)^{n\epsilon} + \frac{\mathcal{B}_{n-1}^{(n)}}{\epsilon^{n-1}} (\mu^2)^{(n-1)\epsilon} + \ldots \frac{\mathcal{B}_1^{(n)}}{\epsilon^{n-1}} \mu^\epsilon + \\
& + \frac{\mathcal{C}_n^{(n)}}{\epsilon^{n-2}} (\mu^2)^{n\epsilon} + \frac{\mathcal{C}_{n-1}^{(n)}}{\epsilon^{n-2}} (\mu^2)^{(n-1)\epsilon} + \ldots \frac{\mathcal{C}_1^{(n)}}{\epsilon^{n-2}} \mu^\epsilon + \\
& + \text{lower pole terms like } \mathcal{D}_k^{(n)}, \mathcal{E}_k^{(n)} \text{ etc.}
\end{aligned}
$$

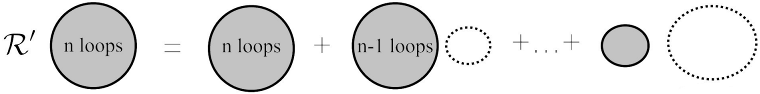
dimensional regularization $d = 4 - 2\epsilon$

2 loop case



$$
\begin{aligned}
\mathcal{R}' G_2 &= \frac{\mathcal{A}_2^{(2)}}{\epsilon^n} (\mu^2)^\epsilon + \frac{\mathcal{A}_1^{(2)}}{\epsilon^n} \mu^\epsilon = \\
&= \frac{\mathcal{A}_1^{(2)} + \mathcal{A}_2^{(2)}}{\epsilon^2} + \color{red}{\frac{\mathcal{A}_1^{(2)} \log \mu + 2\mathcal{A}_2^{(2)} \log \mu}{\epsilon}} + \frac{1}{2} \mathcal{A}_1^{(2)} \log \mu^2 + 2\mathcal{A}_2^{(2)} \log \mu^2 + O(\epsilon)
\end{aligned}
$$

$\mathcal{R}'$-operation



$$
\begin{aligned}
\mathcal{R}'G_n = {} & \frac{\mathcal{A}_n^{(n)}}{\epsilon^n}(\mu^2)^{n\epsilon} + \frac{\mathcal{A}_{n-1}^{(n)}}{\epsilon^n}(\mu^2)^{(n-1)\epsilon} + \dots \frac{\mathcal{A}_1^{(n)}}{\epsilon^n}\mu^\epsilon + \\
& + \frac{\mathcal{B}_n^{(n)}}{\epsilon^{n-1}}(\mu^2)^{n\epsilon} + \frac{\mathcal{B}_{n-1}^{(n)}}{\epsilon^{n-1}}(\mu^2)^{(n-1)\epsilon} + \dots \frac{\mathcal{B}_1^{(n)}}{\epsilon^{n-1}}\mu^\epsilon + \\
& + \frac{\mathcal{C}_n^{(n)}}{\epsilon^{n-2}}(\mu^2)^{n\epsilon} + \frac{\mathcal{C}_{n-1}^{(n)}}{\epsilon^{n-2}}(\mu^2)^{(n-1)\epsilon} + \dots \frac{\mathcal{C}_1^{(n)}}{\epsilon^{n-2}}\mu^\epsilon + \\
& + \text{lower pole terms like } \mathcal{D}_k^{(n)}, \mathcal{E}_k^{(n)} \text{ etc.}
\end{aligned}
$$

We can resolve these system and obtain the reccurence relations

The Local Counter Terms [Kazakov et al'2016]

Due to locality all higher order divergences are related to the lower ones

$$\mathcal{A}_n^{(n)} = (-1)^{n+1}\frac{\mathcal{A}_1^{(n)}}{n}$$ The leading divergences are governed by 1 loop diagrams!

Coefficients of $1/\epsilon^n$

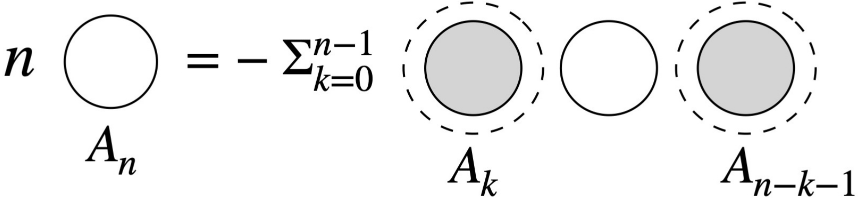
$$\mathcal{B}_n^{(n)} = \left(\frac{2\mathcal{B}_2^{(n)}}{n(n-1)} + \frac{2\mathcal{B}_1^{(n)}}{n}\right)$$ One and two loop diagrams!

