Derivative Expansion, Frustrated Antiferromagnets and O(N)xO(2) models

Nicolás Wschebor, [1]

[1]Instituto de Física, Facultad de Ingeniería, Universidad de la República
J.H.y Reissig 565, 11000 Montevideo, Uruguay,

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Main references, and colleagues

- This talk is the result of the work of several collaborations.
- On the existence of a small parameter in Derivative Expansion:
  - Convergence in the DE and application in the study of Ising critical exponents at order $\mathcal{O}(\partial^6)$:
    [Balog, Chaté, Delamotte, Marohnic, NW, PRL 123 (2019)]
  - Critical exponents at order $\mathcal{O}(\partial^4)$ in $O(N)$ models and error bars:
    [De Polsi, Balog, Tissier, NW, PRE 101 (2020)]
  - Universal amplitude ratios at order $\mathcal{O}(\partial^4)$ in $O(N)$ models:
    [De Polsi, Hernández, NW, PRE 104 (2021)]
  - Quantitative understanding of the regulator dependence:
    [De Polsi, NW, PRE 106 (2022)]
  - Application to $\mathbb{Z}_4$ model:
    [Chlebicki, SÃ¡nchez-Villalobos, Jakubczyk, NW, PRE 106 (2022)]
- Derivative Expansion Applications to $O(N) \times O(2)$ scalar models:
  - Review of LPA/LPA' results:
    [Delamotte, Mouhanna, Tissier, PRB 69 (2004)]
  - $\mathcal{O}(\partial^2)$ contributions:
    [Sánchez-Villalobos, Delamotte, NW, PRE 111 (2025)]

Introduction

- General problem: All-purposes non-perturbative approximations beyond perturbation theory?
- "Functional Renormalization Group" (FRG) is a modern version of Wilson RG.
- In this frame, an approximation scheme denoted "Derivative Expansion" (DE) implemented.
- It was not based on the smallness of a coupling constant.
- We know now that the DE has a very general small parameter $\sim 1/4$.
- Outline:
  - A small parameter has been found for the DE.
  - The $\mathcal{O}(\partial^4)$ were implemented in $O(N)$ models.
  - Controlled error bars were established.
  - This can be applied to non-trivial $O(N) \times O(2)$ models that describe frustrated magnets.

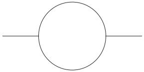
The radious of convergence of momentum expansion (I)

- Let us give a schematic review of the origin of the small parameter of the Derivative Expansion.
- Derivative expansion is expanding in momenta in Fourier space.
- Let us analyze the radious of convergence expansing vertices in momenta.
- Consider first the massive theory with:

  $p \leftrightsquigarrow$ order of magnitude of external momenta.

  $q \leftrightsquigarrow$ momenta circulating in the loop.

  $M \leftrightsquigarrow$ gap of the theory.
- We will consider here small momenta observables: $p \ll M$.
- Let us consider first the perturbative regime.
- Let's present the case of $\Gamma^{(2)}(p)$ (all $\Gamma^{(n)}(p)$ can be studied similarly).
- In models with three-liner interactions and/or uniform external field $\rightarrow$ the multi-particle threshold is at $p^2 \approx -4M^2$.

  $\Gamma^{(2)}(p)$ at one loop $\rightarrow$ 
- Higher vertices include also thresholds at $p^2 \approx -4M^2$.

The radious of convergence of momentum expansion (II)

- In fact, the factor 4 is NOT associated with perturbation theory.
- Only requirement: Absence of ultrarelativistic bound states.
- Consider the expansion

$$\frac{\Gamma^{(2)}(p)}{\Gamma^{(2)}(0)} = 1 + \frac{p^2}{M^2} + \sum_{n=2}^{\infty} c_n \left(\frac{p^2}{M^2}\right)^n$$

