Fermionic fRG with interaction flows: applications to electron-phonon systems

A. Al-Eryani, M. Gievers, K. Fraboulet, Phys. Rev. Research 8, 023029 (2026)

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13th International Conference on the Exact Renormalization Group

September 4, 2026

Introduction

High-temperature superconductivity

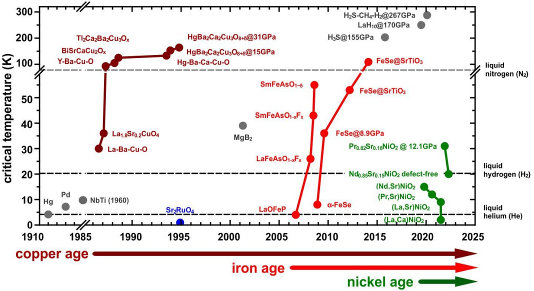

Fig. from www.tuwien.at (Accessed August 25, 2026)

A (selected) overview on the history of superconductivity:

- 1911: Discovery of superconductivity in mercury cooled with liquid helium down to ~ 4.2 K
  [Onnes, Proceedings of the Royal Netherlands Academy of Arts and Sciences (1911)]
- 1957: BCS theory for superconductivity
  [Bardeen, Cooper, Schrieffer, Phys. Rev. 108, 1175 (1957)]
- 1986: Discovery of superconductivity in **cuprates** at ~ 30 K (high T_c)
  [Bednorz, Müller, Z. Phys. B Condensed Matter 64, 189 (1986)]

Importance of electron-phonon coupling in superconductors

Electron-phonon coupling in cuprates:

- Substantial effects observed in experiments (isotope effect, ...)
  [Crawford, Kunchur, Farneth, McCarron III, Poon, PRB 41, 282 (1990)]
  [Yan *et al.*, PNAS 120, e2219491120 (2023)]
  [Chang, Timrov, Park, Zhou, Marzari, Bernardi, PRR 7, L012073 (2025)]
- Phonon-mediated BCS or Eliashberg theory not sufficient to describe superconductivity in cuprates (⚠ fRG relevant!)
  [Sun, Yang, Wang, Wang, PRB 109, L180508 (2024)]
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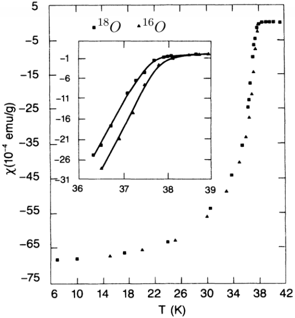
Meissner effect in $La_{2-x}Ba_xCuO_4$



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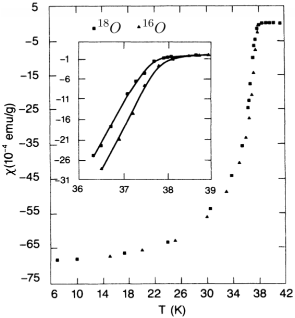

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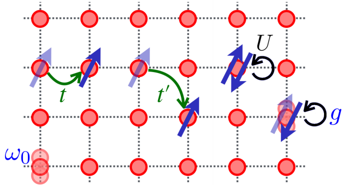
Model example: 2D Hubbard-Holstein model
[Holstein, Ann. Phys. 8, 325 (1959), 8, 343 (1959)]

$$\mathcal{H} = -t \sum_{\langle ij \rangle \sigma} c^\dagger_{i\sigma} c_{j\sigma} - t' \sum_{\ll ij \gg \sigma} c^\dagger_{i\sigma} c_{j\sigma} + U \sum_i c^\dagger_{i\uparrow} c_{i\uparrow} c^\dagger_{i\downarrow} c_{i\downarrow}$$

$$+ \omega_0 \sum_i b^\dagger_i b_i + g \sum_{i\sigma} c^\dagger_{i\sigma} c_{i\sigma} \left(b_i + b^\dagger_i \right)$$



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$$+ \omega_0 \sum_i b^\dagger_i b_i + g \sum_{i\sigma} c^\dagger_{i\sigma} c_{i\sigma} \left(b_i + b^\dagger_i\right)$$

$\Rightarrow$ Integrate out the phonons $b_i^{(\dagger)}$:

$$S[c^\dagger, c] = \sum_{\alpha_1,\alpha_2} c^\dagger_{\alpha_1} G^{-1}_{0,\alpha_1\alpha_2} c_{\alpha_2} + \sum_{\alpha_1,\alpha_2,\alpha_3,\alpha_4} \mathcal{U}_{\alpha_1\alpha_2\alpha_3\alpha_4} c^\dagger_{\alpha_1} c^\dagger_{\alpha_2} c_{\alpha_3} c_{\alpha_4}$$

$$\alpha_i = (\mathbf{k}_i, \nu_i, \sigma_i)$$

with $\quad \mathcal{U}_{\alpha_1\alpha_2\alpha_3\alpha_4} \sim \mathcal{U}(\nu_2 - \nu_1) = U - \dfrac{2\,|g|^2\,\omega_0}{(\nu_2-\nu_1)^2 + \omega_0^2}$

