

Scaling regimes of the 1D Kuramoto-Sivashinsky equation

Liubov GOSTEVA¹, Nicolás WSCHEBOR², Léonie CANET¹

¹ Université Grenoble Alpes, CNRS, LPMMC, Grenoble, France

² Universidad de la República, Montevideo, Uruguay



Kuramoto-Sivashinsky equation in 1D

Kuramoto and Tsuzuki (1975), Sivashinsky (1977)

$$\partial_t h(t, x) = \left(\nu \partial_x^2 - \tau \partial_x^4 \right) h + \frac{\lambda}{2} (\partial_x h)^2$$

Viscosity < 0

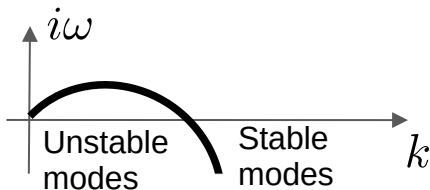
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Dispersion relation:



► The simplest **chaotic** partial differential equation

Brummitt and Sprott (2009)

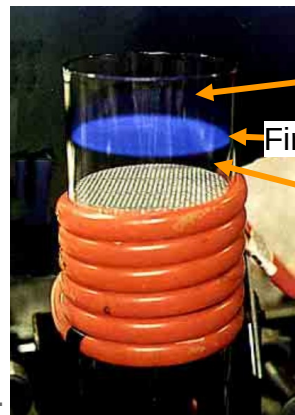
► Describes various phenomena with **instabilities**:

• Reaction-diffusion processes



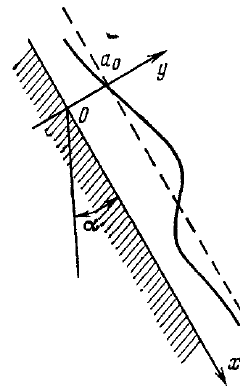
Belousov-Zhabotinsky reaction.
www.isc.meiji.ac.jp

• Flame



Searby (2006)

• Liquid film flowing down an inclined plane



Nepomnyashchii (1974)

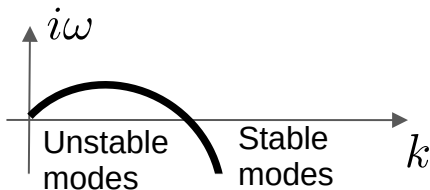
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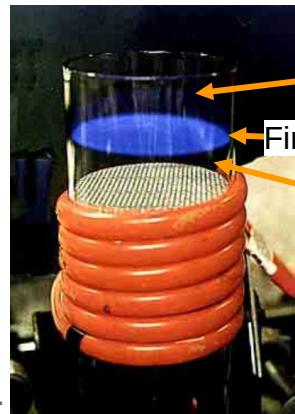
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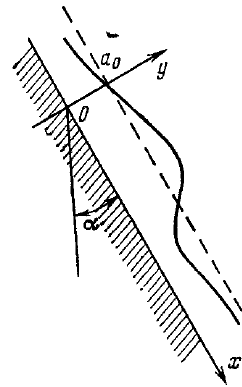
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► Reduced description of phase dynamics in the complex Ginzburg-Landau equation in the phase turbulence regime $\leftrightarrow$ connection to 1D nonequilibrium quantum fluids.

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KS equation falls into the **KPZ universality class**
with dynamical exponent $z = 3/2$ ($t \sim p^{-z}$)

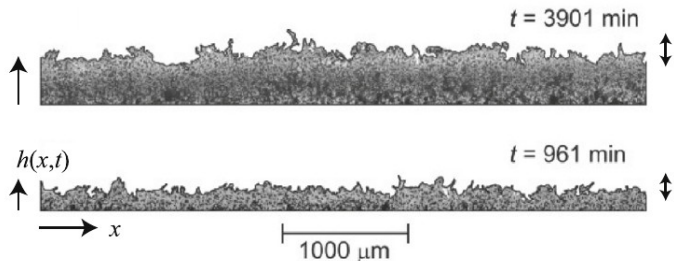
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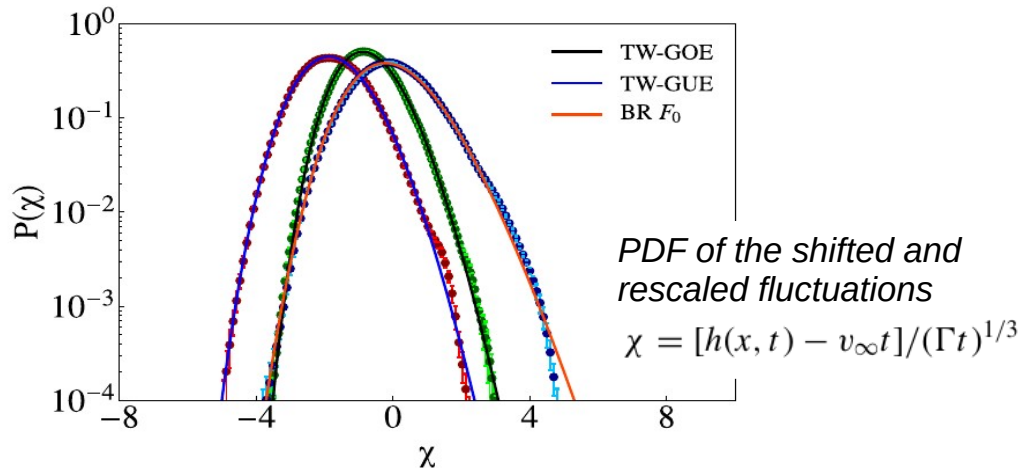
Viscosity > 0

White noise



Growing interfaces example: cancer cell colony. Huergo et al. (2012)

✓ **Numerical evidence:** D.Roy and R.Pandit (2020)



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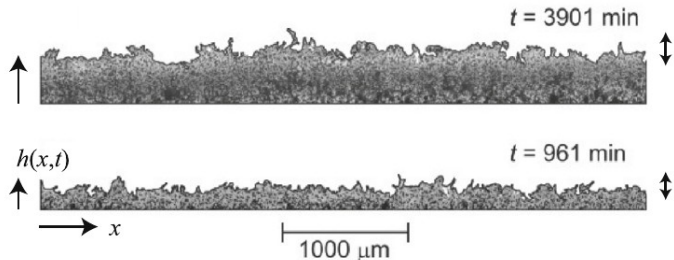
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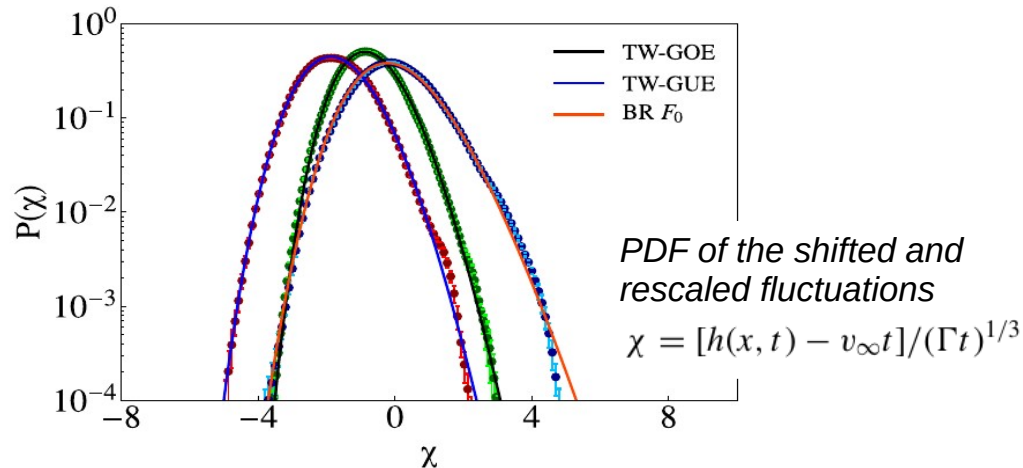
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? **Wilsonian RG with sharp cutoff for the stochastic KS:**
Yakhot (1981); Cuerno and Lauritsen (1995); Ueno, Sakaguchi and Okamura (2005)