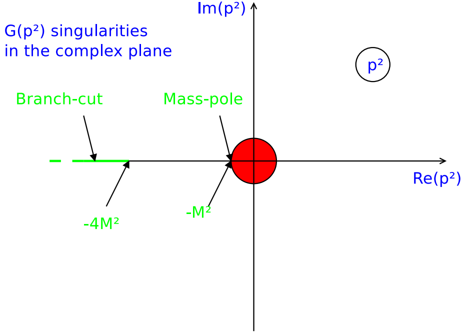
- The coefficients c_n are universal in the critical regime $M \ll \Lambda$.
- How are the coefficients c_n? $\rightarrow$ Already known! (with some precision...)

coefficient	$d=3$, $T>T_c$ HT / ϵ / fixed dim.	$d=3$, $T<T_c$ LT/ ϵ / fixed dim.
c_2	-(3.0 - 7.1)$\times 10^{-4}$	$\simeq -10^{-2}$
c_3	(0.5 - 1.3)$\times 10^{-5}$	$\simeq 4 \times 10^{-3}$
c_4	-(0.3 - 0.6) $\times 10^{-6}$	

Table: Estimates for high-temperature phase for $N=1$. (from [Pelissetto, Vicari, Phys. Rept. 368, 549 (2002)]).

The radious of convergence of momentum expansion (III)

- The radius of convergence of the series can be estimated.
- The Minkowskian extension of the massive model being unitary one knows the position of singularities. (Källén–Lehmann representation).
- To simplify let us consider the connected correlation function $G^{(2)}(p^2)$:



- The radius of convergence is $|p^2| < M^2$.

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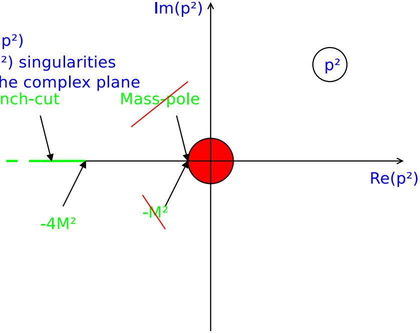

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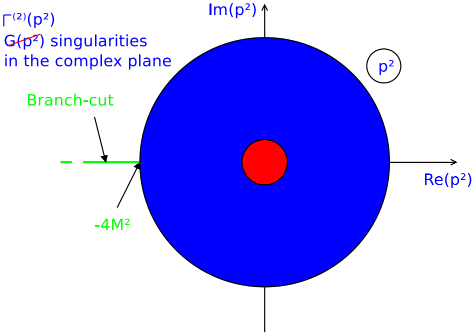


- The radius of convergence is $|p^2| < 4M^2$.

The radious of convergence of momentum expansion (IV)

- Consequently: $c_{n+1} \simeq -\frac{1}{4} c_n$ for large n.
- The $1/4$ is largely model independent. Moreover: the leading coefficients are very small! $(\sim \eta)$.
- So far, this has nothing to do with the Functional Renormalization Group, and these are exact results.
- Problems:
  - What happens in the critical regime? $(M^2 \to 0)$
  - How to decouple fluctuations with momenta $\gg M^2$?
- Both problems are solved in the framework of Functional Renormalization Group.
- Let us focus in zero momentum properties.
  Examples: Equation of state, critical exponents, most universal amplitude ratios, etc.

Functional Renomalization Group (I)

- Exact flow of Γ_k [C.Wetterich, Phys.Lett B301 (1993); U. Ellwanger, Z.Phys. C58 (1993); T. R. Morris, Int.J.Mod.Phys. A9 (1994).]:

$$\partial_k \Gamma_k[\phi] = \frac{1}{2}\int_{x,y} \partial_k R_k(x-y)\big(\Gamma_k^{(2)} + R_k\big)^{-1}_{x,y}$$

- Some properties of Wetterich equation:
  - Only 1PI contributions.
  - Internal momentum q is bounded: $q \lesssim k$;
  - For $q \lesssim k$ behaves as a massive theory with $M \sim k$.
- Approximations are needed in practice.
- Main point: Convenient for implementing approximation schemes beyond perturbation theory.