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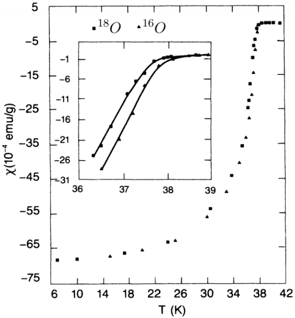

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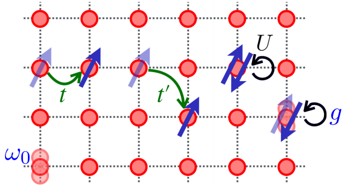
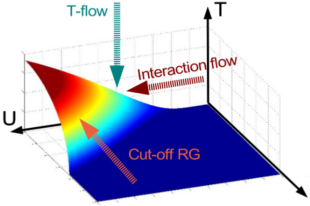


Temperature flow for electron-phonon systems

Main features of the temperature flow (T-flow):
[Honerkamp, Salmhofer, PRB 64, 184516 (2001)]

- Use T as flow parameter
- Start the flow at $T = \infty$ and lower down to T of interest
- Each step of the flow is **physical**



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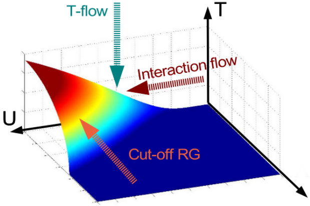

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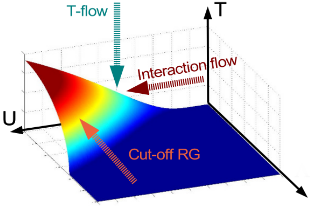

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Original formulation of the T-flow:

- Start from a fermionic model with *frequency-independent* interaction:

$$\mathcal{S}[c^\dagger, c] = T \sum_n c^\dagger(\nu_n) G_0^{-1}(\nu_n) c(\nu_n) + \frac{1}{4} T^3 \sum_{n_1,n_2,n_3} \mathcal{U}\, c^\dagger(\nu_{n_1}) c^\dagger(\nu_{n_2}) c(\nu_{n_3}) c(\nu_{n_1+n_2-n_3})$$

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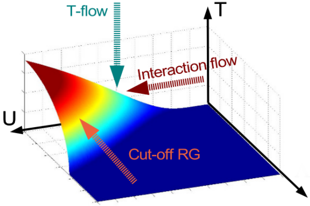

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- Define rescaled fields $\tilde{c}^\dagger(n) = T^{3/4} c^\dagger(\nu_n)$ and $\tilde{c}(n) = T^{3/4} c(\nu_n)$:

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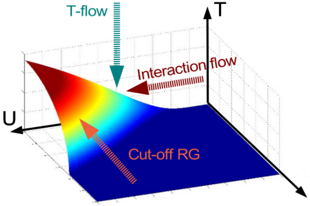

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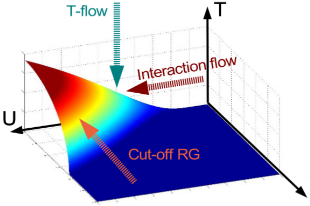

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- Define $G_{0,\Lambda}^{-1} = R_\Lambda G_0^{-1} = \Lambda^{1/2} G_0^{-1}$ with $\Lambda = T$ to use fRG equations with T as flow parameter

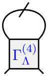
⚠ Only works if the bare interaction $\mathcal{U}$ is *frequency-independent*
For $\mathcal{U} = \mathcal{U}(\nu_{n_1}, \nu_{n_2}, \nu_{n_3})$, a flow parameter in **both** G_0 and $\mathcal{U}$ ($G_0 \to G_{0,\Lambda}$ and $\mathcal{U} \to \mathcal{U}_\Lambda$) is needed

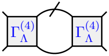
Conventional fermionic fRG with interaction flow

Insert regulator in the bare propagator ($G_0 \to G_{0,\Lambda}$)

[Wetterich, PLB 301, 90 (1993)]

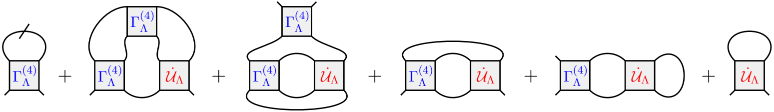
[Metzner, Salmhofer, Honerkamp, Meden, Schönhammer, RMP 84, 299 (2012)]