Problems: Morris (1996)

✗ Flows of the running parameters involve momentum derivatives, e.g.:

$$\partial_s \nu_\kappa = \frac{1}{2} \left[\frac{\partial}{\partial p^2} \partial_s \Gamma_\kappa^{(1,1)}(0, p) \right] \Big|_{p=0}$$

✗ The cutoff is applied only to $G_0(q)$, but not to $G_0(p-q)$

$$\int_{\Lambda(l) \leq |q| \leq \Lambda_0} \frac{dq}{2\pi} \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} |G_0(q, \Omega)|^2 G_0(p-q, \omega-\Omega) \mathcal{F}(p, q)$$

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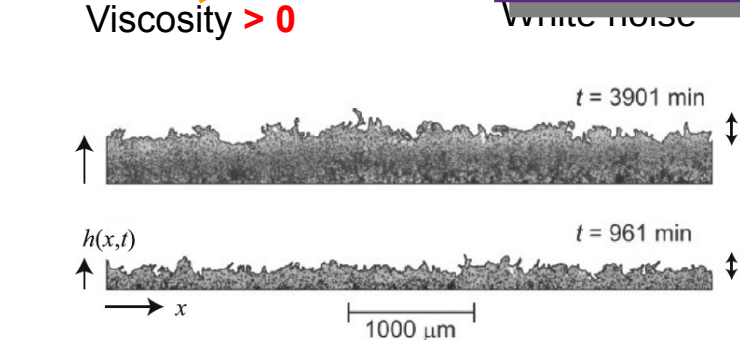
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KS equation falls into the KPZ class with dynamical exponent $z = 2$

Kardar-Parisi-Zhang (KPZ) equation

$$\partial_t h(t, x) = \nu \partial_x^2 h + \frac{\lambda}{2} (\partial_x h)^2 + \xi$$

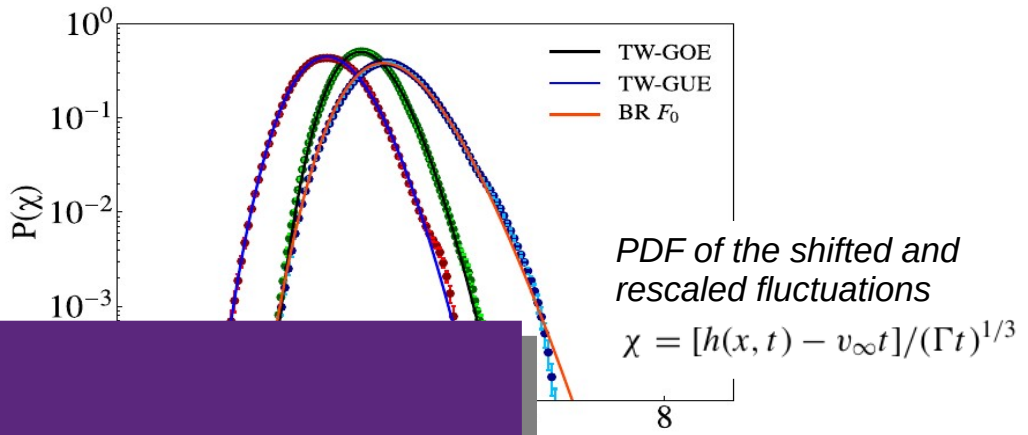
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Part 1.
Revisit the RG approach to the stochastic Kuramoto-Sivashinsky model

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Cutoff for the stochastic KS: Frisch (1995); Ueno, Sakaguchi

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$$\partial_t h = \nu \partial_x^2 h - \tau \partial_x^4 h + \frac{\lambda}{2} (\partial_x h)^2 + \xi, \quad \langle \xi(t, x) \xi(t', x') \rangle = 2 D_{\text{SKS}} \delta(t - t') \delta(x - x')$$

The bare MSRJD action:

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The **simplest ansatz** for the effective action: promoting the parameters to scale-dependent ones

$$\Gamma_\kappa[\varphi, \bar{\varphi}] = \int_{t,x} \bar{\varphi} \left\{ \partial_t \varphi - (\nu_\kappa - \tau_\kappa \partial_x^2) \partial_x^2 \varphi - \frac{\lambda}{2} (\partial_x \varphi)^2 - (D_\kappa - \zeta_\kappa \partial_x^2) \bar{\varphi} \right\}$$

$$\Rightarrow \begin{cases} \partial_s D_\kappa &= -\frac{1}{2} \partial_s \Gamma_\kappa^{(0,2)}(0, 0) \\ \partial_s \nu_\kappa &= \frac{1}{2} \left[\frac{\partial}{\partial p^2} \partial_s \Gamma_\kappa^{(1,1)}(0, p) \right] \Big|_{p=0} \\ \partial_s \zeta_\kappa &= -\frac{1}{4} \left[\frac{\partial}{\partial p^2} \partial_s \Gamma_\kappa^{(0,2)}(0, p) \right] \Big|_{p=0} \\ \partial_s \tau_\kappa &= \frac{1}{24} \left[\frac{\partial}{\partial p^4} \partial_s \Gamma_\kappa^{(1,1)}(0, p) \right] \Big|_{p=0} \end{cases} \quad s \equiv \ln(\kappa/\Lambda)$$

Needed for the
emerging time-
reversal symmetry

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$$r(x) = \begin{cases} +\infty, & 0 \leq x \leq 1, \\ 0, & x > 1. \end{cases}$$

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Results:

(1) in a **sharp cutoff** limit,

- only 2 of 4 equations are defined;
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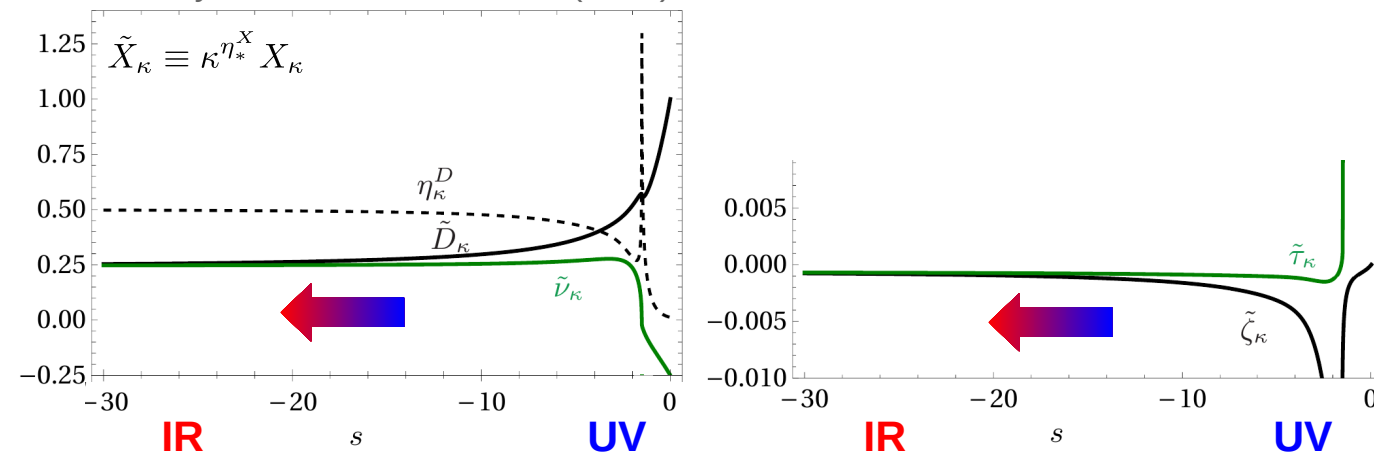
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Litim (2001)

