ERG2026

Scaling regimes of the 1D Kuramoto-Sivashinsky equation

Liubov GOSTEVA[1], Nicolás WSCHEBOR[2], Léonie CANET[1]

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Kuramoto-Sivashinsky equation in 1D

Kuramoto and Tsuzuki (1975), Sivashinsky (1977)

$$\partial_t h(t,x) = \left(\nu \partial_x^2 - \tau \partial_x^4\right) h + \frac{\lambda}{2} (\partial_x h)^2$$

Viscosity **< 0**

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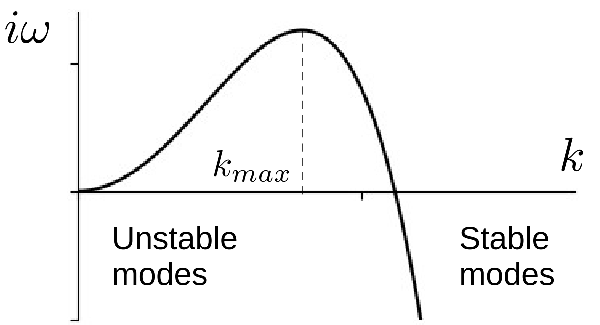
Dispersion relation:



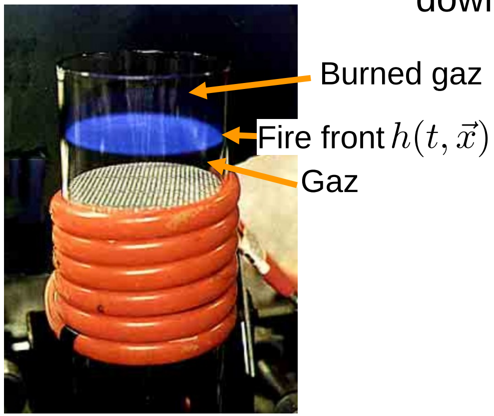
► The simplest **chaotic** partial differential equation with a quadratic or cubic nonlinearity and periodic BC Brummitt and Sprott (2009)

► Describes various phenomena with **instabilities:**

- Reaction-diffusion processes

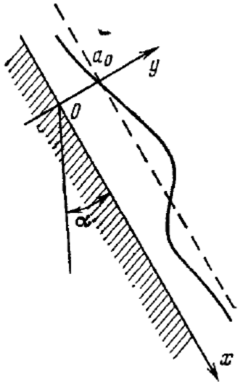
Belousov-Zhabotinsky reaction.
www.isc.meiji.ac.jp

- Flame



Searby (2006)

- Liquid film flowing down an inclined plane



Nepomnyashchii (1974)

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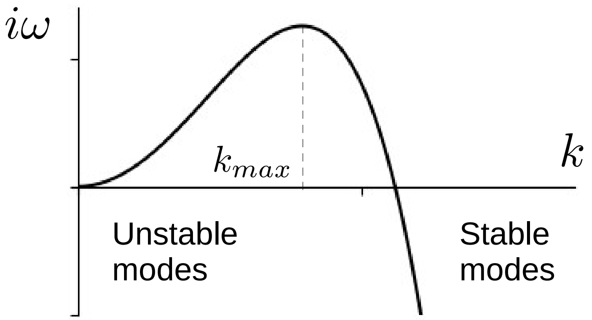

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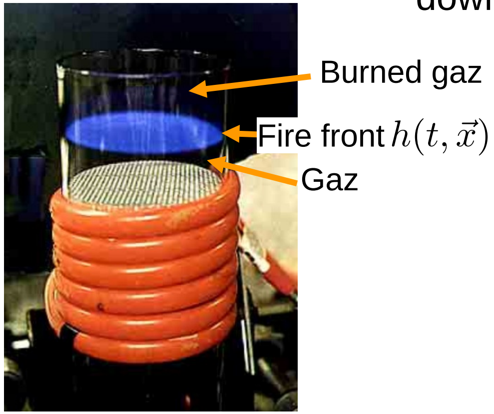

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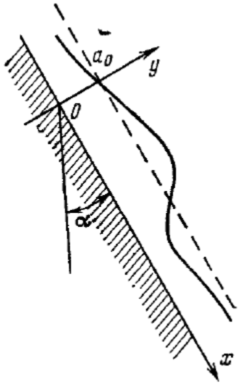

Nepomnyashchii (1974)

► Reduced description of phase dynamics in the complex Ginzburg-Landau equation in the phase turbulence regime

↔ connection to 1D nonequlibrium quantum fluids.

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KS equation falls into the KPZ universality class
with dynamical exponent $z = 3/2 \quad (t \sim p^{-z})$

Kardar-Parisi-Zhang (KPZ) equation:

Kardar, Parisi, and Zhang (1986)

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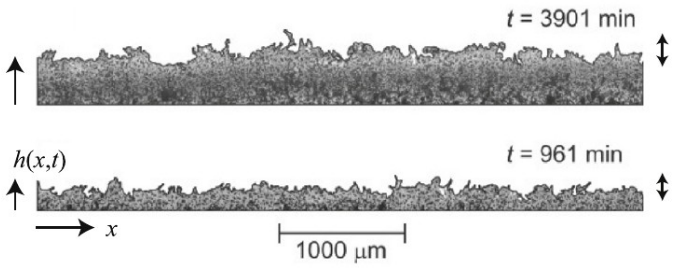
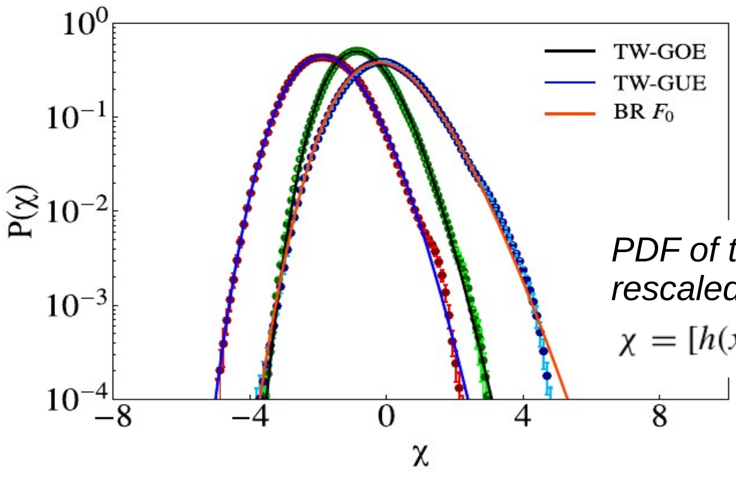
Viscosity > 0

White noise



Growing interfaces example: cancer cell colony. Huergo et al. (2012)

✅ **Numerical evidence:** Roy and Pandit (2020)



PDF of the shifted and rescaled fluctuations

$$\chi = [h(x,t) - v_\infty t]/(\Gamma t)^{1/3}$$

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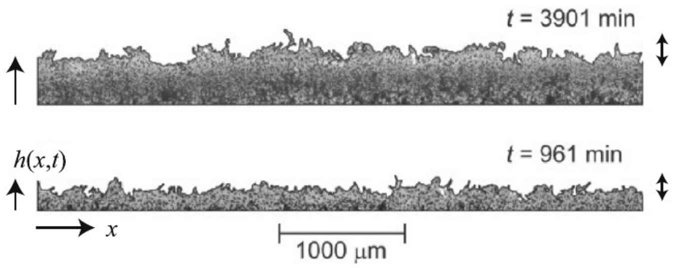