$\partial_\Lambda \Gamma_\Lambda^{(2)} \;=$ 

$\partial_\Lambda \Gamma_\Lambda^{(4)} \;=$ 

Insert regulator in the bare propagator ($G_0 \to G_{0,\Lambda}$) and in the bare interaction ($\mathcal{U} \to \mathcal{U}_\Lambda$) for a fermionic theory with $S[c^\dagger, c] = \sum_{\alpha_1,\alpha_2} c^\dagger_{\alpha_1} G^{-1}_{0,\alpha_1\alpha_2} c_{\alpha_2} + \sum_{\alpha_1,\alpha_2,\alpha_3,\alpha_4} \mathcal{U}_{\alpha_1\alpha_2\alpha_3\alpha_4} c^\dagger_{\alpha_1} c^\dagger_{\alpha_2} c_{\alpha_3} c_{\alpha_4}$:

$\partial_\Lambda \Gamma^{(2)}_\Lambda =$



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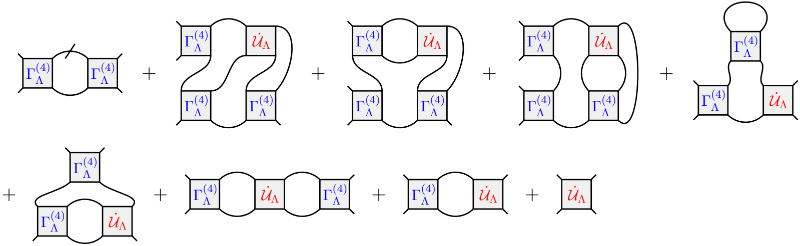

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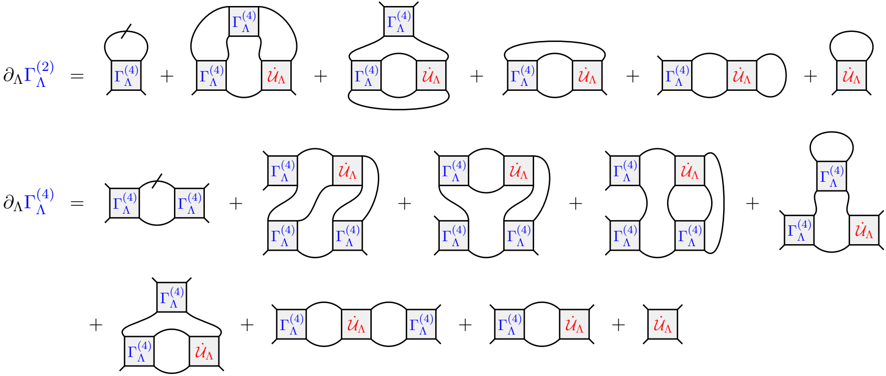

⚠ Several diagrams with nested loops ⇒ Highly **inefficient** to formulate interaction flow in this language!

Multiloop fRG and its bosonization with the single-boson exchange approach

Conventional fRG

Expand Wetterich equation

[Morris, IJMP A09, 2411 (1994)]

$$\Gamma[\psi] = -\ln\left(\int \mathcal{D}c\; e^{-S[c]+Jc}\right) + J\psi$$

$$\Big\Downarrow\; G_0 \to G_{0,\Lambda}$$

$$\partial_\Lambda \Gamma_\Lambda[\psi] = \frac{1}{2}\mathrm{Tr}\left[\left(\Gamma^{(2)}_\Lambda[\psi] + R_\Lambda\right)^{-1} \partial_\Lambda R_\Lambda\right]$$

$$\Big\Downarrow$$

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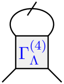

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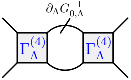

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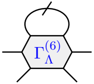

$\partial_\Lambda \Gamma^{(6)}_\Lambda$ = ...

$\vdots$

Conventional fRG vs Multiloop fRG

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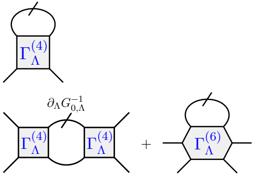

$\partial_\Lambda \Gamma_\Lambda^{(6)} = \quad \ldots$

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Multiloop fRG

Differentiate Schwinger-Dyson, Bethe-Salpeter equations

[Blaizot, Pawlowski, Reinosa, PLB 696, 523 (2011)]
[Kugler, von Delft, PRL 120, 057403 (2018)]

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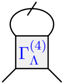

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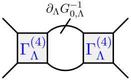

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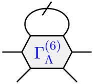

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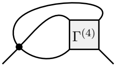
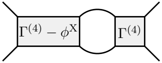
Σ =



, ϕ^{X} =



Conventional fRG vs Multiloop fRG

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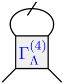
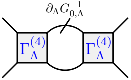
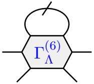
$\partial_\Lambda \Sigma_\Lambda =$ 

$\partial_\Lambda \Gamma_\Lambda^{(4)} =$ 

$+$ 

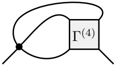
$\partial_\Lambda \Gamma_\Lambda^{(6)} = \quad \ldots$

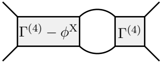
$\vdots$

Multiloop fRG

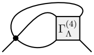
Differentiate Schwinger-Dyson, Bethe-Salpeter equations