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Gosteva, Roy, Wschebor, and Canet (2026)



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- (2) with a **smooth cutoff**,
 - flow equations are defined;
 - KS model flows to the KPZ fixed point in IR;
 - effective viscosity becomes positive and $z=2-\eta=3/2$

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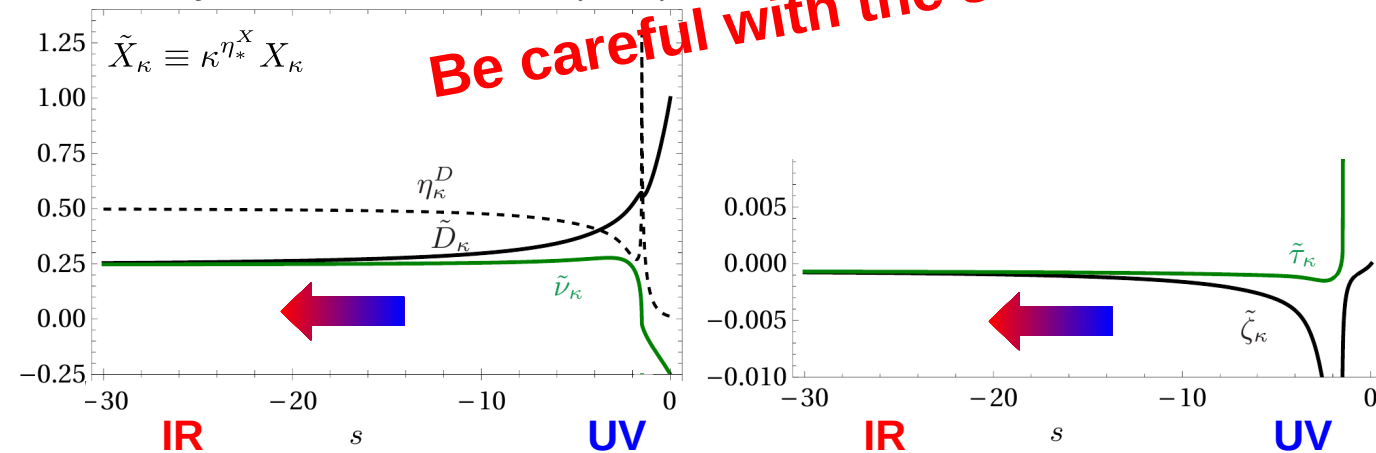
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Part 2.
Scaling regimes of the Kuramoto-
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2. Scaling regimes of the KS equation

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$$z = 3/2$$

Dynamical exponent: $t \sim p^{-z}$

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1D KPZ fixed points:

EW
UV-stable

KPZ
IR-stable

IB
UV-stable

$$g_{\text{KPZ}} \equiv \frac{\lambda^2 D}{\nu^3}$$

$$\lambda = 0$$

$$g_{\text{KPZ}} = 0$$

$$z = 2$$

IR flow

$$z = 3/2$$

$$\nu = 0$$

$$g_{\text{KPZ}} = \infty$$

$$z = 1$$

(diffusive regime)

(ballistic regime)

Forster, Nelson,
and Stephen (1977)

Fontaine, Vercesi,
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Microscopic model

Effective macroscopic description

$$z = 3/2$$

Dynamical exponent: $t \sim p^{-z}$

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KS initial
condition

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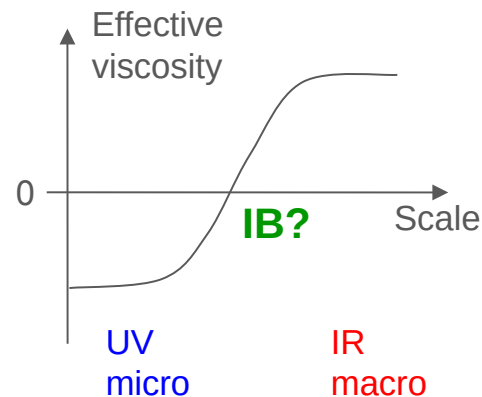
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(diffusive regime)

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- Is $z=1$ regime present in KS?..
- Coexistence of $z=1$, 2 and $3/2$ in the (t, x) plane?..

macro

$$\partial_t h = \nu \partial_x^2 h - \tau \partial_x^4 h + \frac{\lambda}{2} (\partial_x h)^2 + \xi, \quad \langle \xi(t, x) \xi(t', x') \rangle = 2 D_{\text{SKS}} \delta(t - t') \delta(x - x')$$

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$$\mathcal{S}_\Lambda[h, \bar{h}] = \int_{t,x} \bar{h} \left\{ \partial_t h - (\nu - \tau \partial_x^2) \partial_x^2 h - \frac{\lambda}{2} (\partial_x h)^2 - D_{\text{SKS}} \bar{h} \right\}$$

The **NLO ansatz** for the effective action: promoting the parameters to scale-dependent **functions**

$$\Gamma_\kappa[\varphi, \bar{\varphi}] = \int_{t,x} \bar{\varphi} \left\{ \partial_t \varphi - f_\kappa^\nu(\partial_t, \partial_x) \partial_x^2 \varphi - \frac{\lambda}{2} (\partial_x \varphi)^2 - f_\kappa^D(\partial_t, \partial_x) \bar{\varphi} \right\}$$

The ansatz respects the extended symmetries:

- 1) time-gauged shift invariance
- 2) time-gauged Galilean invariance
- 3) emergent time-reversal symmetry

Flow equations

$$\partial_s \hat{f}_\kappa^{D,\nu}(\hat{\omega}, \hat{p}) = (\eta_\kappa^{D,A} + \hat{p} \partial_{\hat{p}} + (2 - \eta^A) \hat{\omega} \partial_{\hat{\omega}}) \hat{f}_\kappa^{D,\nu}(\hat{\omega}, \hat{p}) + \hat{I}_\kappa^{D,\nu}(\hat{\omega}, \hat{p})$$

for the dimensionless functions

$$\hat{f}_\kappa^D(\hat{\omega}, \hat{p}) \equiv f_\kappa^D(\omega, p) / D_\kappa,$$

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$$\hat{p} \equiv p / \kappa, \quad \hat{\omega} \equiv \omega / (\kappa^2 A_\kappa)$$

2. Scaling regimes of the KS equation

The NLO approximation

Kloss, Canet, and Wschebor (2012)

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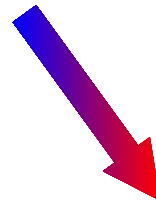
$$\partial_s \hat{f}_\kappa^{D,\nu}(\hat{\omega}, \hat{p}) = (\eta_\kappa^{D,A} + \hat{p} \partial_{\hat{p}} + (2 - \eta^A) \hat{\omega} \partial_{\hat{\omega}}) \hat{f}_\kappa^{D,\nu}(\hat{\omega}, \hat{p}) + \hat{I}_\kappa^{D,\nu}(\hat{\omega}, \hat{p})$$

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Solve numerically and obtain
the IR fixed point

