

Fermionic fRG with interaction flows: applications to electron-phonon systems

A. Al-Eryani, M. Gievers, K. Fraboulet, Phys. Rev. Research 8, 023029 (2026)

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13th International Conference on the Exact Renormalization Group

September 4, 2026

Introduction

High-temperature superconductivity

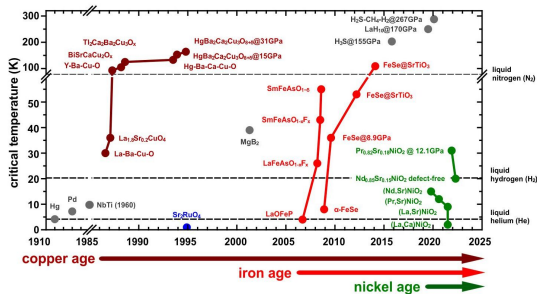


Fig. from www.tuwien.at (Accessed August 25, 2026)

A (selected) overview on the history of superconductivity:

- ▶ 1911: Discovery of superconductivity in mercury cooled with liquid helium down to ~ 4.2 K [Onnes, Proceedings of the Royal Netherlands Academy of Arts and Sciences (1911)]
- ▶ 1957: BCS theory for superconductivity [Bardeen, Cooper, Schrieffer, Phys. Rev. 108, 1175 (1957)]
- ▶ 1986: Discovery of superconductivity in **cuprates** at ~ 30 K (high T_c) [Bednorz, Müller, Z. Phys. B Condensed Matter 64, 189 (1986)]

Importance of electron-phonon coupling in superconductors

Electron-phonon coupling in cuprates:

- Substantial effects observed in experiments (isotope effect, ...)
[Crawford, Kunchur, Farneth, McCarron III, Poon, PRB 41, 282 (1990)]
[Yan *et al.*, PNAS 120, e2219491120 (2023)]
[Chang, Timrov, Park, Zhou, Marzari, Bernardi, PRR 7, L012073 (2025)]
- Phonon-mediated BCS or Eliashberg theory not sufficient to describe superconductivity in cuprates (Δ fRG relevant!)
[Sun, Yang, Wang, Wang, PRB 109, L180508 (2024)]
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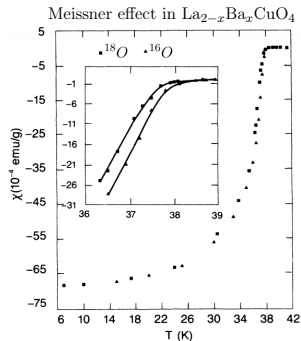


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Model example: 2D Hubbard-Holstein model

[Holstein, Ann. Phys. 8, 325 (1959), 8, 343 (1959)]

$$\mathcal{H} = -t \sum_{\langle ij \rangle \sigma} c_{i\sigma}^\dagger c_{j\sigma} - t' \sum_{\langle\langle ij \rangle\rangle \sigma} c_{i\sigma}^\dagger c_{j\sigma} + U \sum_i c_{i\uparrow}^\dagger c_{i\uparrow} c_{i\downarrow}^\dagger c_{i\downarrow} \\ + \omega_0 \sum_i b_i^\dagger b_i + g \sum_{i\sigma} c_{i\sigma}^\dagger c_{i\sigma} (b_i + b_i^\dagger)$$

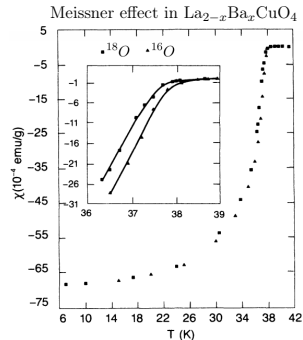
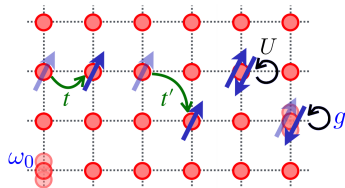


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$\Rightarrow$ Integrate out the phonons $b_i^{(\dagger)}$:

$$S[c^\dagger, c] = - \sum_{\alpha_1, \alpha_2} c_{\alpha_1}^\dagger G_{0, \alpha_1 \alpha_2}^{-1} c_{\alpha_2} + \sum_{\alpha_1, \alpha_2, \alpha_3, \alpha_4} \mathcal{U}_{\alpha_1 \alpha_2 \alpha_3 \alpha_4} c_{\alpha_1}^\dagger c_{\alpha_2}^\dagger c_{\alpha_3} c_{\alpha_4} \\ \alpha_i = (\mathbf{k}_i, \nu_i, \sigma_i)$$

$$\text{with } \mathcal{U}_{\alpha_1 \alpha_2 \alpha_3 \alpha_4} \sim \mathcal{U}(\nu_2 - \nu_1) = U - \frac{2|g|^2 \omega_0}{(\nu_2 - \nu_1)^2 + \omega_0^2}$$