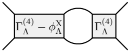
[Blaizot, Pawlowski, Reinosa, PLB 696, 523 (2011)]
[Kugler, von Delft, PRL 120, 057403 (2018)]

$\Sigma =$ 

, $\phi^{\mathrm{X}} =$ 

$$\Downarrow\; G_0 \to G_{0,\Lambda}$$

$\partial_\Lambda \Sigma_\Lambda = \partial_\Lambda\Big($ 

$\Big)$, $\partial_\Lambda \phi_\Lambda^{\mathrm{X}} = \partial_\Lambda\Big($ 

$\Big)$

Conventional fRG vs Multiloop fRG

Conventional fRG

Expand Wetterich equation

[Morris, IJMP A09, 2411 (1994)]

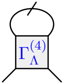
$$\Gamma[\psi] = -\ln\left(\int \mathcal{D}c\; e^{-S[c]+Jc}\right) + J\psi$$

$$\Downarrow \; G_0 \to G_{0,\Lambda}$$

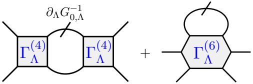
$$\partial_\Lambda \Gamma_\Lambda[\psi] = \frac{1}{2}\mathrm{Tr}\left[\left(\Gamma^{(2)}_\Lambda[\psi] + R_\Lambda\right)^{-1}\partial_\Lambda R_\Lambda\right]$$

$$\Downarrow$$

$\partial_\Lambda \Sigma_\Lambda =$



$\partial_\Lambda \Gamma^{(4)}_\Lambda =$



$\partial_\Lambda \Gamma^{(6)}_\Lambda = \;\ldots$

$\vdots$

Multiloop fRG

Differentiate Schwinger-Dyson, Bethe-Salpeter equations

[Blaizot, Pawlowski, Reinosa, PLB 696, 523 (2011)]
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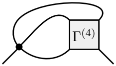
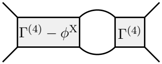
$\Sigma =$

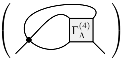


, $\phi^{\mathrm{X}} =$

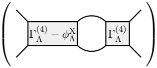


$$\Downarrow \; G_0 \to G_{0,\Lambda}$$

$\partial_\Lambda \Sigma_\Lambda = \partial_\Lambda$

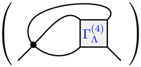


, $\partial_\Lambda \phi^{\mathrm{X}}_\Lambda = \partial_\Lambda$

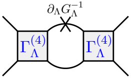


$$\Downarrow \; \partial_\Lambda \phi^{\mathrm{X}}_\Lambda = \sum_{\ell=1}^{\infty} \partial_\Lambda \phi^{\mathrm{X}(\ell)}_\Lambda$$

$\partial_\Lambda \Sigma_\Lambda = \partial_\Lambda$



$\partial_\Lambda \phi^{(1)\mathrm{X}}_\Lambda =$



$\partial_\Lambda \phi^{(2)\mathrm{X}}_\Lambda = \;\ldots$

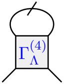
$\vdots$

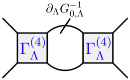
Conventional fRG vs Multiloop fRG

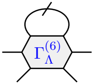
Conventional fRG

Expand Wetterich equation

[Morris, IJMP A09, 2411 (1994)]

$\partial_\Lambda \Sigma_\Lambda =$ 

$\partial_\Lambda \Gamma^{(4)}_\Lambda =$ 

$+$ 

$\partial_\Lambda \Gamma^{(6)}_\Lambda = \quad \ldots$

$\vdots$

Multiloop fRG

Differentiate Schwinger-Dyson, Bethe-Salpeter equations

[Blaizot, Pawlowski, Reinosa, PLB 696, 523 (2011)]
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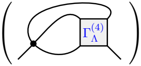
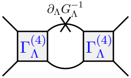
$\partial_\Lambda \Sigma_\Lambda = \partial_\Lambda$ 

$\partial_\Lambda \phi^{(1)\mathrm{X}}_\Lambda =$ 

$\partial_\Lambda \phi^{(2)\mathrm{X}}_\Lambda = \quad \ldots$

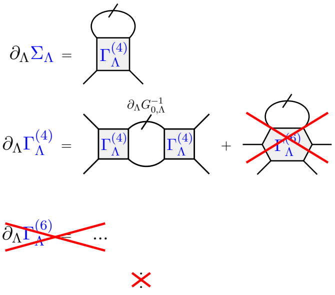
$\vdots$

Conventional fRG vs Multiloop fRG

Conventional fRG

Expand Wetterich equation

[Morris, IJMP A09, 2411 (1994)]



Multiloop fRG

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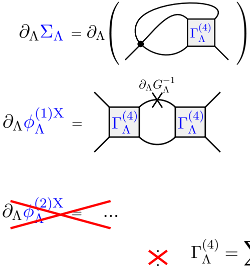
[Blaizot, Pawlowski, Reinosa, PLB 696, 523 (2011)]
[Kugler, von Delft, PRL 120, 057403 (2018)]