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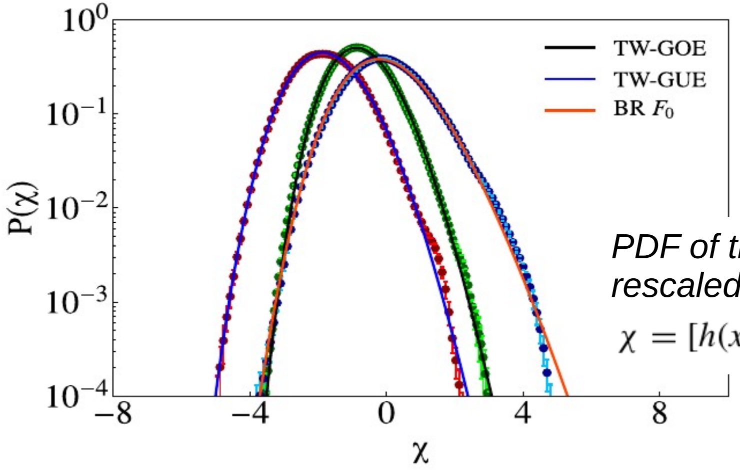

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❓**Wilsonian RG with sharp cutoff for the stochastic KS:**
Yakhot (1981); Cuerno and Lauritsen (1995); Ueno, Sakaguchi and Okamura (2005)
Problems: Morris (1996)
❌ Flows of the running paramters involve momentum derivatives, e.g.:

$$\partial_s \nu_\kappa = \frac{1}{2}\left[\frac{\partial}{\partial p^2}\partial_s \Gamma_\kappa^{(1,1)}(0,p)\right]\bigg|_{p=0}$$

❌ The cutoff is applied only to $G_0(q)$, but not to $G_0(p\text{-}q)$

$$\int_{\Lambda(l)\le|q|\le\Lambda_0}\frac{dq}{2\pi}\int_{-\infty}^{\infty}\frac{d\Omega}{2\pi}|G_0(q,\Omega)|^2 G_0(p-q,\omega-\Omega)\mathcal{F}(p,q)$$

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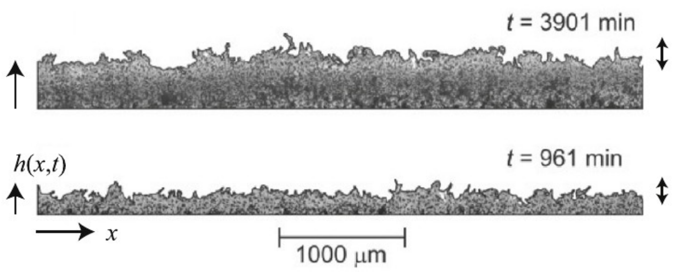

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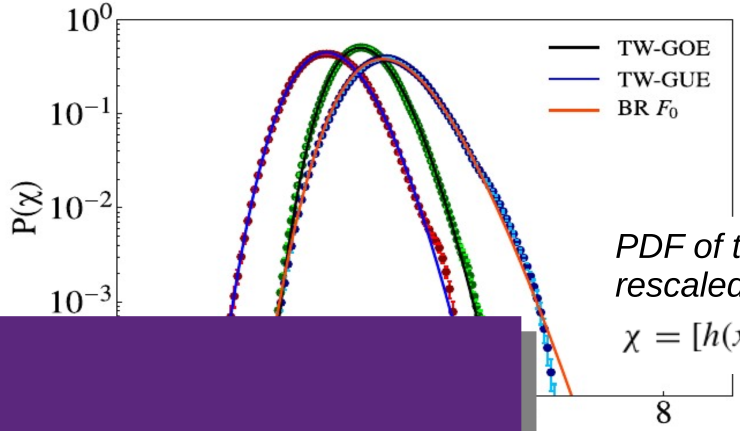

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Part 1.
Revisit the RG approach to the stochastic Kuramoto-Sivashinsky model

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1. Revisit the RG approach to the stochastic KS: FRG + sharp / smooth cutoff

$$\partial_t h = \nu\partial_x^2 h - \tau\partial_x^4 h + \tfrac{\lambda}{2}(\partial_x h)^2 + \xi, \quad \langle \xi(t,x)\xi(t',x')\rangle = 2D_{\mathrm{SKS}}\delta(t-t')\delta(x-x')$$

The bare MSRJD action: Martin, Siggia and Rose (1973); Janssen (1976); De Dominicis (1976)

$$\mathcal{S}_\Lambda[h,\bar{h}] = \int_{t,x} \bar{h}\Big\{\partial_t h - (\nu - \tau\partial_x^2)\partial_x^2 h - \frac{\lambda}{2}(\partial_x h)^2 - D_{\mathrm{SKS}}\bar{h}\Big\}$$

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The **simplest ansatz** for the effective action: promoting the parameters to scale-dependent ones

$$\Gamma_\kappa[\varphi,\bar\varphi] = \int_{t,x} \bar\varphi\Big\{\partial_t\varphi - (\nu_\kappa - \tau_\kappa\partial_x^2)\partial_x^2\varphi - \frac{\lambda}{2}(\partial_x\varphi)^2 - (D_\kappa - \zeta_\kappa\partial_x^2)\bar\varphi\Big\}$$

Needed for the emerging time-reversal symmetry

$$\Rightarrow \begin{cases} \partial_s D_\kappa &= -\frac{1}{2}\partial_s\Gamma_\kappa^{(0,2)}(0,0) \\ \partial_s \nu_\kappa &= \frac{1}{2}\left[\frac{\partial}{\partial p^2}\partial_s\Gamma_\kappa^{(1,1)}(0,p)\right]\Big|_{p=0} \\ \partial_s \zeta_\kappa &= -\frac{1}{4}\left[\frac{\partial}{\partial p^2}\partial_s\Gamma_\kappa^{(0,2)}(0,p)\right]\Big|_{p=0} \\ \partial_s \tau_\kappa &= \frac{1}{24}\left[\frac{\partial}{\partial p^4}\partial_s\Gamma_\kappa^{(1,1)}(0,p)\right]\Big|_{p=0} \end{cases} \qquad s \equiv \ln(\kappa/\Lambda)$$

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$$r(x) = \begin{cases} +\infty, & 0 \le x \le 1, \\ 0, & x > 1. \end{cases}$$

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Results:

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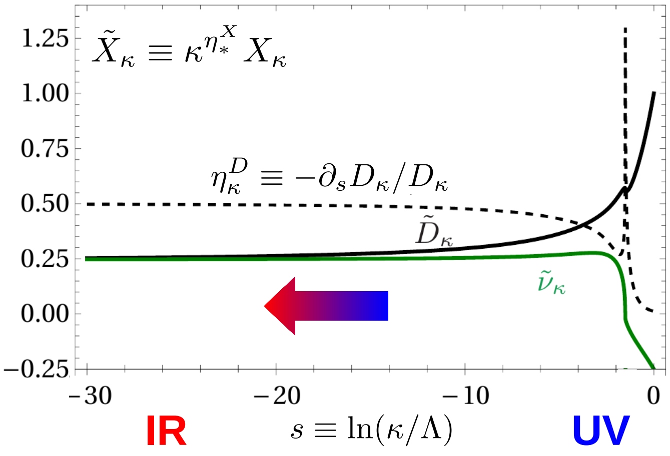
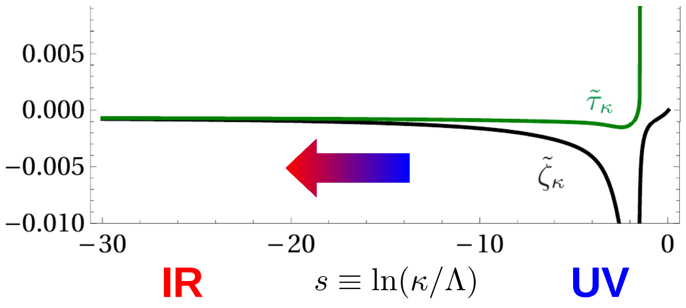
Gosteva, Wschebor, and Canet (2026)