Coefficients of $1/\epsilon^{n-1}$

$\bar{\mathcal{A}}_n^{(n)} \equiv \mathcal{A}_n^{(n)}$, Coefficient of the leading logarithms $\log^n \mu^2$

$\bar{\mathcal{B}}_n^{(n)} = (-1)^n \left(\mathcal{B}_1^{(n)} + \frac{2}{(n-1)}\mathcal{B}_2^{(n)}\right)$ Coefficient of the subleading logarithms $\log^{n-1} \mu^2$

Reccurence relations [Kazakov et al'2016]



- ⋆ They can be promoted to the RG equations for the scattering amplitudes, effective potential, etc which sum up the leading divergences (logarithms) and to find out the high energy/field behaviour
- ⋆ In renormalizable theories A_n is a constant and this relation is reduced to the algebraic one
- ⋆ In non-renormalizable theories for the scattering amplitudes A_n depends on kinematics and one has to integrate through the one loop diagrams
- ⋆ For the effective potential A_n depends on the fields and one has to differentiate

Effective Potential

The effective potential is defined as part of the effective action without derivatives.
The generating functional for the Green functions in the path integral formalism

$$Z(J) = \int \mathcal{D}\phi \texttt{exp} \left(i \int d^4x \mathcal{L}(\phi, d\phi) + J\phi \right),$$

take the functional $W(J) = -i\texttt{log}Z(J)$ responsible for generating connected correlation functions.
The quantum effective action is obtained using the Legendre transformation of $W(J)$

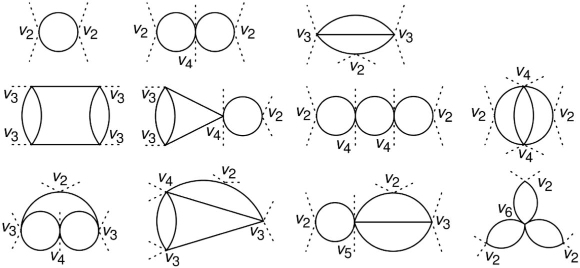
$$\Gamma(\phi) = W(J) - \int d^4x J(x)\phi(x),$$

where the classical field $\phi(x)$ is defined as the solution to $\phi(x) = \frac{\delta W(J)}{\delta J(x)}$.

Effective Potential in Scalar Theory [Kazakov, Iakhibbaev, Tolkachev, JHEP 04 (2023)]

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - g V_0(\phi),$$

$$v_2(\phi) \equiv \frac{d^2 V_0(\phi)}{d\phi^2}, \quad v_n(\phi) \equiv \frac{d^n V_0(\phi)}{d\phi^n}$$



one has to take the 1PI vacuum diagrams with the propagator containing an infinite number of insertions of v_2. The vertices are also generated by expansion of $V_0(\phi + \hat{\phi})$ over $\hat{\phi}$.

The effective potential in a power series over the coupling g

$$V_{eff} = g \sum_{n=0}^{\infty} (-g)^n V_n.$$

Leading reccurence relation [Kazakov, Iakhibbaev, Tolkachev, JHEP 04 (2023)]

The recurrence relation looks like

$$n\Delta V_n = \frac{1}{4}\sum_{k=0}^{n-1} D_2\Delta V_k D_2\Delta V_{n-1-k}, \quad n \geq 1,$$

Using the recurrence relation one can generate the leading divergence in every order of PT pure algebraically. One can then define the sum

$$\Sigma_A(z,\phi) = \sum_{n=0}^{\infty}(-z)^n\Delta V_n(\phi), \quad z = \frac{g}{\epsilon}$$

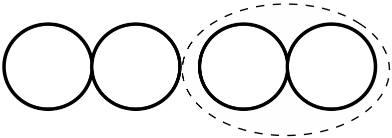
RG pole equation

$$\boxed{\frac{d\Sigma_A}{dz} = -\frac{1}{4}(D_2\Sigma_A)^2, \quad \Sigma(0,\phi) = V_0(\phi)}.$$