Functional Renomalization Group (II)

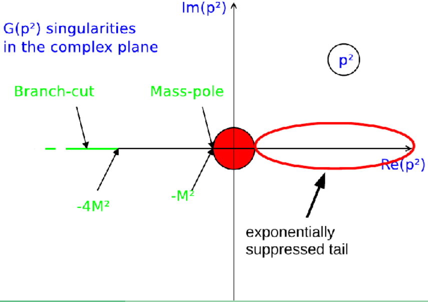
- The regulated theory behaves for $q \lesssim k$ as a massive theory with a mass: $M_k \sim k$.
- Derivative Expansion parameter is

$$\lambda = \frac{q^2}{4M_k^2} \lesssim \frac{k^2}{4M_k^2} \sim \frac{1}{4}$$

- Caveat: This applies to the bulk of the integrals not the tails.
- With a typical regulator:

$$R_k(q) \propto \frac{q^2}{\exp(q^2/k^2) - 1}, \qquad R_k(q) \propto k^2 \exp(-q^2/k^2), \ldots$$

- Tails are exponentially suppressed.
- However, at really high orders can induce a non-convergent asymptotic behavior of the Derivative Expansion.
  [T. Morris, ArXiv: 2607.18397]



The Derivative Expansion at work (I)

- Derivative Expansion: expanding vertices in momenta $\Leftrightarrow$ expanding in derivatives in direct space.
- Usually, DE is presented as an ansatz for Γ_k.
- Examples: In the Ising universality class the order $\mathcal{O}(\partial^6)$ have been studied:

$$\begin{aligned}
\Gamma_k[\phi] \; &= \int d^d x \Big[U_k(\phi) + \tfrac{1}{2} Z_k(\phi)(\partial\phi)^2 \\
&+ \tfrac{1}{2} W_k^a(\phi)(\partial_\mu\partial_\nu\phi)^2 + \tfrac{1}{2}\phi W_k^b(\phi)(\partial^2\phi)(\partial\phi)^2 \\
&+ \tfrac{1}{2} W_k^c(\phi)\left((\partial\phi)^2\right)^2 + \tfrac{1}{2}\tilde{X}_k^a(\phi)(\partial_\mu\partial_\nu\partial_\rho\phi)^2 \\
&+ \tfrac{1}{2}\phi\tilde{X}_k^b(\phi)(\partial_\mu\partial_\nu\phi)(\partial_\nu\partial_\rho\phi)(\partial_\mu\partial_\rho\phi) \\
&+ \tfrac{1}{2}\phi\tilde{X}_k^c(\phi)\left(\partial^2\phi\right)^3 + \tfrac{1}{2}\tilde{X}_k^d(\phi)\left(\partial^2\phi\right)^2(\partial\phi)^2 \\
&+ \tfrac{1}{2}\tilde{X}_k^e(\phi)(\partial\phi)^2(\partial_\mu\phi)(\partial^2\partial_\mu\phi) + \tfrac{1}{2}\tilde{X}_k^f(\phi)(\partial\phi)^2(\partial_\mu\partial_\nu\phi)^2 \\
&+ \tfrac{1}{2}\phi\tilde{X}_k^g(\phi)\left(\partial^2\phi\right)\left((\partial\phi)^2\right)^2 + \tfrac{1}{96}\tilde{X}_k^h(\phi)\left((\partial\phi)^2\right)^3 \Big]. \qquad (1)
\end{aligned}$$

[Balog, Chaté, Delamotte, Marohnic, NW, PRL 123 (2019)]

The Derivative Expansion at work (II)

- A typical Hamiltonian for $O(N)$ universality class is:

$$H[\varphi] = \int d^d x \left\{ \frac{1}{2}\partial_\mu\varphi^a\partial_\mu\varphi^a + \frac{r}{2}\varphi^a\varphi^a + \frac{u}{4!}(\varphi^a\varphi^a)^2 \right\}$$

- In the $O(N)$ universality class the order $\mathcal{O}(\partial^4)$ have been studied.