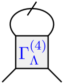
$$\Gamma^{(4)}_\Lambda = \sum_{X} \phi^X_\Lambda + \underbrace{\cancel{\mathcal{I}^{2PI}_\Lambda}}_{\mathcal{U}}$$

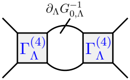
Conventional fRG vs Multiloop fRG

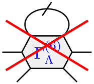
Conventional fRG

Expand Wetterich equation

[Morris, IJMP A09, 2411 (1994)]

$\partial_\Lambda \Sigma_\Lambda =$ 

$\partial_\Lambda \Gamma^{(4)}_\Lambda =$ 

$+$ 

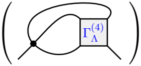
$\cancel{\partial_\Lambda \Gamma^{(6)}_\Lambda = \dots}$

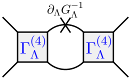
$\cancel{\vdots}$

Multiloop fRG

Differentiate Schwinger-Dyson, Bethe-Salpeter equations

[Blaizot, Pawlowski, Reinosa, PLB 696, 523 (2011)]
[Kugler, von Delft, PRL 120, 057403 (2018)]

$\partial_\Lambda \Sigma_\Lambda = \partial_\Lambda$ 

$\partial_\Lambda \phi^{(1)\mathrm{X}}_\Lambda =$ 

$\cancel{\partial_\Lambda \phi^{(2)\mathrm{X}}_\Lambda = \dots}$

$\cancel{\vdots}$ $\Gamma^{(4)}_\Lambda = \sum_{\mathrm{X}} \phi^{\mathrm{X}}_\Lambda + \underbrace{\cancel{\mathcal{I}^{\mathrm{2PI}}_\Lambda}}_{\mathcal{U}}$

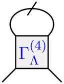
- **1-loop truncation** yields same flow equations for the 4-point vertex, with $\partial_\Lambda G^{-1}_{0,\Lambda} \leftrightarrow \partial_\Lambda G^{-1}_\Lambda$ (Katanin substitution)

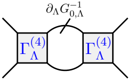
Conventional fRG vs Multiloop fRG

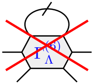
Conventional fRG

Expand Wetterich equation

[Morris, IJMP A09, 2411 (1994)]

$\partial_\Lambda \Sigma_\Lambda =$ 

$\partial_\Lambda \Gamma^{(4)}_\Lambda =$ 

$+$ 

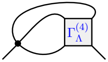
$\cancel{\partial_\Lambda \Gamma^{(6)}_\Lambda = }\ \ldots$

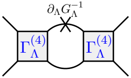
$\cancel{\therefore}$

Multiloop fRG

Differentiate Schwinger-Dyson, Bethe-Salpeter equations

[Blaizot, Pawlowski, Reinosa, PLB 696, 523 (2011)]
[Kugler, von Delft, PRL 120, 057403 (2018)]

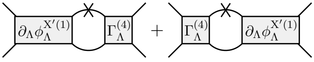
$\partial_\Lambda \Sigma_\Lambda = \partial_\Lambda \Big($  $\Big)$

$\partial_\Lambda \phi^{(1)\mathrm{X}}_\Lambda =$ 

$\cancel{\partial_\Lambda \phi^{(2)\mathrm{X}}_\Lambda = }\ \ldots$

$\cancel{\therefore}$ $\quad \Gamma^{(4)}_\Lambda = \sum_{\mathrm{X}} \phi^{\mathrm{X}}_\Lambda + \underbrace{\cancel{\mathcal{I}^{\mathrm{2PI}}_\Lambda}}_{\mathcal{U}}$

- **1-loop truncation** yields same flow equations for the 4-point vertex, with $\partial_\Lambda G^{-1}_{0,\Lambda} \leftrightarrow \partial_\Lambda G^{-1}_{\Lambda}$ (Katanin substitution)
- Much more affordable to go **beyond 1-loop truncation** with multiloop fRG than with conventional fRG

$\partial_\Lambda \phi^{(2)\mathrm{X}}_\Lambda = \sum_{\mathrm{X}' \neq \mathrm{X}} \Big($  $\Big)$

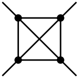
Bosonization with the single-boson exchange approach

Single-boson exchange (SBE) decomposition of the 4-point vertex:

[Krien, Valli, Capone, PRB 100, 155149 (2019)]

$$\Gamma^{(4)} = \sum_{\mathrm{X}} \phi^{\mathrm{X}} + \mathcal{I}^{\mathrm{2PI}}, \qquad \text{with} \quad \mathcal{I}^{\mathrm{2PI}} =$$