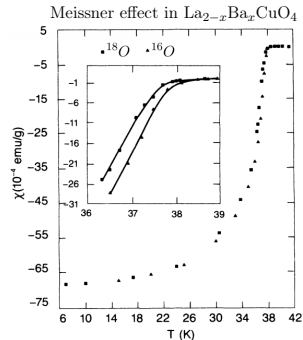
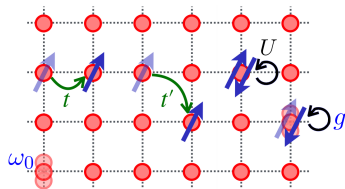


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Temperature flow for electron-phonon systems

Main features of the temperature flow (T -flow):

[Honerkamp, Salmhofer, PRB 64, 184516 (2001)]

- ▶ Use T as flow parameter
- ▶ Start the flow at $T = \infty$ and lower down to T of interest
- ▶ Each step of the flow is **physical**

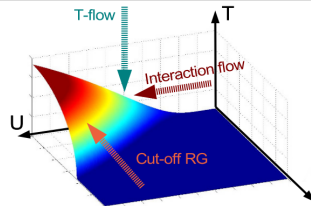


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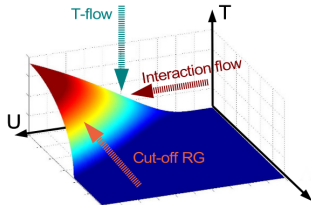


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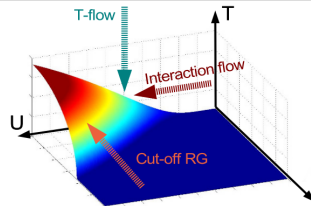


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Original formulation of the T -flow:

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$$\mathcal{S}[c^\dagger, c] = T \sum_n c^\dagger(\nu_n) G_0^{-1}(\nu_n) c(\nu_n) + \frac{1}{4} T^3 \sum_{n_1, n_2, n_3} \mathcal{U} c^\dagger(\nu_{n_1}) c^\dagger(\nu_{n_2}) c(\nu_{n_3}) c(\nu_{n_1+n_2-n_3})$$

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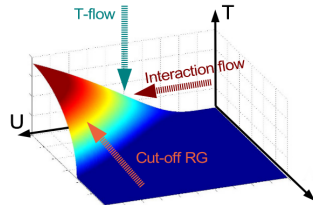


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- Define rescaled fields $\tilde{c}^\dagger(n) = T^{3/4} c^\dagger(\nu_n)$ and $\tilde{c}(n) = T^{3/4} c(\nu_n)$:

$$\mathcal{S}[c^\dagger, c] = \sum_n \tilde{c}^\dagger(n) \underbrace{T^{-1/2} G_0^{-1}(\nu_n)}_{G_{0,T}^{-1}(\nu_n)} \tilde{c}(n) + \frac{1}{4} \sum_{n_1, n_2, n_3} \mathcal{U} \tilde{c}^\dagger(n_1) \tilde{c}^\dagger(n_2) \tilde{c}(n_3) \tilde{c}(n_1 + n_2 - n_3)$$

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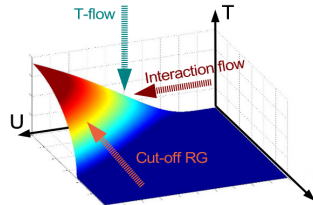


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- Define $G_{0,\Lambda}^{-1} = R_\Lambda G_0^{-1} = \Lambda^{1/2} G_0^{-1}$ with $\Lambda = T$ to use fRG equations with T as flow parameter

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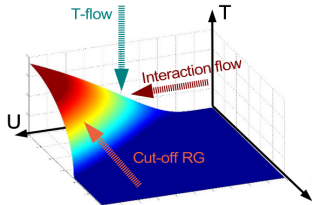


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- Define $G_{0,\Lambda}^{-1} = R_\Lambda G_0^{-1} = \Lambda^{1/2} G_0^{-1}$ with $\Lambda = T$ to use fRG equations with T as flow parameter