$$\begin{aligned}\Gamma_k^{\partial^4}[\phi] = \int_x \Big\{ & U_k(\rho) + \frac{1}{2}Z_k(\rho)\big(\partial_\mu\phi^a\big)^2 + \frac{1}{4}Y_k(\rho)\big(\partial_\mu\rho\big)^2 \\ & + \frac{W_1(\rho)}{2}\big(\partial_\mu\partial_\nu\phi^a\big)^2 + \frac{W_2(\rho)}{2}\big(\phi^a\partial_\mu\partial_\nu\phi^a\big)^2 \\ & + W_3(\rho)\partial_\mu\rho\partial_\nu\phi^a\partial_\mu\partial_\nu\phi^a + \frac{W_4(\rho)}{2}\phi^b\partial_\mu\phi^a\partial_\nu\phi^a\partial_\mu\partial_\nu\phi^b \\ & + \frac{W_5(\rho)}{2}\varphi^a\partial_\mu\rho\partial_\nu\rho\partial_\mu\partial_\nu\varphi^a + \frac{W_6(\rho)}{4}\Big(\big(\partial_\mu\varphi^a\big)^2\Big)^2 \\ & + \frac{W_7(\rho)}{4}\big(\partial_\mu\phi^a\partial_\nu\phi^a\big)^2 + \frac{W_8(\rho)}{2}\partial_\mu\phi^a\partial_\nu\varphi^a\partial_\mu\rho\partial_\nu\rho \\ & + \frac{W_9(\rho)}{2}\big(\partial_\mu\varphi^a\big)^2\big(\partial_\nu\rho\big)^2 + \frac{W_{10}(\rho)}{4}\Big(\big(\partial_\mu\rho\big)^2\Big)^2 + \mathcal{O}(\partial^6)\Big\}.\end{aligned}$$

[De Polsi, Balog, Tissier, NW, PRE 101 (2020)]

The Derivative Expansion at work (III)

- The DE can be employed to calculate many properties of the model.
- It is limited to momenta $p^2 \lesssim k^2, m$.
- In the critical regime it is limited to vertices or their derivatives at zero momenta.
- The "small parameter" has been tested:
  - Ising model at order $\mathcal{O}(\partial^6)$ [De Polsi, Balog, Tissier, NW, PRE 101 (2020)].
  - $O(N)$ model at order $\mathcal{O}(\partial^4)$ [De Polsi, Balog, Tissier, NW, PRE 101 (2020)].
  - Not only exponents. Also Universal Amplitude Ratios (UAR) [De Polsi, Hernández, NW, PRE 104 (2021)]
  - More precise than 6 and 7 loops.
  - In some exponents and most UAR : most precise in the literature.
  - Also applied to more involved models.
  - Let us consider $O(N) \times O(2) \leftrightarrow$ Frustrated magnets. [Delamotte, Mouhanna, Tissier, PRB 69 (2004); Sánchez-Villalobos, Delamotte, NW, PRE 111 (2025)]

Frustrated magnets and $O(N) \times O(2)$ models (I)

- Stacked triangular Antiferromagnets (STA) with $O(N)$ symmetry are frustrated magnets.
- They are described by $O(N) \times O(2)$ models.
  (see, for example, [Delamotte, Mouhanna, Tissier, PRB 69 (2004)])
- Described by two $N-$component vectors with Hamiltonian:

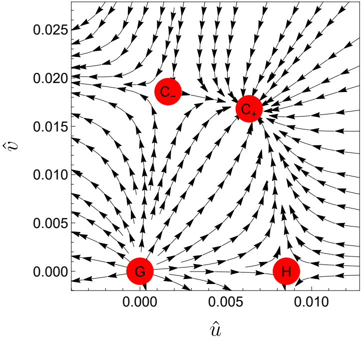
$$H = \int d^d x \Big\{ \frac{1}{2}(\partial_\mu \vec{\varphi}_1)^2 + \frac{1}{2}(\partial_\mu \vec{\varphi}_2)^2 + \frac{r}{2}\big(\vec{\varphi}_1^2 + \vec{\varphi}_2^2\big) + \frac{u}{4!}\big(\vec{\varphi}_1^2 + \vec{\varphi}_2^2\big)^2 + \frac{v}{4!}\Big(\frac{1}{2}(\vec{\varphi}_1^2 - \vec{\varphi}_2^2)^2 + 2(\vec{\varphi}_1 \cdot \vec{\varphi}_2)^2\Big)\Big\}$$