2. Scaling regimes of the KS equation

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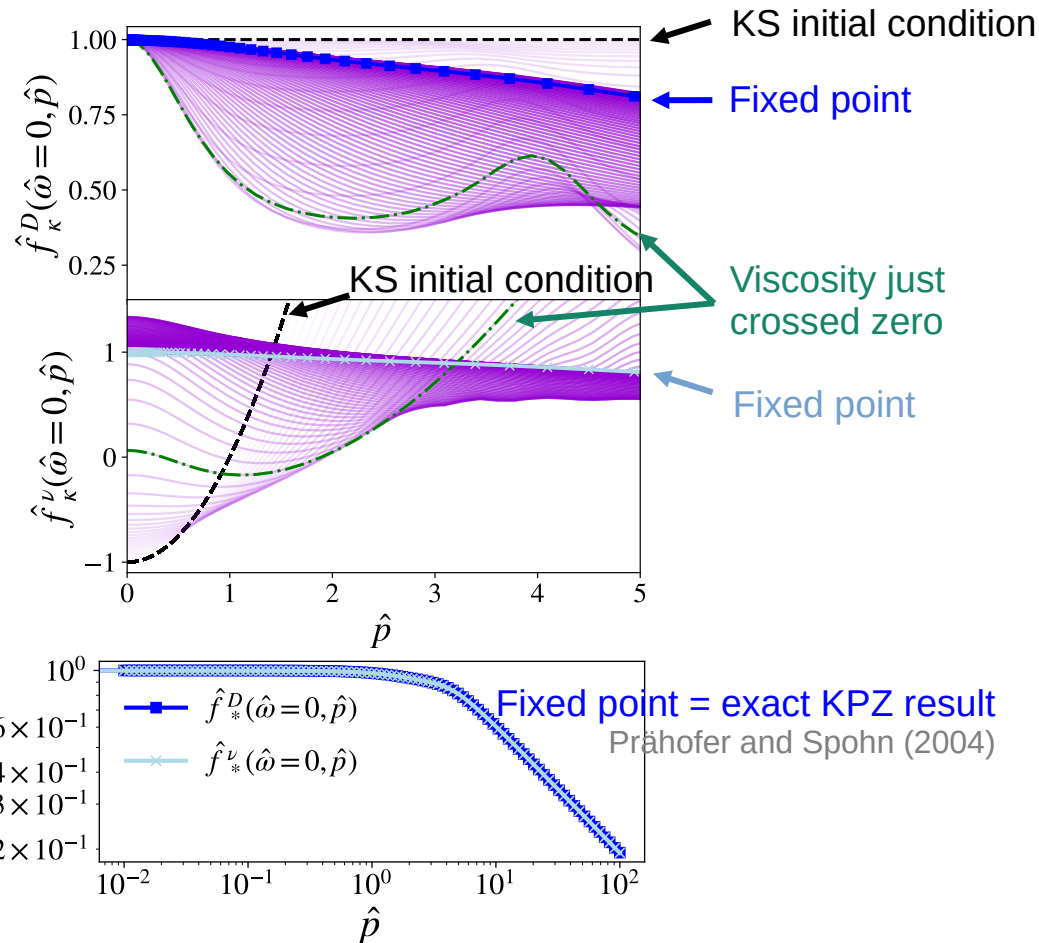
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Result for the dimensionless functions:



Solve numerically and obtain the IR fixed point

2. Scaling regimes of the KS equation

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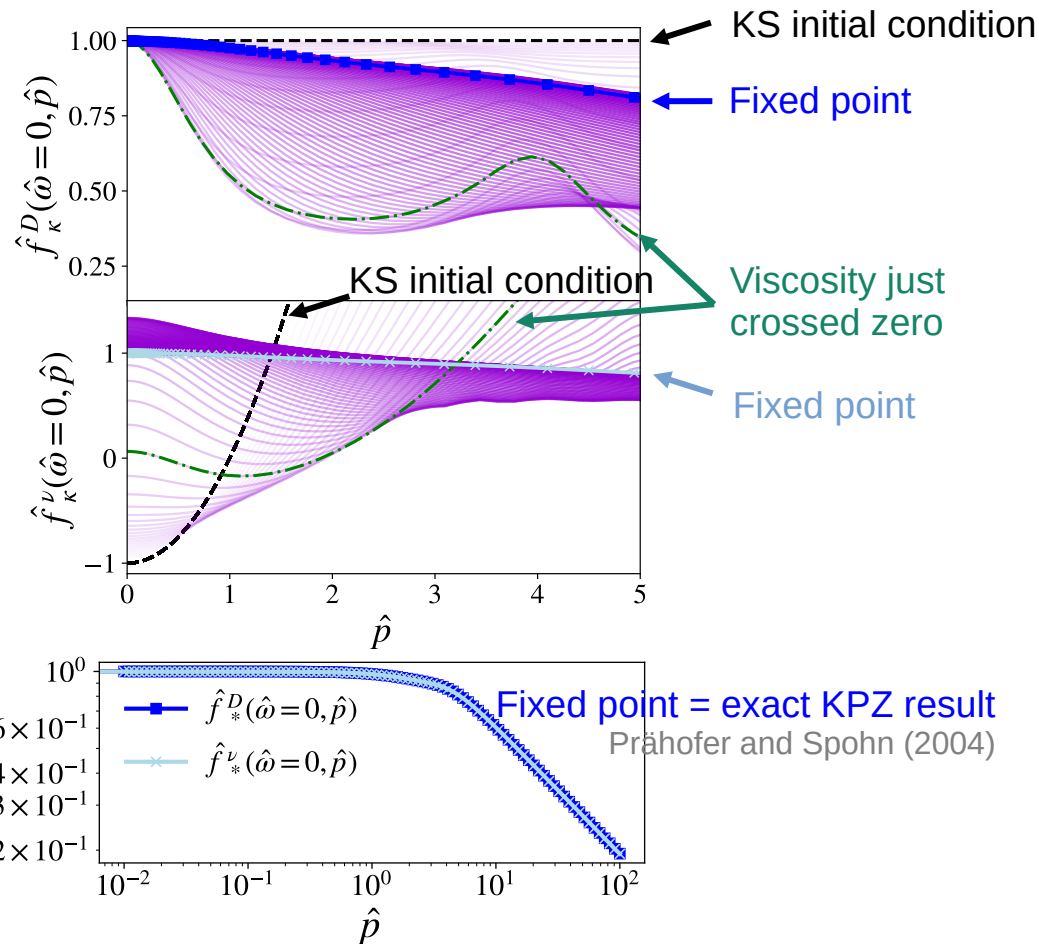
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Result for the dimensionless functions:



Solve numerically and obtain the IR fixed point

But what about the intermediate scales?

2. Scaling regimes of the KS equation

Exact equation

C. Wetterich (1993)

$$\partial_\kappa \Gamma_\kappa = \frac{1}{2} \text{tr} \int_{\omega, \mathbf{q}} \partial_\kappa \mathcal{R}_\kappa \mathcal{G}_\kappa$$



NLO ansatz

(for small momenta)

$$\partial_s \hat{f}_\kappa^{D, \nu}(\hat{\omega}, \hat{p}) = \dots$$

The *two grids* scheme

Benitez, Blaizot, Chaté, Delamotte, Méndez-Galain, and Wschebor (2009)
Gosteva, Wschebor, and Canet (2025)

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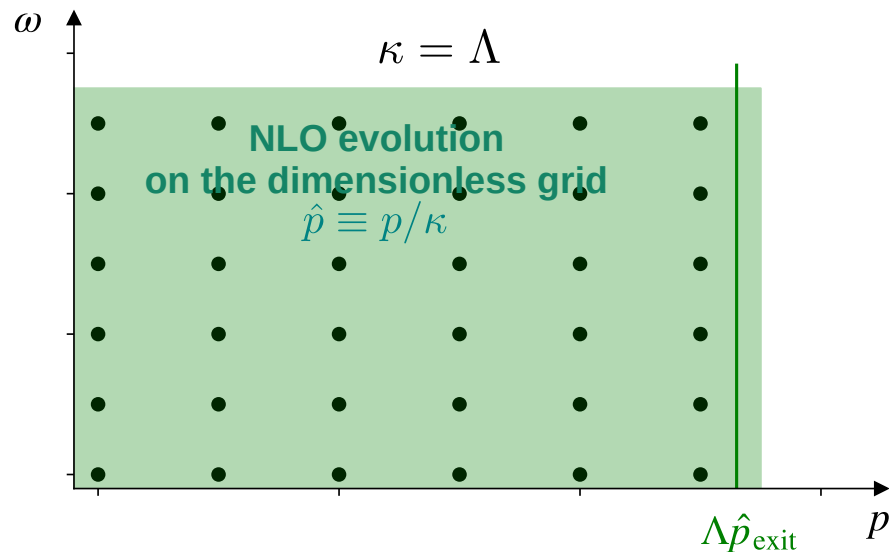