To get the effective potential one has just make the substitution

$$V_{eff}(g,\phi) = g\Sigma_A(z,\phi)|_{z\to -\frac{g}{16\pi^2}\log gv_2/\mu^2}.$$

Linearity of equations



$$\left(\frac{A}{\epsilon^2}+\frac{B}{\epsilon}\right)\left(\frac{A'}{\epsilon^2}+\frac{B'}{\epsilon}\right)=$$
$$=\frac{AA'}{\epsilon^4}+\frac{AB'+A'B}{\epsilon^3}+\frac{BB'}{\epsilon^2}$$

$$\frac{\partial}{\partial z}\bar{\alpha}=\beta(\bar{\alpha})=\beta_0\bar{\alpha}^2+\beta_1\bar{\alpha}^3+\beta_2\bar{\alpha}^4+\beta_3\bar{\alpha}^5+O(\bar{\alpha}^6)$$

$$\left(\mu^2\frac{d}{d\mu^2}+\beta(\alpha)\frac{d}{d\alpha}-\gamma(\alpha)\right)\Gamma\left(\frac{Q^2}{\mu^2},\alpha\right)=0$$

Subleading and higher order terms are always linear! see D. Tolkachev's talk

Subleading reccurence relation for the poles

[Kazakov, Iakhibbaev, Mukhaeva, Tolkachev, Phys.Rev.D 113 (2026)]

$$\frac{\partial^2\Sigma_B}{\partial z^2} = -2\frac{1}{4}\frac{\partial}{\partial z}\left\{2D_2\Sigma_A D_2\Sigma_B - \left(D_2\Sigma_A\right)^2 D_2\Sigma_G\right\} -$$
$$-2\left\{\frac{1}{8}(D_3\Sigma_A)^2 D_2\Sigma_B - 2\frac{1}{8}(D_3\Sigma_A)^2 D_2\Sigma_A D_2\Sigma_B + 2\frac{1}{8}D_3\Sigma_A D_3\Sigma_B D_2\Sigma_A + \right.$$
$$+2\frac{1}{8}D_3\Sigma_A D_3\Sigma_B D_2\Sigma_A - \frac{1}{8}(D_3\Sigma_A)^2 D_2\Sigma_A + \frac{1}{8}(D_2\Sigma_A)^2 D_4\Sigma_B -$$
$$\left. -2\frac{1}{8}(D_2\Sigma_A)^2 D_4\Sigma_A D_2\Sigma_G + 2\frac{1}{8}D_2\Sigma_A D_2\Sigma_B D_4\Sigma_A\right\},$$
$$\frac{\partial^2\Sigma_G}{\partial z^2} = \frac{2}{24}(D_3\Sigma_A)^2$$

Subleading reccurence relation for the logarithms

[Kazakov, Iakhibbaev, Mukhaeva, Tolkachev, Phys.Rev.D 113 (2026)]

$$
\begin{aligned}
\frac{\partial}{\partial z}\bar{\Sigma}_B = \frac{\partial}{\partial z}\left\{2D_2\Sigma_A D_2\Sigma_B - (D_2\Sigma_A)^2 D_2\Sigma_G\right\} - \\
-2\left\{\frac{1}{8}(D_3\Sigma_A)^2 D_2\Sigma_B - 2\frac{1}{8}(D_3\Sigma_A)^2 D_2\Sigma_A D_2\Sigma_B + 2\frac{1}{8}D_3\Sigma_A D_3\Sigma_B D_2\Sigma_A + \right. \\
+2\frac{1}{8}D_3\Sigma_A D_3\Sigma_B D_2\Sigma_A - \frac{1}{8}(D_3\Sigma_A)^2 D_2\Sigma_A + \frac{1}{8}(D_2\Sigma_A)^2 D_4\Sigma_B - \\
\left. -2\frac{1}{8}(D_2\Sigma_A)^2 D_4\Sigma_A D_2\Sigma_G + 2\frac{1}{8}D_2\Sigma_A D_2\Sigma_B D_4\Sigma_A\right\}.
\end{aligned}
$$

Summary

- ✓ Based on BPHZ procedure and Bogoliubov-Parasiuk theorem the new method was developed that involves summing UV counterterms and the corresponding logarithms for arbitrary interactions
- ✓ Unlike the well-known RG equations, the resulting equations are linear
- ✓ As an example, a scalar potential of arbitrary form was considered
- ✓ The leading and sub-leading equations have been derived

Thank you for attention!