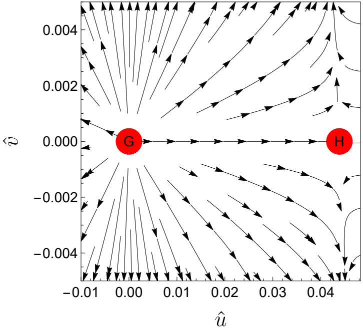
  with $v > 0$.
- Behavior in $d = 4 - \epsilon$:
  i) For $N > N_c^+(d)$, four fixed points. C_+ is critical $\rightarrow$ Second order.
  ii) For $N < N_c^+(d)$, two fixed points. No one is critical $\rightarrow$ First order.

$$N_c^+(d) = 21.8 - 23.4\epsilon + O(\epsilon^2),$$

Frustrated magnets and $O(N) \times O(2)$ models (II)

- Flow in (u, v) plane in $d = 3.9 \simeq 4$:





$N = 32 > N_c(d = 3.9)$
Second order

$N = 3 < N_c(d = 3.9)$
First order

Frustrated magnets and $O(N) \times O(2)$ models (III)

Problem: Realistic values $N = 2, 3$
First or Second order in d=3?

- Previous results:

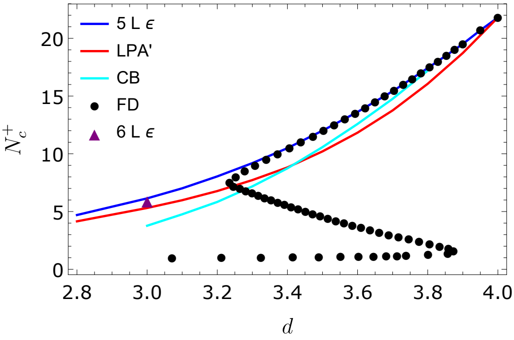


Figure based on [Delamotte, Dudka, Holovatch, Mouhanna, PRB 82 (2010)]

- Long term controversy between FRG and fixed-dimension perturbation theory.
- Difficulty: FRG results were LPA/LPA'. $\rightarrow$ no error bars.

$\mathcal{O}(\partial^2)$ corrections in $O(N) \times O(2)$ models (I)

- The effective potencial is a function of two invariants:

$$\rho = \frac{1}{2}\left(\vec{\varphi}_1^2 + \vec{\varphi}_2^2\right)$$

and

$$\tau = \frac{1}{2}(\vec{\varphi}_1^2 - \vec{\varphi}_2^2)^2 + 2(\vec{\varphi}_1 \cdot \vec{\varphi}_2)^2$$

- Solving the full potential in LPA/LPA' is already a difficult numerical problem

[Yabunaka, Delamotte, J.Stat.Mech. (2025)]

$$\Gamma_k^{LPA'}[\vec{\phi}] = \int_x \left\{ U_k(\rho, \tau) + \frac{Z_k}{2}\left(\partial_\mu \vec{\phi}_i \cdot \partial_\mu \vec{\phi}_i\right) \right\}.$$

$\mathcal{O}(\partial^2)$ corrections in $O(N) \times O(2)$ models (II)