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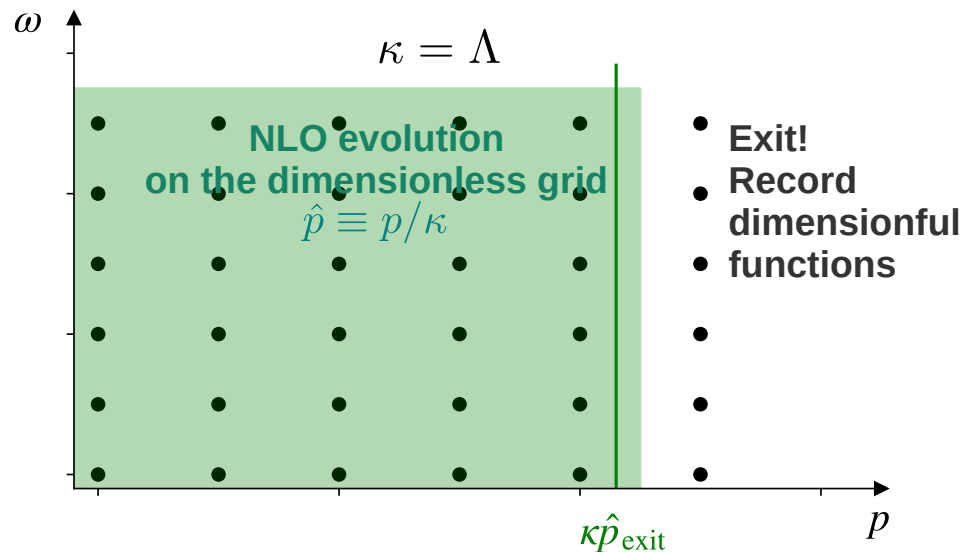
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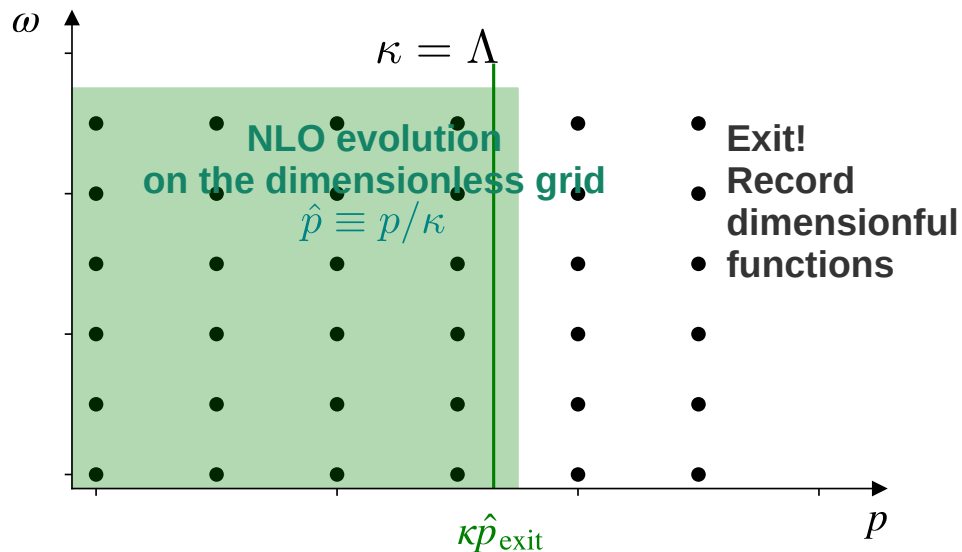
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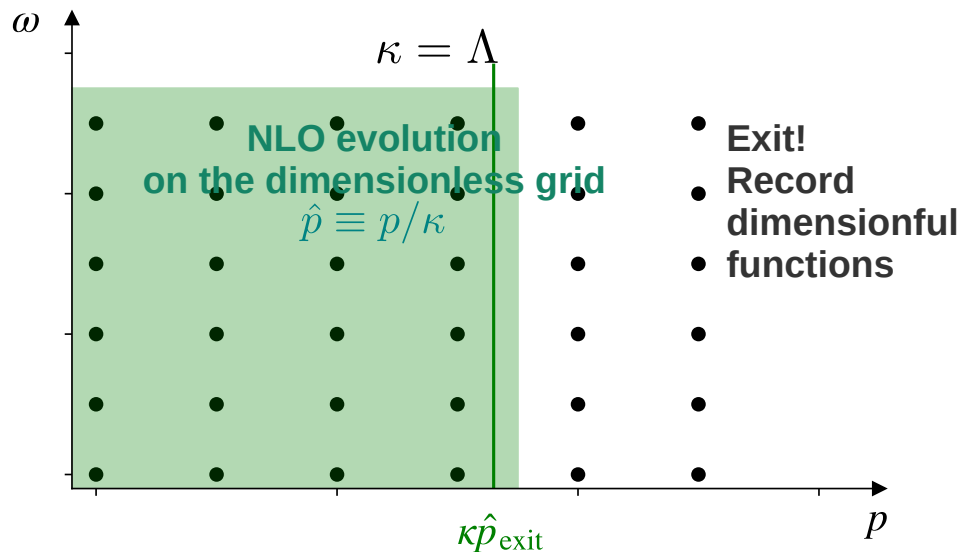
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2-point connected correlation function

$$C(\omega, p) \equiv \mathcal{F}[\langle h(t, x) h(0, 0) \rangle_c]$$

can be found from $f_\kappa^{\nu, D}$:

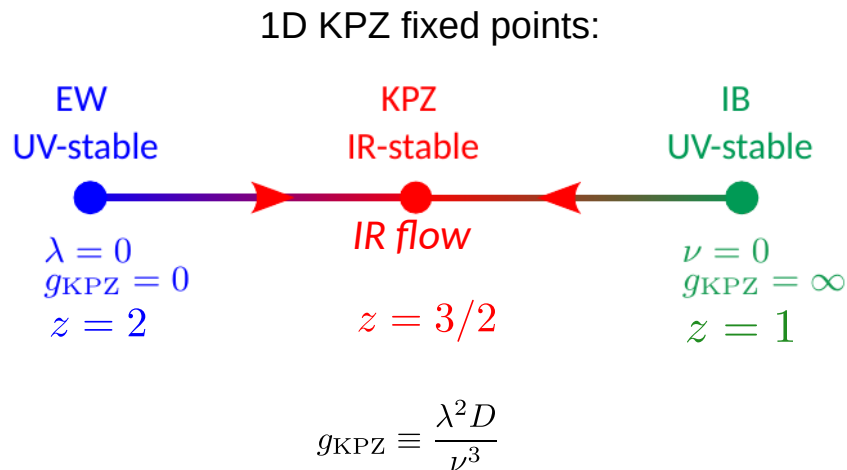
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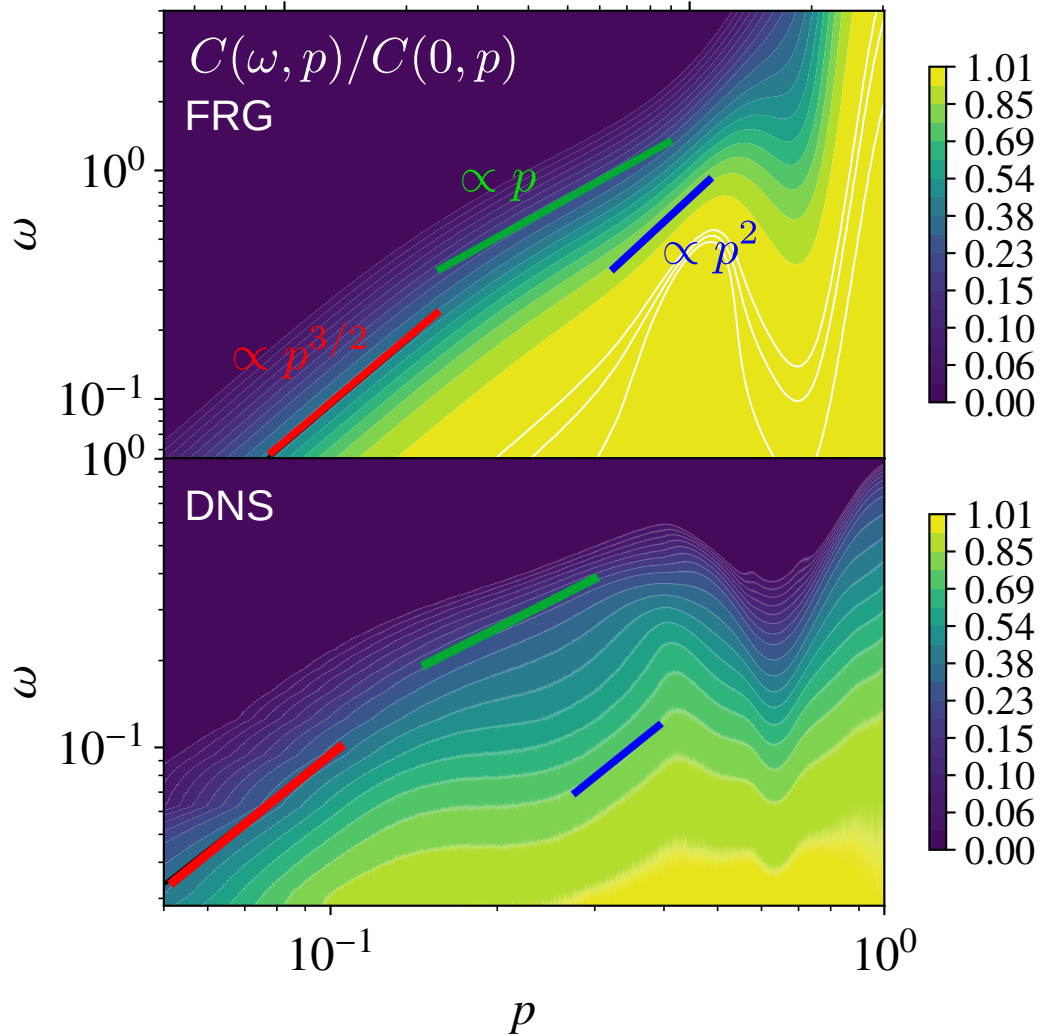
Gosteva, Roy, Wschebor, and Canet (2026)

Result:

FRG vs Direct Numerical Simulations (DNS)



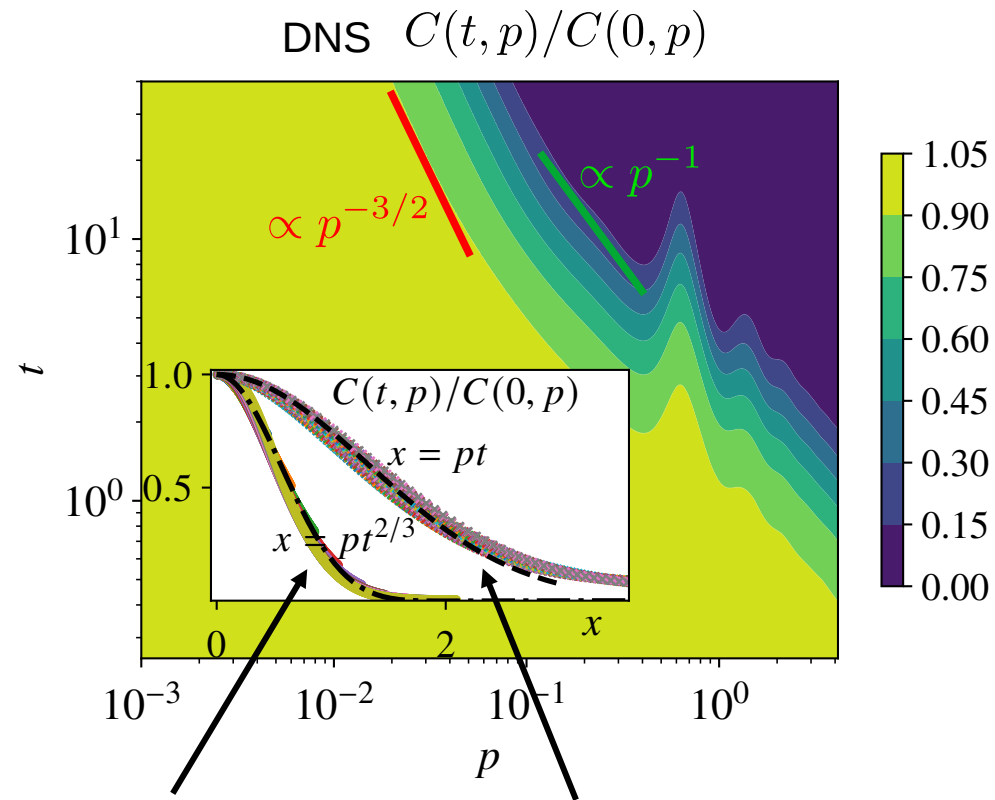
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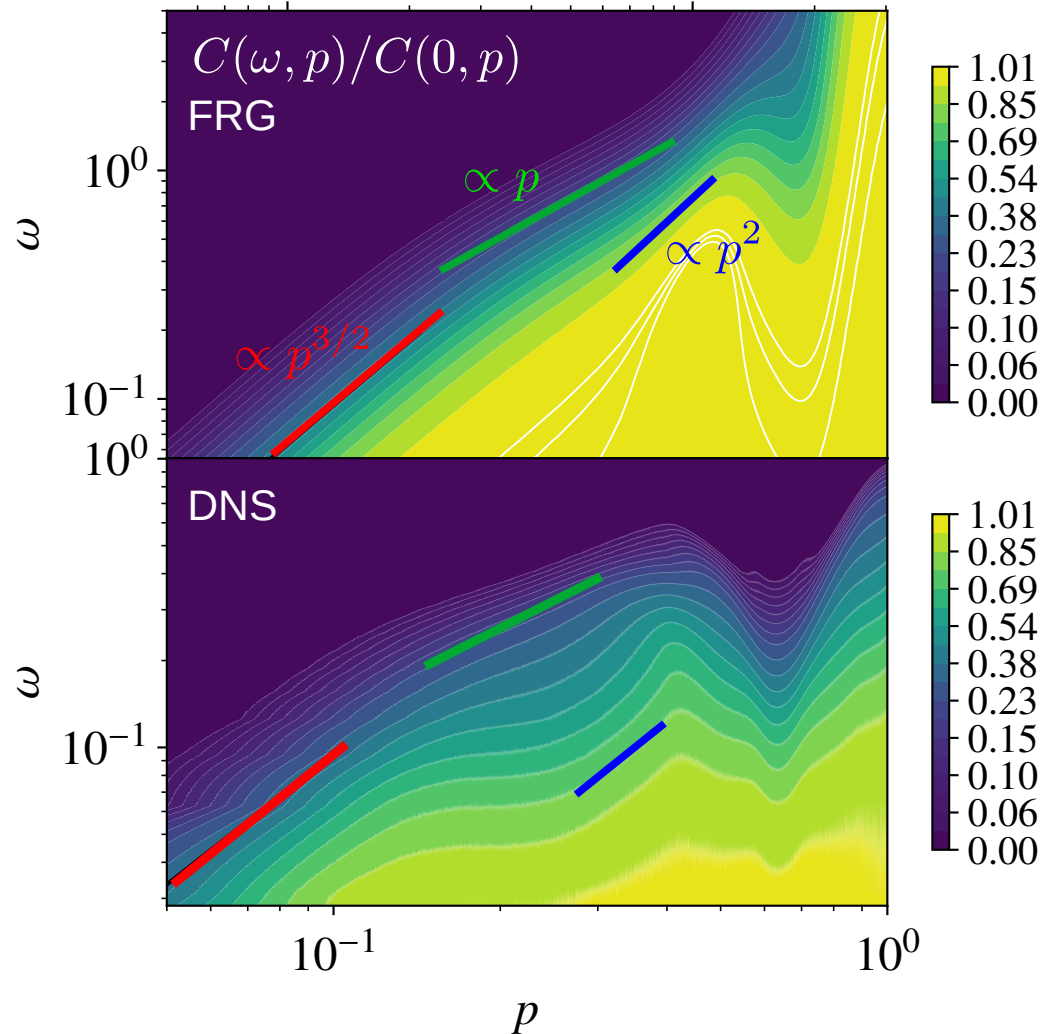
Gosteva, Roy, Wschebor, and Canet (2026)