$+$



$+ \ldots$

only 2PI diagrams

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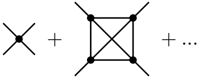
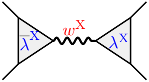


only 2PI diagrams

- Heart of the SBE decomposition:

$$\phi^{\mathrm{X}}(Q,k,k') =$$



$$+ M^{\mathrm{X}}(Q,k,k') - \mathcal{U}$$

$\bar{\lambda}^{\mathrm{X}}(Q,k) w^{\mathrm{X}}(Q) \lambda^{\mathrm{X}}(Q,k')$

w^{X} : bosonic propagator
λ^{X} : Yukawa coupling
M^{X} : rest function

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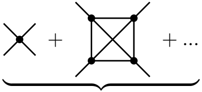

only 2PI diagrams

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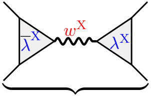

$$+ M^{\mathrm{X}}(Q,k,k') - \mathcal{U}$$

$\bar{\lambda}^{\mathrm{X}}(Q,k) w^{\mathrm{X}}(Q) \lambda^{\mathrm{X}}(Q,k')$

w^{X} : bosonic propagator
λ^{X} : Yukawa coupling
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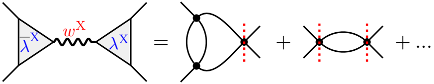
- SBE part of ϕ^{X} only contains **$\mathcal{U}$-reducible diagrams**:



Single-boson exchange (SBE) decomposition of the 4-point vertex:
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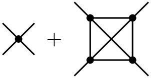
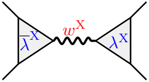


$+ \ldots$

only 2PI diagrams

- Heart of the SBE decomposition:

$$\phi^{\mathrm{X}}(Q,k,k') =$$



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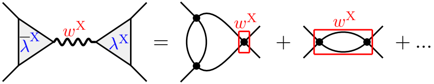

Bosonization with the single-boson exchange approach

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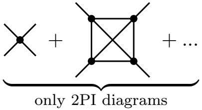

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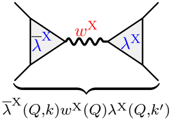

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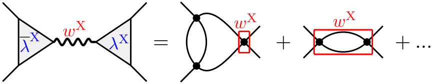
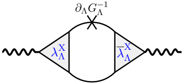
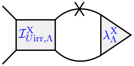


- (Multiloop) fRG flow equations can be formulated for w^{X}, λ^{X} and M^{X}:
[Gievers, Walter, Ge, von Delft, Kugler, EPJB 95, 108 (2022)]

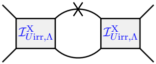
$$\partial_\Lambda w_\Lambda^{\mathrm{X}(1)} =$$



$$, \quad \partial_\Lambda \lambda_\Lambda^{\mathrm{X}(1)} =$$



$$, \quad \partial_\Lambda M_\Lambda^{\mathrm{X}(1)} =$$



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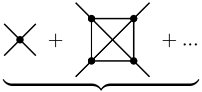
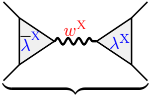


only 2PI diagrams

- Heart of the SBE decomposition:

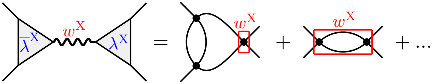
$$\phi^{\mathrm{X}}(Q,k,k') =$$



$$\underbrace{\qquad}_{\overline{\lambda}^{\mathrm{X}}(Q,k)\,w^{\mathrm{X}}(Q)\,\lambda^{\mathrm{X}}(Q,k')} + M^{\mathrm{X}}(Q,k,k') - \mathcal{U}$$

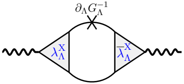
w^{X} : bosonic propagator
λ^{X} : Yukawa coupling
M^{X} : rest function

- SBE part of ϕ^{X} only contains **$\mathcal{U}$-reducible diagrams**:

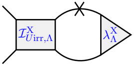


- (Multiloop) fRG flow equations can be formulated for w^{X}, λ^{X} and M^{X}:
[Gievers, Walter, Ge, von Delft, Kugler, EPJB 95, 108 (2022)]

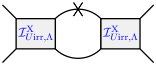
$$\partial_\Lambda w_\Lambda^{\mathrm{X}(1)} =$$



$$, \quad \partial_\Lambda \lambda_\Lambda^{\mathrm{X}(1)} =$$



$$, \quad \partial_\Lambda M_\Lambda^{\mathrm{X}(1)} =$$



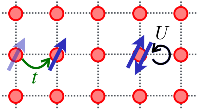
⚠ $w^{\mathrm{X}}(Q)$ and $\lambda^{\mathrm{X}}(Q,k)$ much simpler objects than $\phi^{\mathrm{X}}(Q,k,k')$ ⇒ Room for **efficient approximations**

Multiloop SBE fRG results for the 2D Hubbard model

2D Hubbard model Hamiltonian:

[Hubbard, PRSL A 276, 238 (1963)]
[Gutzwiller, PRL 10, 159 (1963)]
[Kanamori, PTP 30, 275 (1963)]

$$\mathcal{H} = -t \sum_{\langle ij \rangle \sigma} c^\dagger_{i\sigma} c_{j\sigma} + U \sum_i c^\dagger_{i\uparrow} c_{i\uparrow} c^\dagger_{i\downarrow} c_{i\downarrow}$$



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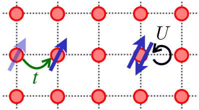

Multiloop SBE fRG results without rest functions ($M^{\mathrm{X}}(Q, k, k') = 0$):

[Fraboulet, Al-Eryani, Heinzelmann, Kauch, Andergassen, arXiv:2512.11190]

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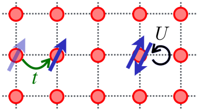

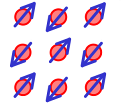

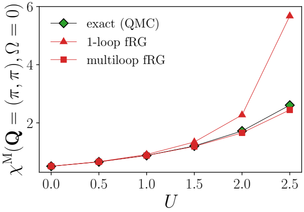

Parameters: $T = 0.2t$, half filling

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[Fraboulet, Al-Eryani, Heinzelmann, Kauch, Andergassen, arXiv:2512.11190]

- Possible to reach **quantitative accuracy** at loop convergence

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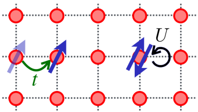

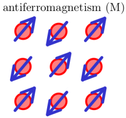

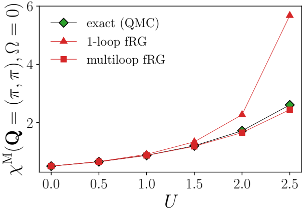

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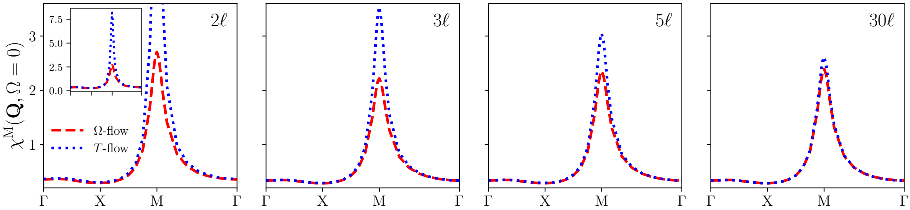
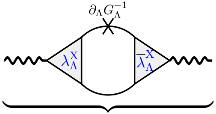
- ▶ Possible to reach **quantitative accuracy** at loop convergence
- ▶ **Weak regulator dependence** observed at loop convergence



Parameters: $U = 2.5t$, $T = 0.2t$, half filling

Interaction flow with multiloop SBE fRG flow equations

Multiloop SBE fRG equations **barely** modified when adding a regulator also in the bare interaction ($\mathcal{U} \to \mathcal{U}_\Lambda$):
[Al-Eryani, Gievers, Fraboulet, PRR 8, 023029 (2026)]

$\partial_\Lambda w_\Lambda^{\mathrm{X}(1)} =$ 

$- \; w_\Lambda^{\mathrm{X}} \big(\partial_\Lambda \mathcal{U}_\Lambda^{-1}\big) w_\Lambda^{\mathrm{X}}$

$w_\Lambda^{\mathrm{X}} \lambda_\Lambda^{\mathrm{X}} \big(\partial_\Lambda \Pi_\Lambda^{\mathrm{X}}\big) \overline{\lambda}_\Lambda^{\mathrm{X}} w_\Lambda^{\mathrm{X}}$

- New terms involving $\partial_\Lambda \mathcal{U}_\Lambda$ essentially in $\partial_\Lambda w_\Lambda^{\mathrm{X}(1)}$
  ⚠ $w_\Lambda^{\mathrm{X}}(Q)$ much simpler than $\lambda_\Lambda^{\mathrm{X}}(Q,k)$ or $M_\Lambda^{\mathrm{X}}(Q,k,k')$

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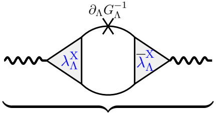

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$$\left(w^{\mathrm{X}}\right)^{-1} = \mathcal{U}^{-1} - P^{\mathrm{X}} \;\; \Rightarrow \;\; \partial_\Lambda \left(w_\Lambda^{\mathrm{X}}\right)^{-1} = \partial_\Lambda \mathcal{U}_\Lambda^{-1} - P_\Lambda^{\mathrm{X}}$$

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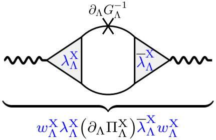

$$- \; w_\Lambda^{\mathrm{X}} \left(\partial_\Lambda \mathcal{U}_\Lambda^{-1}\right) w_\Lambda^{\mathrm{X}}$$

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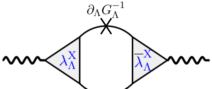

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Use this generalized fRG approach to formulate a ***T*-flow** for a model with **retarded interaction** (Anderson-Holstein impurity)