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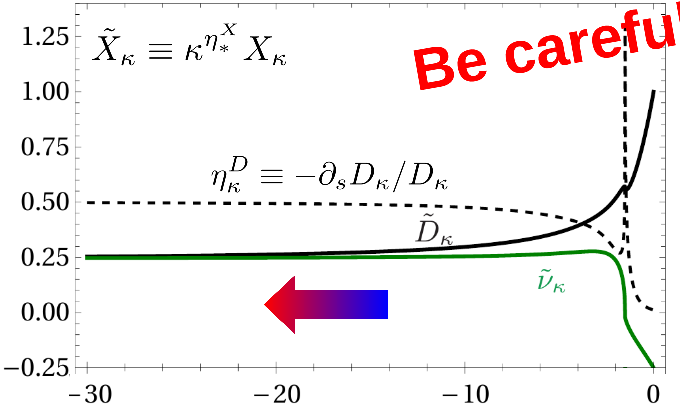
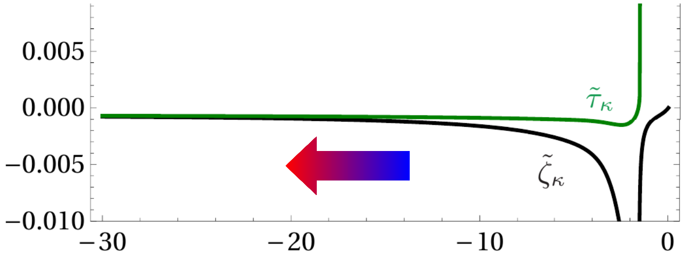
Gosteva, Wschebor, and Canet (2026)

Be careful with the sharp cutoff!



IR $s \equiv \ln(\kappa/\Lambda)$ UV



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Part 2.
Scaling regimes of the Kuramoto-Sivashinsky equation

2. Scaling regimes of the KS equation

Kardar-Parisi-Zhang (KPZ) equation

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Scale

Microscopic model

Effective macroscopic description

$z = 3/2$

Dynamical exponent: $t \sim p^{-z}$

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1D KPZ fixed points:

$$g_{\text{KPZ}} \equiv \frac{\lambda^2 D}{\nu^3}$$

EW	KPZ	IB
UV-stable	IR-stable	UV-stable

EW	KPZ	IB
$\lambda = 0$		$\nu = 0$
$g_{\text{KPZ}} = 0$		$g_{\text{KPZ}} = \infty$
$z = 2$	$z = 3/2$	$z = 1$

(diffusive regime)

(ballistic regime)

Forster, Nelson, and Stephen (1977)

Fontaine, Vercesi, Brachet, and Canet (2023)

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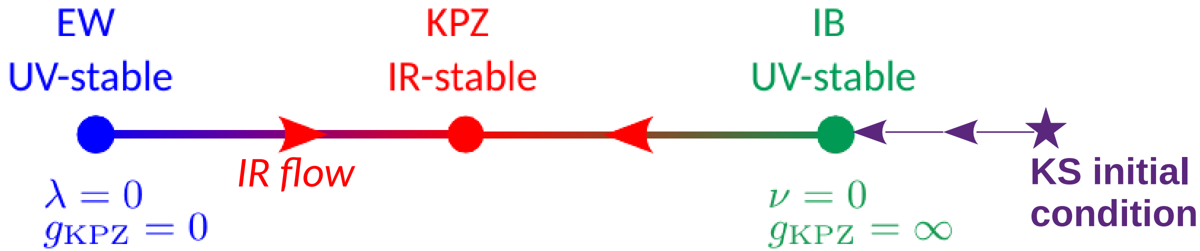


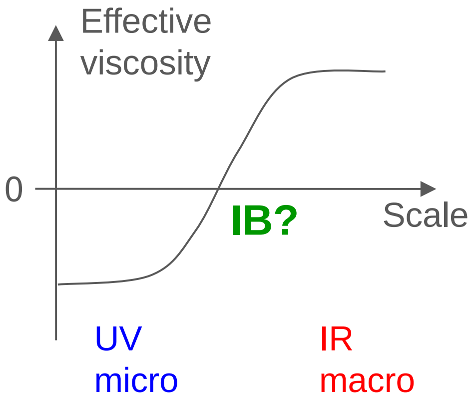
EW: $\lambda = 0$, $g_{\text{KPZ}} = 0$, $z = 2$ (diffusive regime)

KPZ: $z = 3/2$

IB: $\nu = 0$, $g_{\text{KPZ}} = \infty$, $z = 1$ (ballistic regime)

Forster, Nelson, and Stephen (1977)

Fontaine, Vercesi, Brachet, and Canet (2023)



2. Scaling regimes of the KS equation

$$\partial_t h(t,x) = \left(\nu\partial_x^2 - \tau\partial_x^4\right)h + \frac{\lambda}{2}(\partial_x h)^2$$

Viscosity **< 0**

Kardar-Parisi-Zhang (KPZ) equation

$$\partial_t h(t,x) = \nu_{\text{eff}}\partial_x^2 h + \frac{\lambda}{2}(\partial_x h)^2 + \eta(t,x)$$

Effective viscosity **> 0**

$$\langle \eta(t,x)\eta(t',x')\rangle = D_{\text{eff}}\delta(t-t')\delta(x-x')$$

Scale

Microscopic model

Effective macroscopic description

$z = 3/2$

Dynamical exponent: $t \sim p^{-z}$

1D KPZ fixed points:

$$g_{\text{KPZ}} \equiv \frac{\lambda^2 D}{\nu^3}$$

EW
UV-stable

KPZ
IR-stable

IB
UV-s

IR flow

$\lambda = 0$
$g_{\text{KPZ}} = 0$
$z = 2$

(diffusive regime)

$z = 3/2$

$$\underbrace{\hspace{6em}}$$

Forster, Nelson, and Stephen (1977)

Fontaine, Brachet, and Canet

macro

- Is **z=1** regime present in KS?..
- Coexistence of **z=1**, **2** and **3/2** in the (t,x) plane?..

The NLO approximation

Kloss, Canet, and Wschebor (2012)

$$\partial_t h = \nu\partial_x^2 h - \tau\partial_x^4 h + \tfrac{\lambda}{2}(\partial_x h)^2 + \xi, \quad \langle\xi(t,x)\xi(t',x')\rangle = 2D_{\mathrm{SKS}}\delta(t-t')\delta(x-x')$$

The bare MSRJD action:

$$\mathcal{S}_\Lambda[h,\bar{h}] = \int_{t,x} \bar{h}\Big\{\partial_t h - (\nu - \tau\partial_x^2)\partial_x^2 h - \frac{\lambda}{2}(\partial_x h)^2 - D_{\mathrm{SKS}}\bar{h}\Big\}$$

The **NLO ansatz** for the effective action: promoting the parameters to scale-dependent ***functions***

$$\Gamma_\kappa[\varphi,\bar{\varphi}] = \int_{t,x} \bar{\varphi}\Big\{\partial_t\varphi - f_\kappa^\nu(\partial_t,\partial_x)\partial_x^2\varphi - \frac{\lambda}{2}(\partial_x\varphi)^2 - f_\kappa^D(\partial_t,\partial_x)\bar{\varphi}\Big\}$$

The ansatz respects the extended symmetries:
1) time-gauged shift invariance
2) time-gauged Galilean invariance
3) emergent time-reversal symmetry

Flow equations

$$\partial_s \hat{f}_\kappa^{D,\nu}(\hat{\varpi},\hat{p}) = \big(\eta_\kappa^{D,A} + \hat{p}\partial_{\hat{p}} + (2-\eta^A)\hat{\varpi}\partial_{\hat{\varpi}}\big)\hat{f}_\kappa^{D,\nu}(\hat{\varpi},\hat{p}) + \hat{I}_\kappa^{D,\nu}(\hat{\varpi},\hat{p})$$

for the dimensionless functions

$$\hat{f}_\kappa^D(\hat{\omega},\hat{p}) \equiv f_\kappa^D(\omega,p)/D_\kappa\,,$$

$$\hat{f}_\kappa^\nu(\hat{\omega},\hat{p}) \equiv f_\kappa^\nu(\omega,p)/A_\kappa\,,\ A_\kappa \equiv |\nu_\Lambda|\kappa^{-\eta^A}$$

$$\hat{p} \equiv p/\kappa\,, \quad \hat{\omega} \equiv \omega/(\kappa^2 A_\kappa)$$