Collapses to the exact KPZ result

Collapses to the exact large- p solution

Prähofer and Spohn (2004)



Conclusions

1. RG approach revisited

- Be careful with the sharp cutoff

2. UV and IR scaling regimes of the KS equation

- FRG + the *two grids* scheme provides the correlation function in a wide range of momentum and frequency comparable to DNS
- We defined positions of all the scaling regimes of the KS equation: $z=1$, 2 and $3/2$

Articles:

L. Gosteva, D. Roy, N. Wschebor, and L. Canet, arxiv:2605.23364 (2026)

L. Gosteva, N. Wschebor, and L. Canet, arXiv:2607.15784 (2026)

Codes:

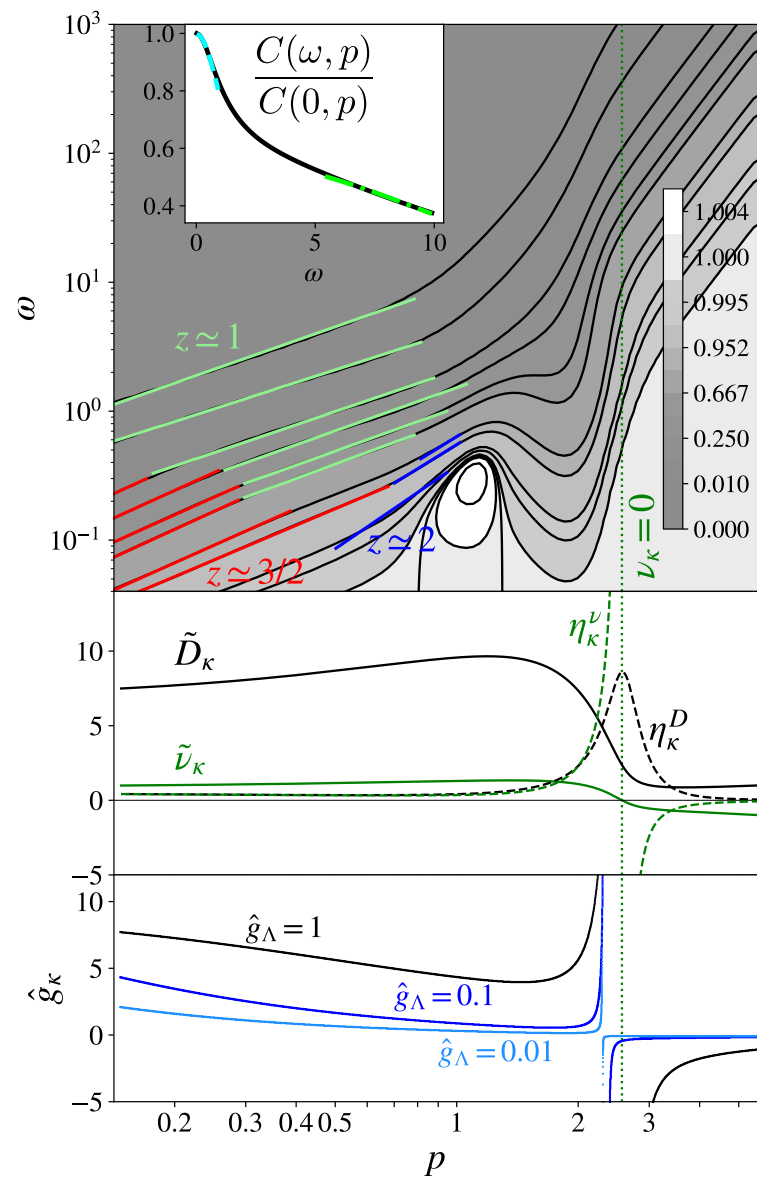
github.com/lghost/FRG



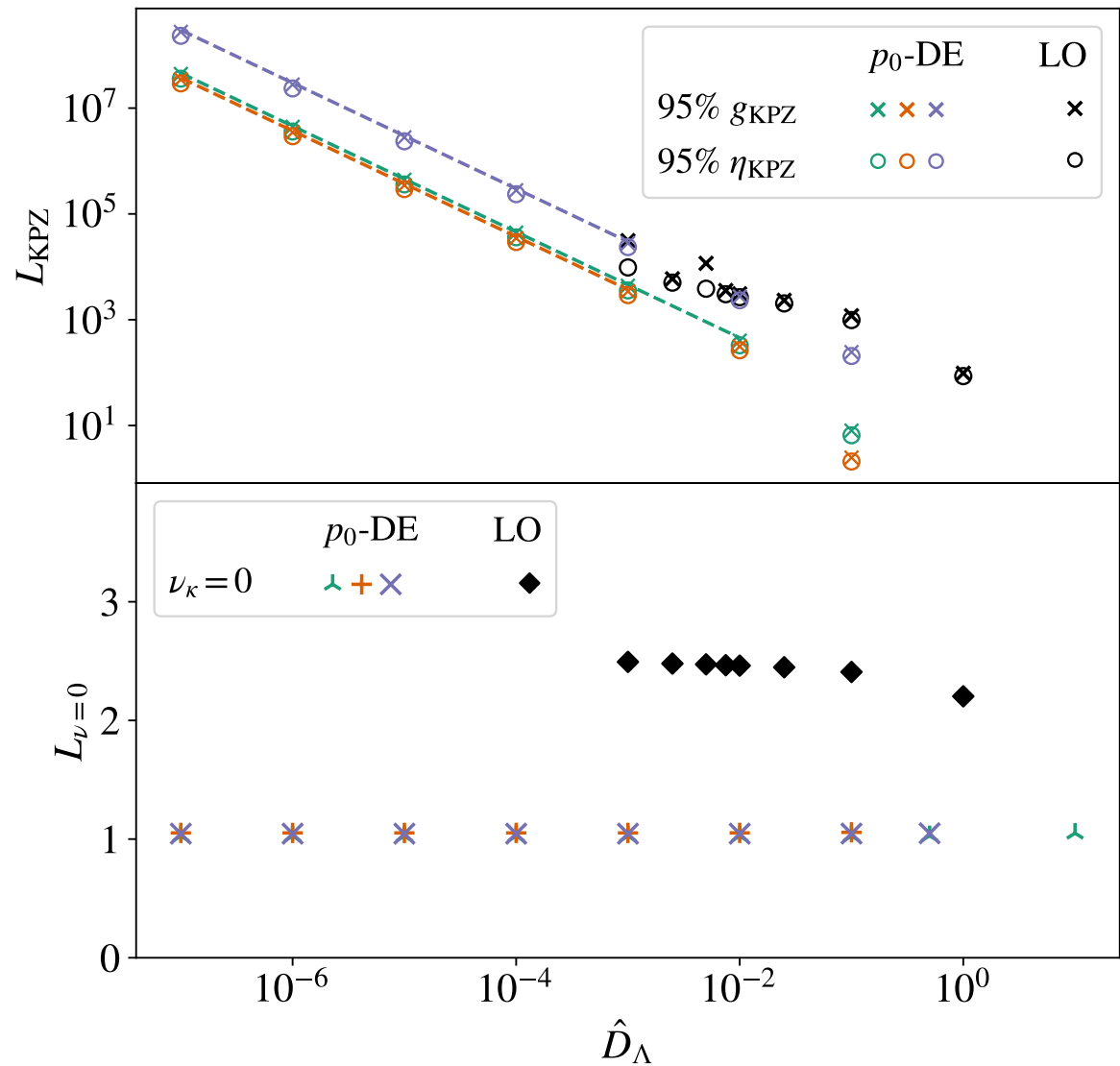
Backup slides

2. Scaling regimes of the

scheme



3. Zero noise limit



2. Scaling regimes of the KS equation

The *two grids* scheme