- At order $\mathcal{O}(\partial^2)$ there are many functions of ρ and τ.
- Full $\mathcal{O}(\partial^2)$ ansatz ($S_{ab} = \vec{\phi}_a \cdot \vec{\phi}_b$)):

$$\begin{aligned}
\Gamma_k^{\mathcal{O}(\partial^2)}[\vec{\phi}] = \int_x \Big\{ & U_k(\rho,\tau) + \frac{Z_k(\rho,\tau)}{2}\left(\partial_\mu \vec{\phi}_i \cdot \partial_\mu \vec{\phi}_i\right) \\
& + \frac{Y_k(\rho,\tau)}{4}\partial_\mu\rho\partial_\mu\rho + \frac{W^{(1)}(\rho,\tau)}{4}\left(\vec{\phi}_1 \cdot \partial_\mu\vec{\phi}_2 - \vec{\phi}_2 \cdot \partial_\mu\vec{\phi}_1\right)^2 \\
& + \frac{W^{(2)}(\rho,\tau)}{4}\Big[\left(\vec{\phi}_1 \cdot \partial_\mu\vec{\phi}_2 + \vec{\phi}_2 \cdot \partial_\mu\vec{\phi}_1\right)^2 \\
& + \left(\vec{\phi}_1 \cdot \partial_\mu\vec{\phi}_1 - \vec{\phi}_2 \cdot \partial_\mu\vec{\phi}_2\right)^2\Big] + \frac{W^{(3)}(\rho,\tau)}{4} S_{ab}\partial_\mu\vec{\phi}_a \cdot \partial_\mu\vec{\phi}_b \\
& + \frac{W^{(4)}(\rho,\tau)}{4} S_{ac}(\vec{\phi}_a \cdot \partial_\mu\vec{\phi}_b)(\vec{\phi}_b\partial_\mu\vec{\phi}_c) \\
& + \frac{W^{(5)}(\rho,\tau)}{4} S_{ac}(\vec{\phi}_b \cdot \partial_\mu\vec{\phi}_a)(\vec{\phi}_b\partial_\mu\vec{\phi}_c) \\
& + \frac{W^{(6)}(\rho,\tau)}{4} S_{ab}(\vec{\phi}_b \cdot \partial_\mu\vec{\phi}_a) S_{cd}(\vec{\phi}_c \cdot \partial_\mu\vec{\phi}_d) \Big\}.
\end{aligned} \tag{2}$$

$\mathcal{O}(\partial^2)$ corrections in $O(N) \times O(2)$ models (III)

- The full $\mathcal{O}(\partial^2)$ order is too expensive.
- We considered only blue functions:

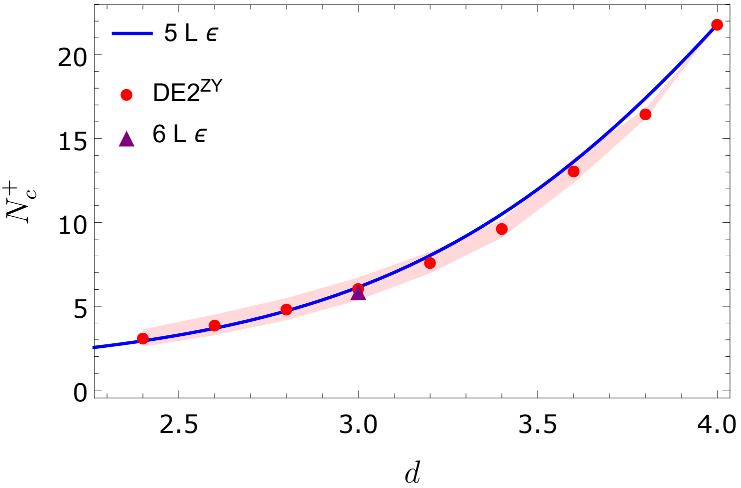
$$\Gamma_k^{ZY}[\vec{\phi}] = \int_x \left\{ U_k(\rho,\tau) + \frac{Z_k(\rho,\tau)}{2}\left(\partial_\mu \vec{\phi}_i \cdot \partial_\mu \vec{\phi}_i\right) + \frac{Y_k(\rho,\tau)}{4} \partial_\mu \rho \partial_\mu \rho \right\}.$$

- It is exact for the $O(2N)$-invariant and Gaussian fixed points.
- We assume that these terms are the dominant ones also for C_+ and C_- fixed points.
- As done for $O(N)$ models: we employ de Principle of Minimal Sensitivity (PMS).
- We used the regulator:

$$R_k(q) = \alpha \frac{Z_k q^2}{\exp(q^2/k^2) - 1}$$

$\mathcal{O}(\partial^2)$ corrections in $O(N) \times O(2)$ models (IV)