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Solve numerically and obtain the IR fixed point

2. Scaling regimes of the KS equatior

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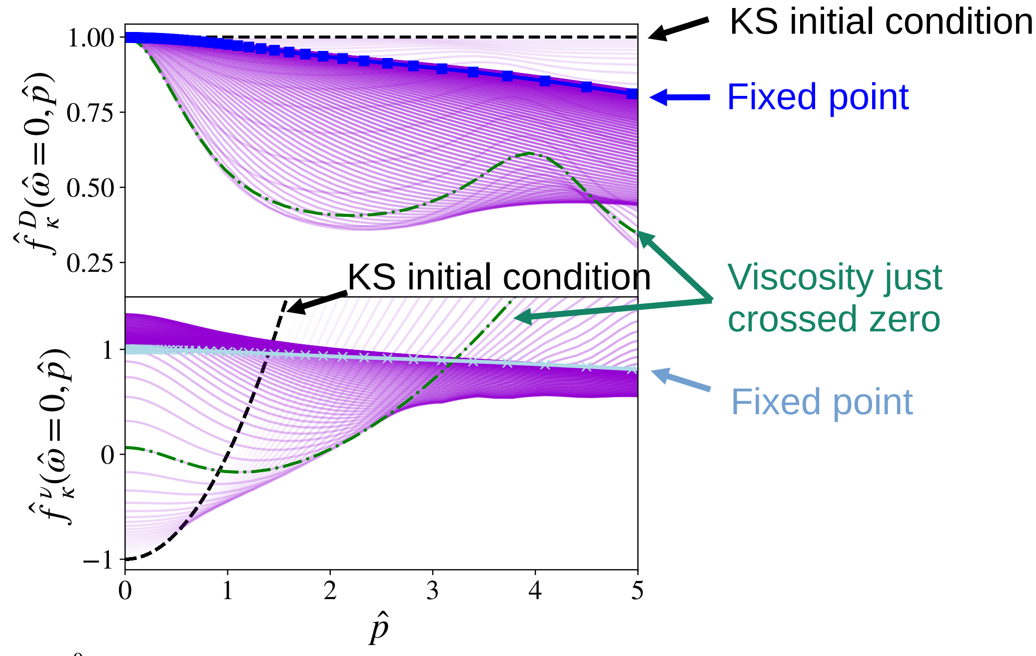
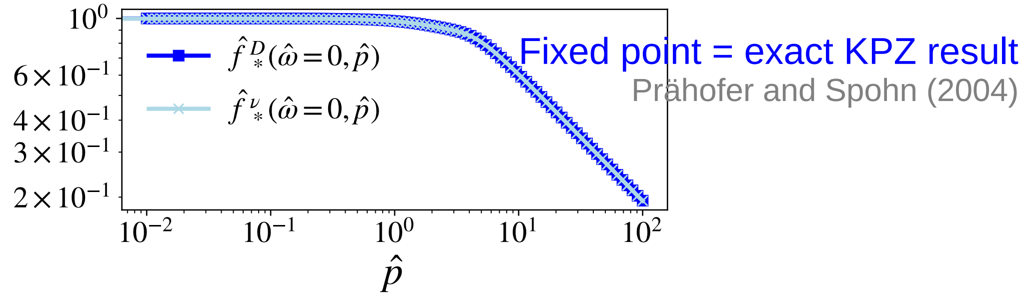
Result for the dimensionless functions:





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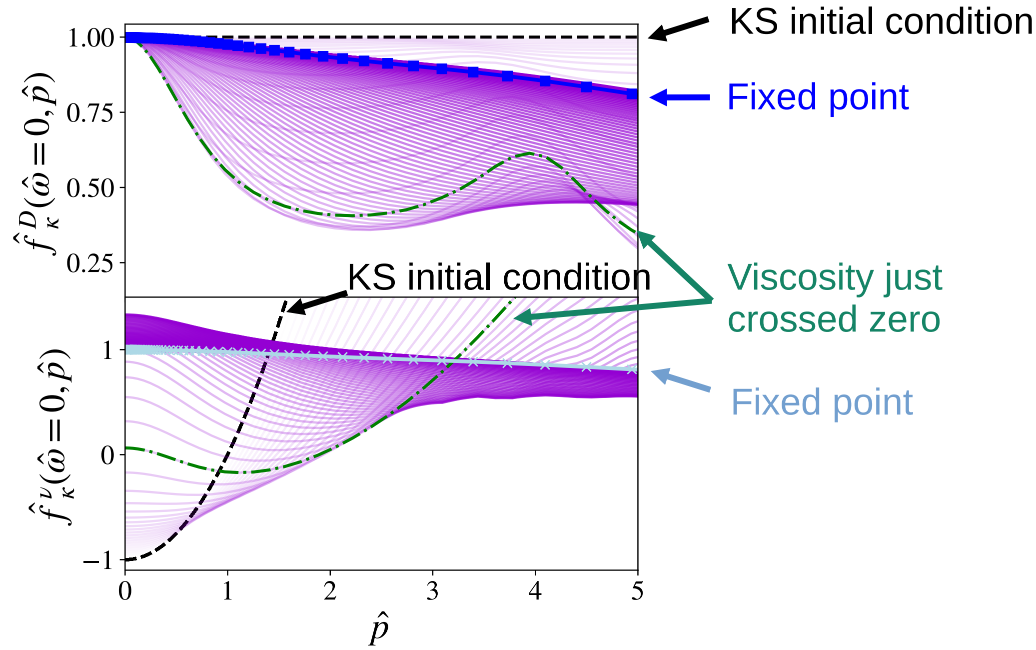

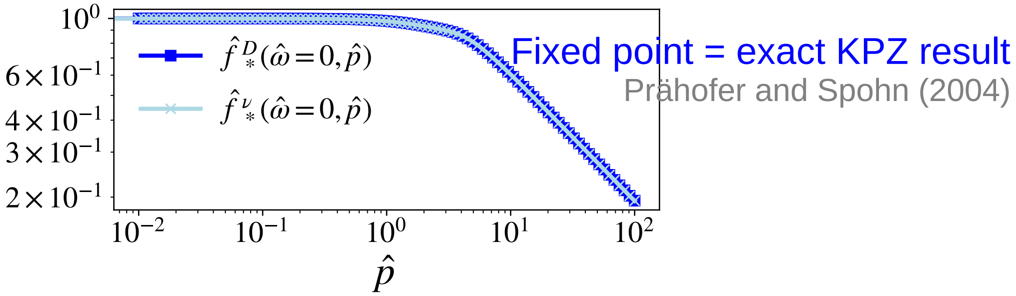

Solve numerically and obtain the IR fixed point

But what about the intermediate scales?