Exact FRG equation $\partial_\kappa \Gamma_\kappa = \frac{1}{2} \text{tr} \int_{\omega, \mathbf{q}} \partial_\kappa \mathcal{R}_\kappa \mathcal{G}_\kappa$
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The “small- p ” equations
 (valid for small momenta)
NLO ansatz

$$\partial_s \hat{f}_\kappa^{D, \nu}(\hat{\omega}, \hat{p}) = \dots$$

The “large- p ” equations
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No ansatz needed!

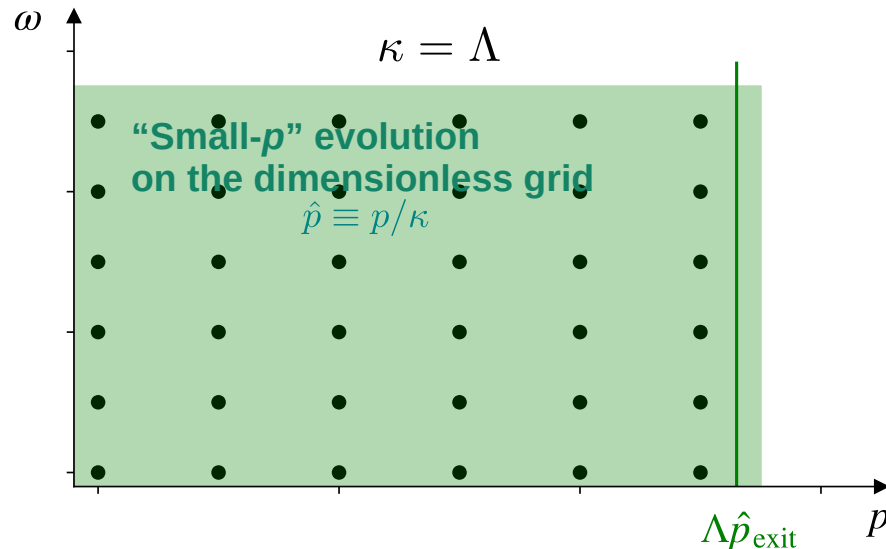
$$\partial_s C_\kappa(\varpi, p) = p^2 \int_\omega \frac{C_\kappa(\omega + \varpi, p) - C_\kappa(\varpi, p)}{\omega^2} \mathcal{J}_\kappa(\omega)$$

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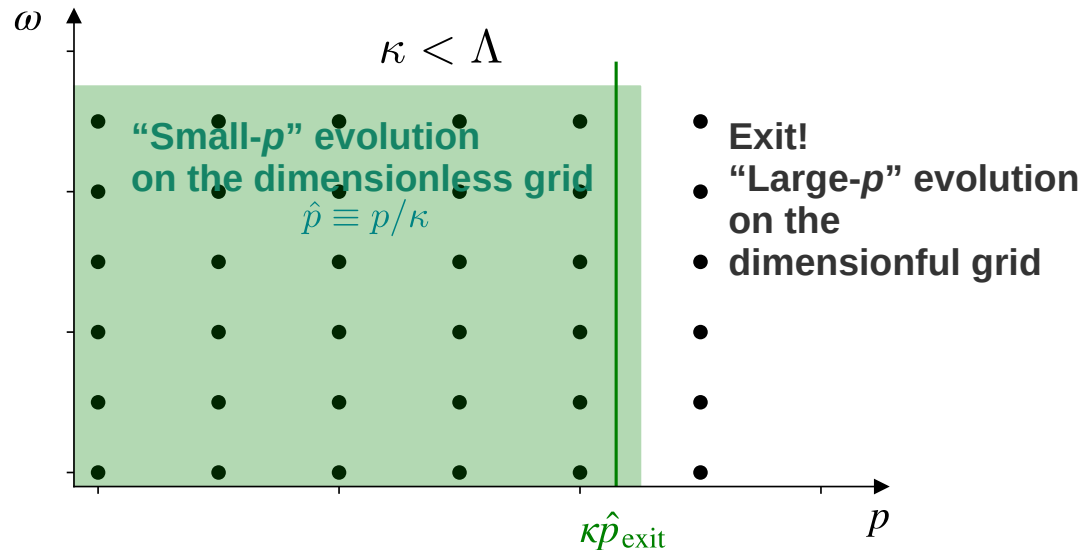
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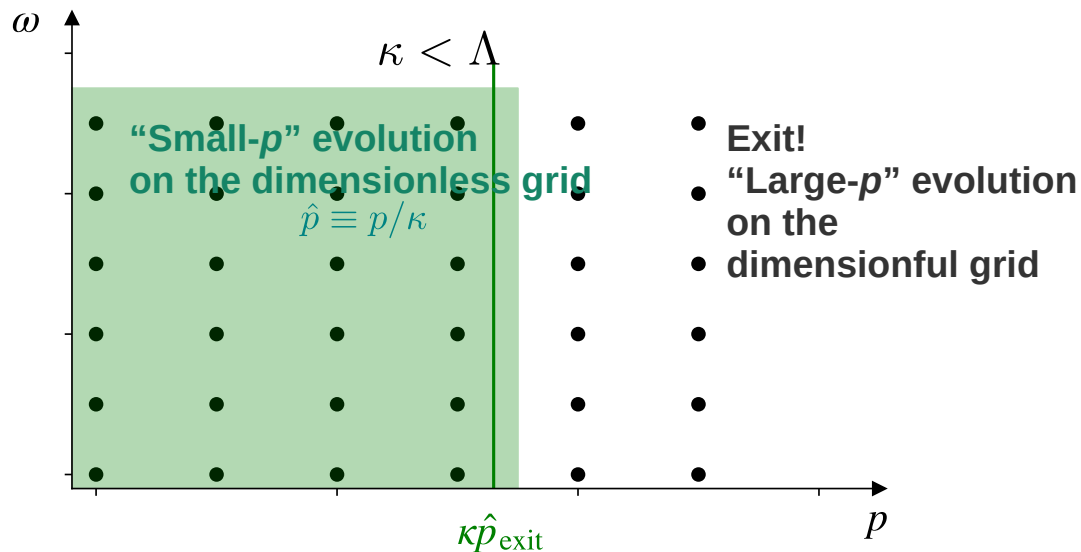
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