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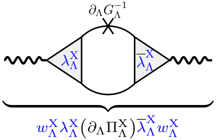
$\partial_\Lambda w_\Lambda^{\mathrm{X}(1)} =$  $- \; w_\Lambda^{\mathrm{X}} \left(\partial_\Lambda \mathcal{U}_\Lambda^{-1}\right) w_\Lambda^{\mathrm{X}}$

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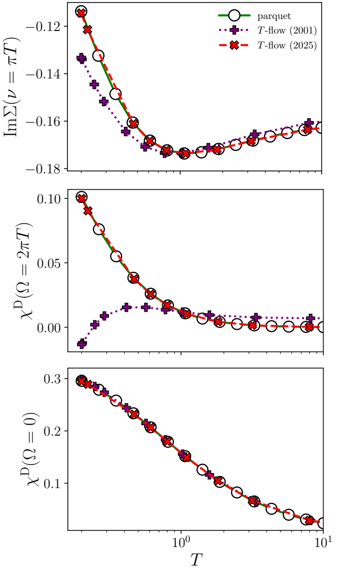
Use this generalized fRG approach to formulate a ***T*-flow** for a model with **retarded interaction** (Anderson-Holstein impurity)

$\Rightarrow$ Get back the parquet approximation ($\mathcal{I}^{2\mathrm{PI}} = \mathcal{U}$) at each step of the flow when loop convergence is reached



Parameters: $U = 1.4$, $\omega_0 = 1$, $V_H = 0.5$, half filling

Conclusion

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  - Use $\mathcal{U}_\Lambda$ to fix useful constraints during the flow (Ward identities, ...)
  - Extended DMFT + fRG for non-local interactions
    [Taranto, Andergassen, Bauer, Held, Katanin, Metzner, Rohringer, Toschi, PRL 112, 196402 (2014)]
    [Vilardi, Taranto, Metzner, PRB 99, 104501 (2019)]
    [Katanin, PRB 99, 115112 (2019)]
    [Bonetti, Toschi, Hille, Andergassen, Vilardi, PRR 4, 013034 (2022)]

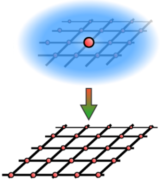

Fig. from Bonetti *et al.*, PRR 4, 013034 (2022)

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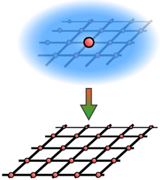


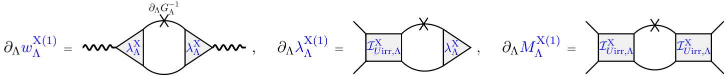
Fig. from Bonetti *et al.*, PRR 4, 013034 (2022)

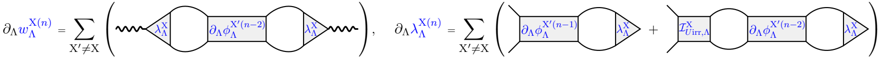
- fRG solver publicly available at Al-Eryani, Patricolo, Fraboulet, arXiv:2604.04232

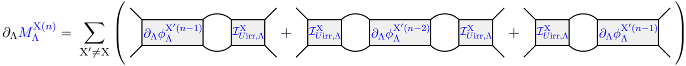
Backup slides

Multiloop SBE fRG equations:

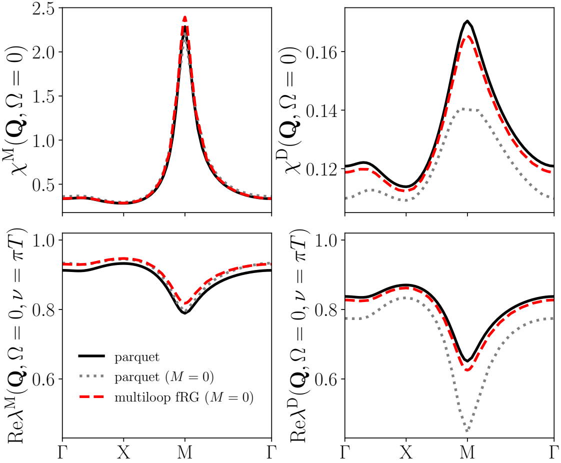
[Gievers, Walter, Ge, von Delft, Kugler, EPJB 95, 108 (2022)]







$$\partial_\Lambda w_\Lambda^X = \sum_{\ell=1}^{\infty} \partial_\Lambda w_\Lambda^{X(\ell)}, \quad \partial_\Lambda \lambda_\Lambda^X = \sum_{\ell=1}^{\infty} \partial_\Lambda \lambda_\Lambda^{X(\ell)}, \quad \partial_\Lambda M_\Lambda^X = \sum_{\ell=1}^{\infty} \partial_\Lambda M_\Lambda^{X(\ell)}$$



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2D Hubbard model, $U = 2.5t$, $T = 0.2t$, half filling