The *two grids* scheme

Benitez, Blaizot, Chaté, Delamotte, Méndez-Galain, and Wschebor (2009)
Gosteva, Wschebor, and Canet (2025)

Exact equation
C. Wetterich (1993)

$$\partial_\kappa \Gamma_\kappa = \frac{1}{2}\mathrm{tr}\int_{\omega,\mathbf{q}} \partial_\kappa \mathcal{R}_\kappa\, \mathcal{G}_\kappa$$

$\downarrow$

NLO ansatz
(for small momenta)

$$\partial_s \hat{f}_\kappa^{D,\nu}(\hat{\varpi},\hat{p}) = \ldots$$

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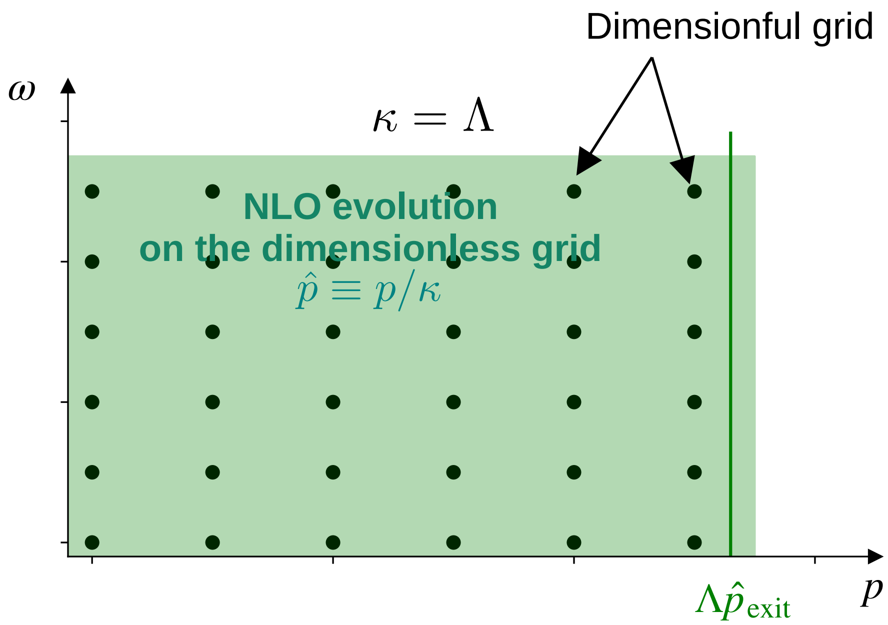

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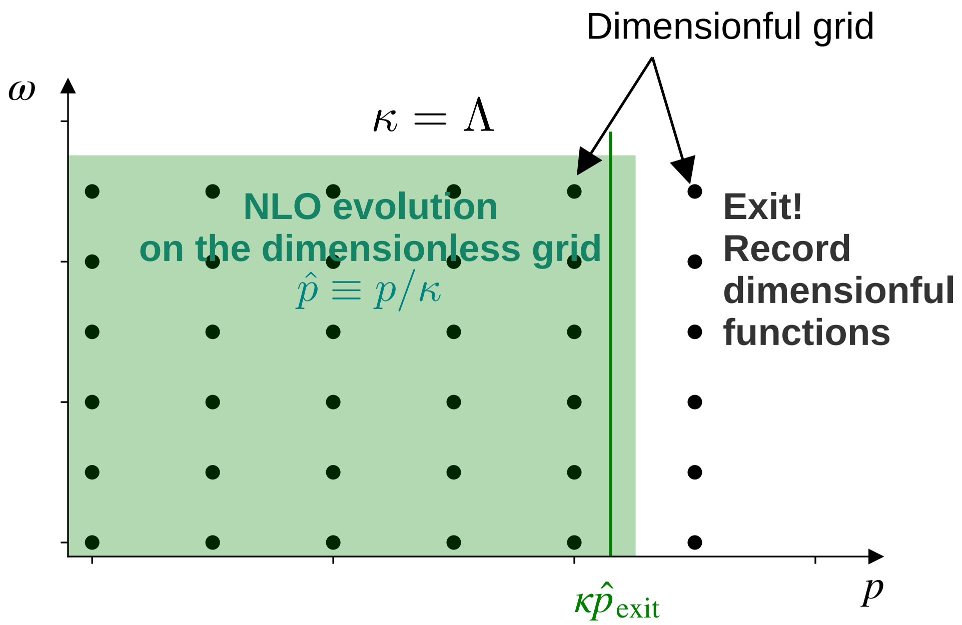

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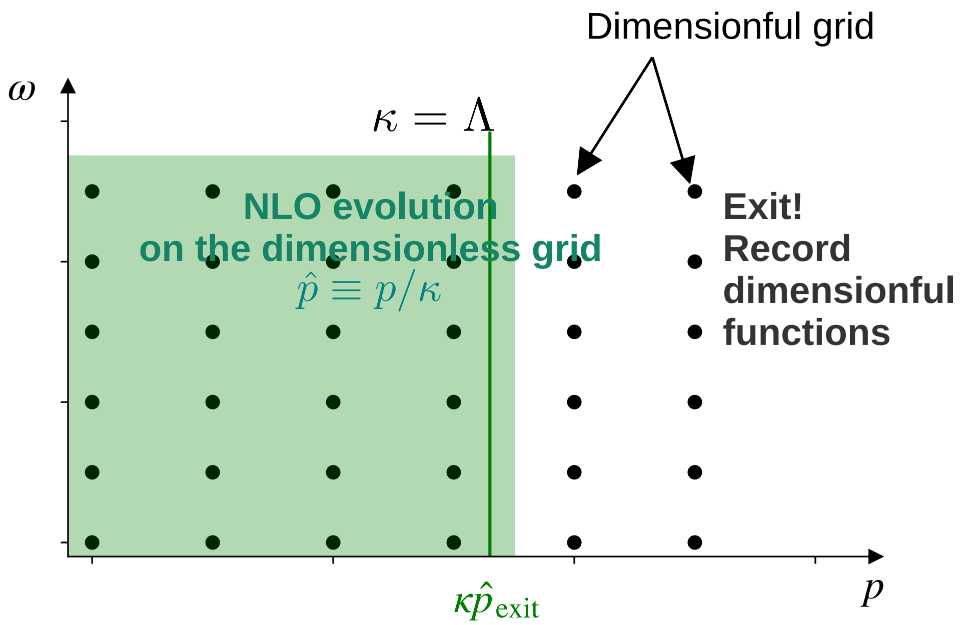

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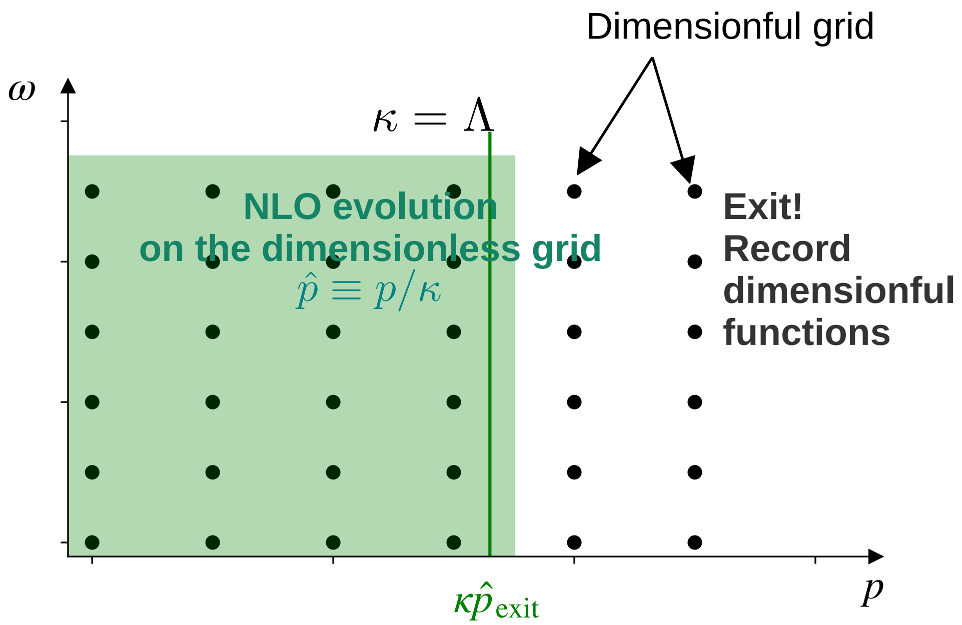

2-point connected correlation function

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can be found from $f_\kappa^{\nu,D}$:

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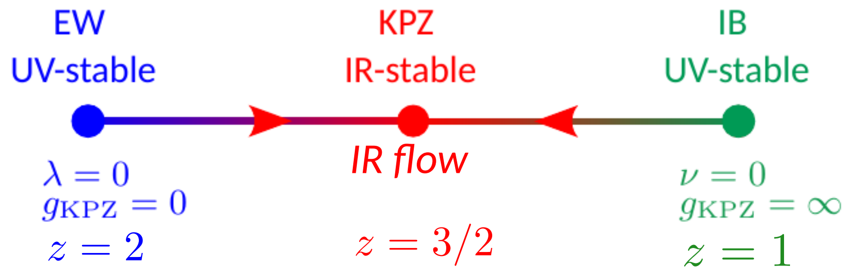
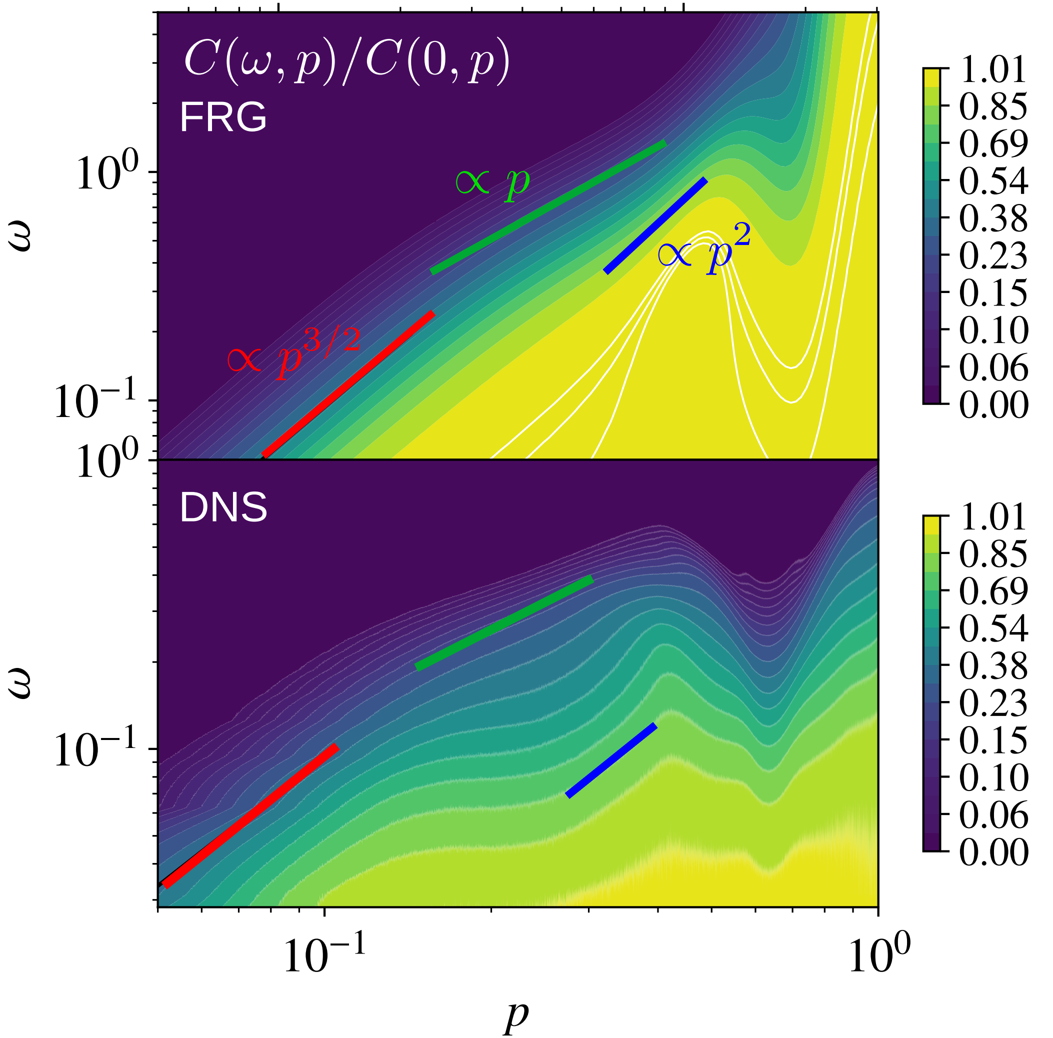
Gosteva, Roy, Wschebor, and Canet (2026)

Result:
FRG vs Direct Numerical Simulations (DNS)

1D KPZ fixed points:



$$g_{\rm KPZ} \equiv \frac{\lambda^2 D}{\nu^3}$$



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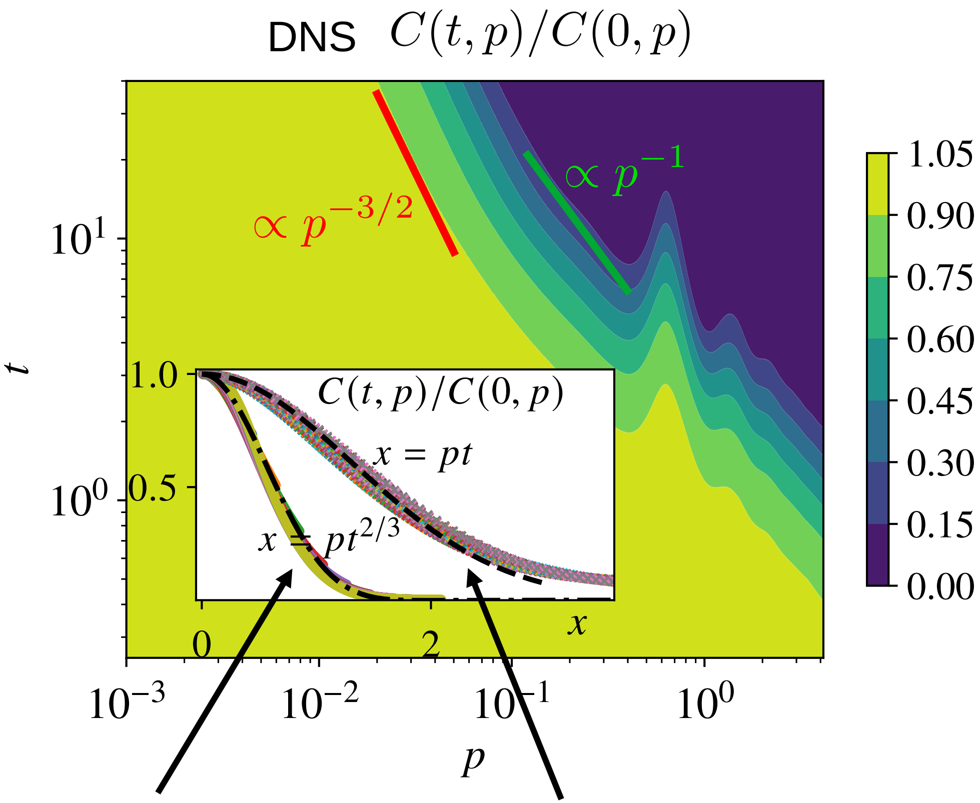
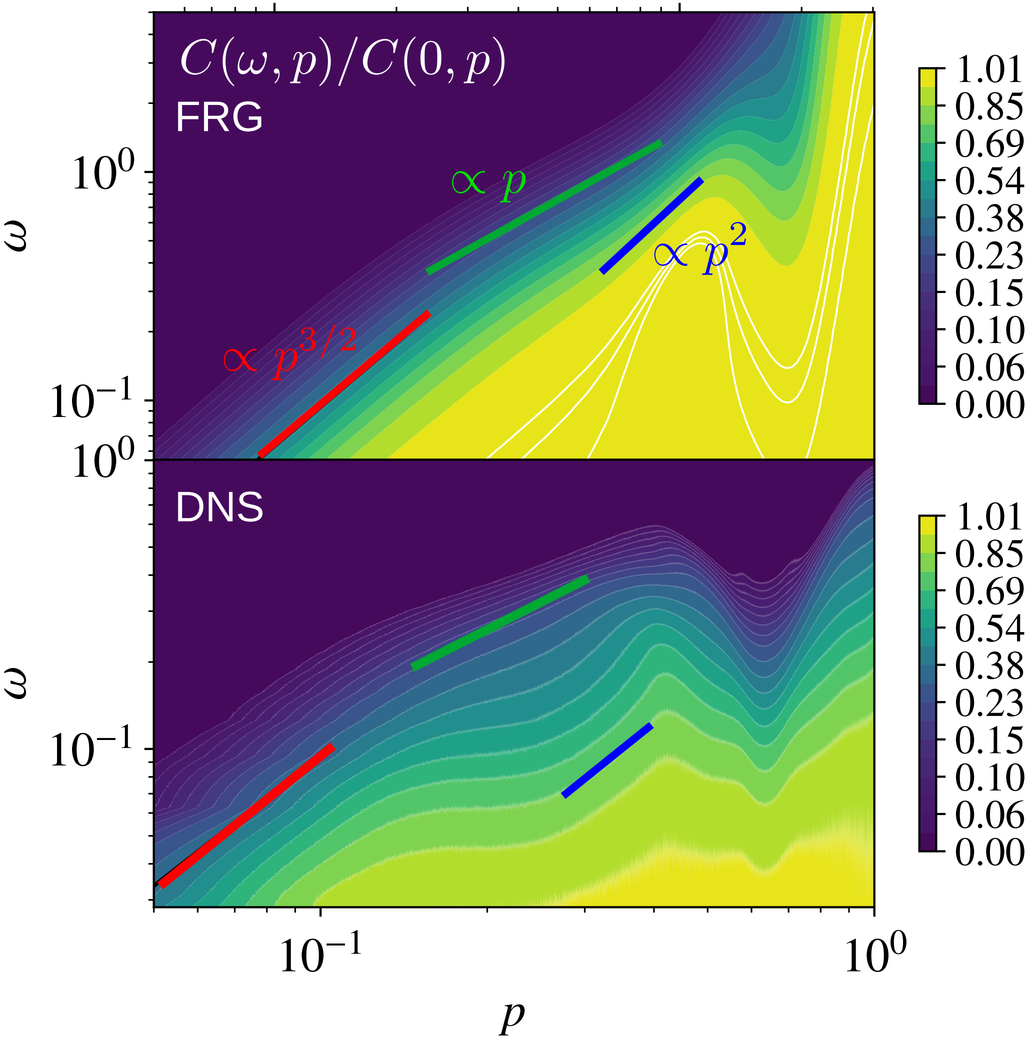