- Result for the $N_c^+(d)$ (with error bars!):



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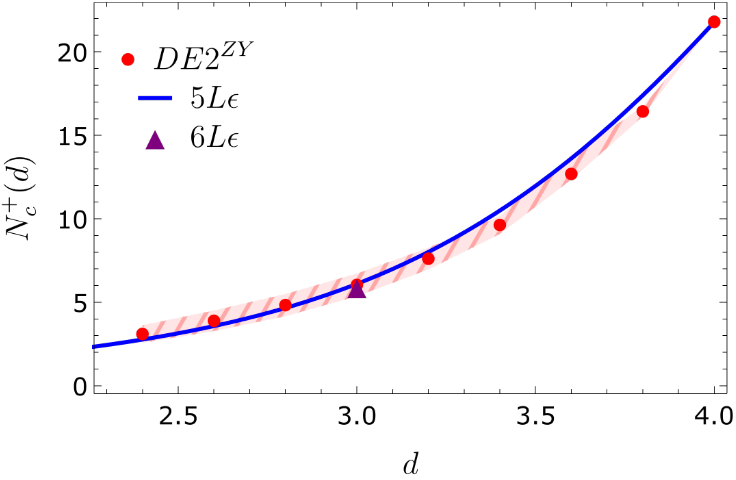

$\mathcal{O}(\partial^2)$ corrections in $O(N) \times O(2)$ models (V)

- Example in the 2d order region for $d = 3$.
- Exponents for $O(7) \times O(2)$ (with error bars!):

Table: Results at LPA and $DE2^{ZY}$ for $N = 7$ in $d = 3$ + other methods.

	ν	η	ω_1	ω_2
LPA	0.80	0	0.85	0.56
$DE2^{ZY}$	0.71(5)	0.047(24)	0.89(2)	0.30(12)
LPA' [1]	0.74	0.039		0.48
MC [2]	0.739(7)	0.030(17)		
6-loop, $d = 3$ [3]	0.68(2)		0.83(2)	0.23(5)
5-loop, $\overline{MS}$ [4]	0.68(4)	0.047(15)	0.8(2)	0.5(2)
5-loop, $\epsilon-$exp [5]	0.71(4)		0.84(3)	0.33(10)
6-loop, $\epsilon-$exp [6]	0.713(8)	0.045(3)	0.812(13)	0.34(2)
large$-N$ [7,8]	0.70(4)	0.053(2)	0.77(12)	0.54(23)

[1] Delamotte, Dudka, Mouhanna, Yabunaka, PRB93 (2016); [2] Sorokin, 2205.07199; [3] Calabrese, Parruccini, Sokolov, PRB68 (2003); [4] Calabrese, Parruccini, Pelissetto, Vicari, PRB70 (2004); [5] Calabrese, Parruccini, Nucl.Phys.B679 (2004); [6] Kompaniets, Kudlis, Sokolov, Nucl.Phys.B950 (2020); [7] Pelissetto, Rossi, Vicari, Nucl.Phys.B607 (2001); [8] Gracey, Nucl.Phys.B644 (2002).

Conclusions

- In the last ten years a qualitative change in the results of the Derivative Expansion has taken place.
- For low momenta: robust and accurate approximation scheme that, moreover, becomes precise at large enough orders.
- The existence of a small parameter makes it a controlled approximation for a very broad set of models.
- In many cases we became the most precise field-theoretical method.
- It works for non-trivial models as $O(N) \times O(2)$.
- What is the perspectives for the few years?
  - DE only works for small momenta.
  - The same analysis can be extended for arbitrary momenta. In that case, one needs the more sophisticated approximations [Blaizot, Méndez-Galain, NW, Phys. Lett B 632 (2006)].
  - In the DE a huge set of applications can be implemented in the next few years with error bars.
  - Most of the results presented here are not rigorous. Can we made them mathematically rigorous?
  - And what about asymptoticity?