Collapses to the exact KPZ result

Prähofer and Spohn (2004)

Collapses to the exact large-p solution



Conclusions

1. RG approach revisited

- Be careful with the sharp cutoff

2. UV and IR scaling regimes of the KS eqaution

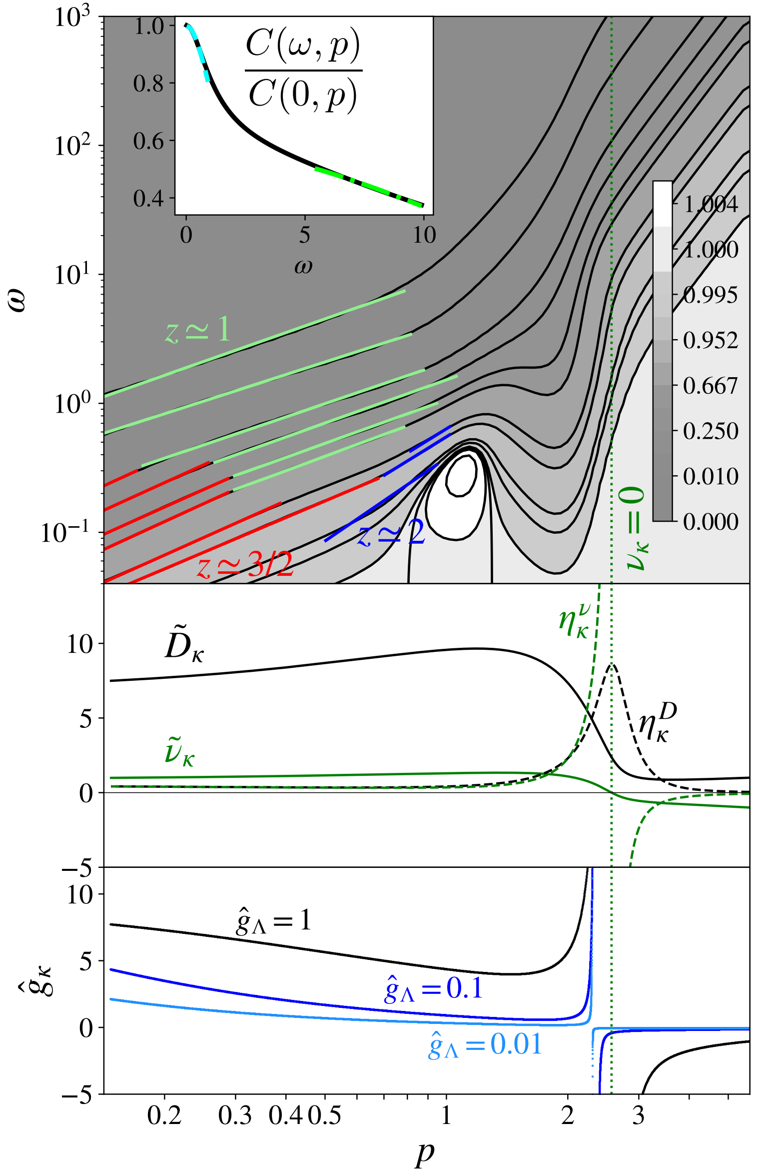
- FRG + the *two grids* scheme provides the correlation function in a wide range of momentum and frequency comparable to DNS
- We defined positions of all the scaling regimes of the KS equation: **z=1**, **2** and **3/2**

Articles:
L. Gosteva, D. Roy, N. Wschebor, and L. Canet, arxiv:2605.23364 (2026)
L. Gosteva, N. Wschebor, and L. Canet, arXiv:2607.15784 (2026)

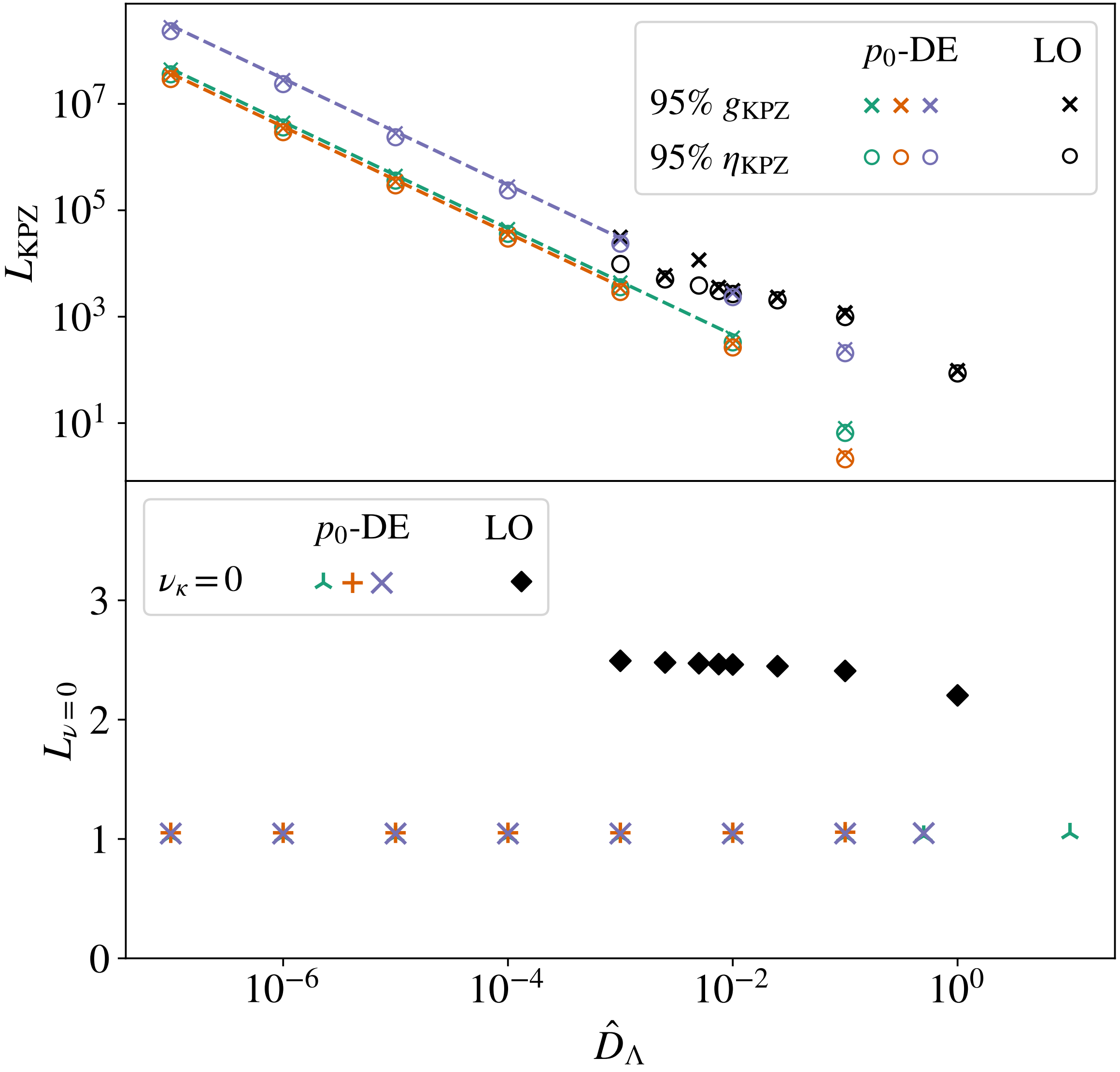
Codes for the NLO:
github.com/lgost/FRG

Backup slides

2. Scaling regimes of the … scheme



3. Zero noise limit



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Exact FRG equation $\partial_\kappa \Gamma_\kappa = \frac{1}{2}\mathrm{tr}\int_{\omega,\mathbf{q}} \partial_\kappa \mathcal{R}_\kappa\, \mathcal{G}_\kappa$
C. Wetterich (1993)

The "small-*p*" equations
(valid for small momenta)
NLO ansatz

$$\partial_s \hat{f}^{D,\nu}_\kappa(\hat{\varpi},\hat{p}) = \ldots$$

The "large-*p*" equations
(valid for large momenta)
No ansatz needed!

$$\partial_s C_\kappa(\varpi,p) = p^2 \int_\omega \frac{C_\kappa(\omega+\varpi,p) - C_\kappa(\varpi,p)}{\omega^2}\, \mathcal{J}_\kappa(\omega)$$

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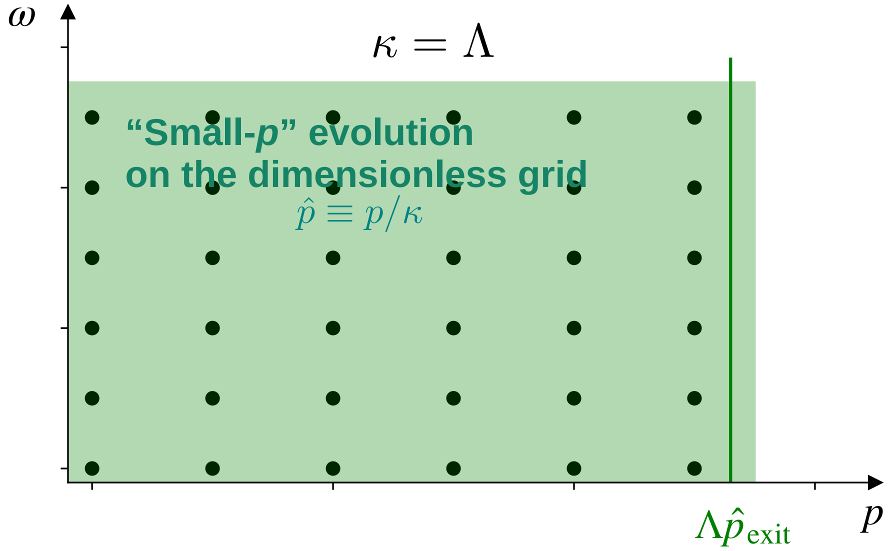

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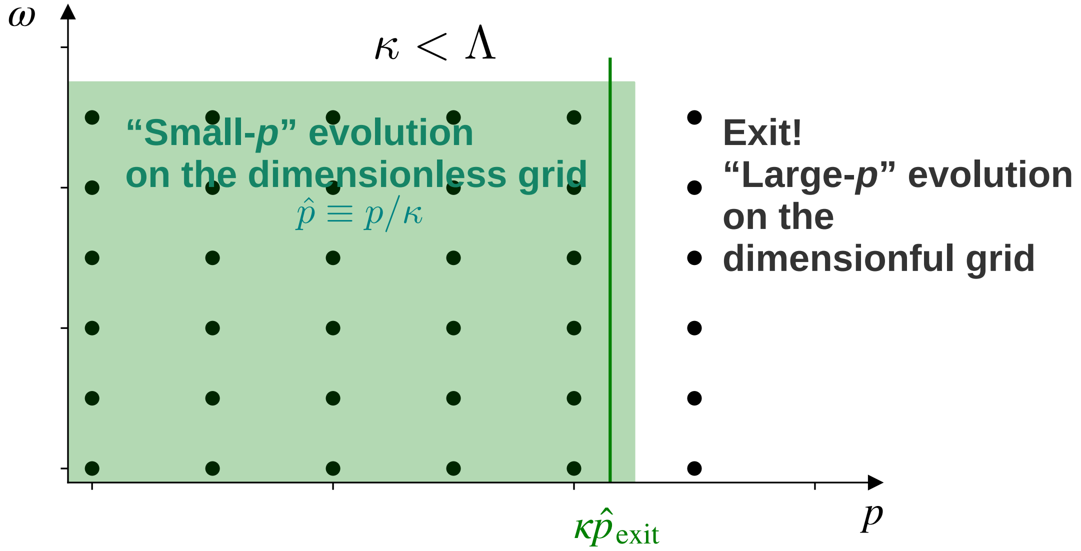

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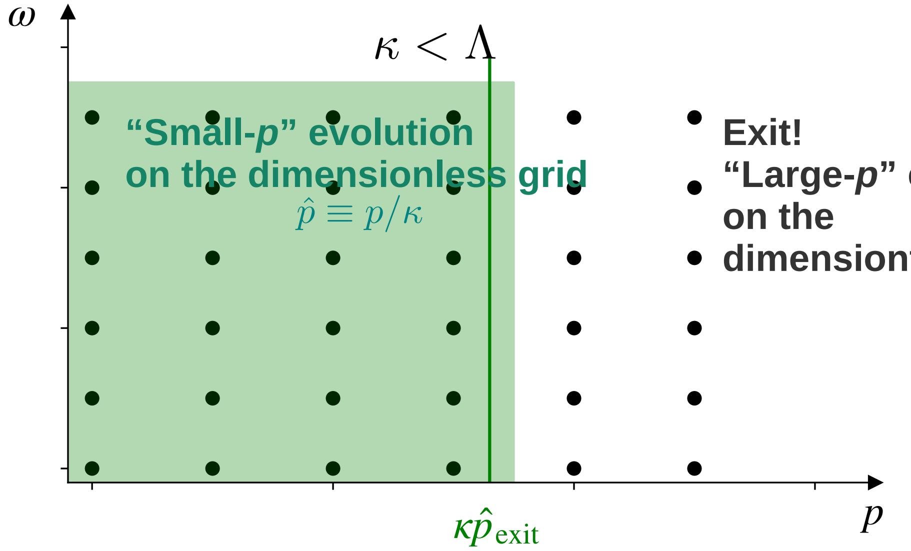

Exit!
"Large-*p*" evolution on the dimensionful grid

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