⚠ Only works if the bare interaction $\mathcal{U}$ is *frequency-independent*

For $\mathcal{U} = \mathcal{U}(\nu_{n_1}, \nu_{n_2}, \nu_{n_3})$, a flow parameter in **both** G_0 and $\mathcal{U}$ ($G_0 \rightarrow G_{0,\Lambda}$ and $\mathcal{U} \rightarrow \mathcal{U}_\Lambda$) is needed

Conventional fermionic fRG

Conventional fermionic fRG equations (Wetterich equation + vertex expansion):

[Wetterich, PLB 301, 90 (1993)]

[Morris, IJMP A09, 2411 (1994)]

[Metzner, Salmhofer, Honerkamp, Meden, Schönhammer, RMP 84, 299 (2012)]

$$\begin{aligned}\partial_\Lambda \Gamma_\Lambda^{(2)} &= \text{Diagram 1} \\ \partial_\Lambda \Gamma_\Lambda^{(4)} &= \text{Diagram 2} + \text{Diagram 3} \\ \partial_\Lambda \Gamma_\Lambda^{(6)} &= \dots \\ &\vdots\end{aligned}$$

The diagrams represent the flow equations for the two-particle, four-particle, and six-particle vertices. Diagram 1 shows a two-particle vertex $\Gamma_\Lambda^{(2)}$ with a self-energy loop. Diagram 2 shows a four-particle vertex $\Gamma_\Lambda^{(4)}$ with a self-energy loop and a four-particle vertex $\Gamma_\Lambda^{(4)}$ with a self-energy loop. Diagram 3 shows a six-particle vertex $\Gamma_\Lambda^{(6)}$ with a self-energy loop.

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$$\begin{aligned}
 \partial_\Lambda \Gamma_\Lambda^{(2)} &= \text{Diagram: A square loop with a vertical line through the center, representing a self-energy insertion.} \\
 \partial_\Lambda \Gamma_\Lambda^{(4)} &= \text{Diagram: Two square loops connected by a horizontal line, with a vertical line through the center of the horizontal line, representing a four-point vertex correction.} + \text{Diagram: A hexagon loop with a vertical line through the center, representing a six-point vertex correction.} \\
 \partial_\Lambda \Gamma_\Lambda^{(6)} &= \dots \\
 &\vdots
 \end{aligned}$$

- Regulator introduced only in the bare propagator ($G_0 \rightarrow G_{0,\Lambda}$)

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$$\begin{aligned}
 \partial_{\Lambda} \Gamma_{\Lambda}^{(2)} &= \text{Diagram: a square loop with a bubble on top, labeled } \Gamma_{\Lambda}^{(4)} \\
 \partial_{\Lambda} \Gamma_{\Lambda}^{(4)} &= \text{Diagram: two squares connected by a bubble, labeled } \partial_{\Lambda} G_{0,\Lambda}^{-1} \text{ and } \Gamma_{\Lambda}^{(4)} \text{ on each square} + \text{Diagram: a hexagon with a bubble on top, labeled } \Gamma_{\Lambda}^{(6)} \\
 \partial_{\Lambda} \Gamma_{\Lambda}^{(6)} &= \dots \\
 &\vdots
 \end{aligned}$$

- Regulator introduced only in the bare propagator ($G_0 \rightarrow G_{0,\Lambda}$)
- Expand the effective action w.r.t. the field as $\Gamma_{\Lambda}[\bar{\psi}, \psi] = \sum_n \Gamma_{\Lambda}^{(n)}(\bar{\psi}\psi)^n$ and extract flow equations for the derivatives $\partial_{\Lambda} \Gamma_{\Lambda}^{(n)}$ from the Wetterich equation

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$$\begin{aligned}
 \partial_\Lambda \Gamma_\Lambda^{(2)} &= \text{Diagram: a box labeled } \Gamma_\Lambda^{(4)} \text{ with a loop on top} \\
 \partial_\Lambda \Gamma_\Lambda^{(4)} &= \text{Diagram: two boxes labeled } \Gamma_\Lambda^{(4)} \text{ connected by a loop, with } \partial_\Lambda G_{0,\Lambda}^{-1} \text{ on the loop} \\
 &\quad + \text{Diagram: a box labeled } \Gamma_\Lambda^{(6)} \text{ with a loop on top, crossed out with a red X} \\
 \cancel{\partial_\Lambda \Gamma_\Lambda^{(6)}} &= \dots
 \end{aligned}$$

✗

- ▶ Regulator introduced only in the bare propagator ($G_0 \rightarrow G_{0,\Lambda}$)
- ▶ Expand the effective action w.r.t. the field as $\Gamma_\Lambda[\bar{\psi}, \psi] = \sum_n \Gamma_\Lambda^{(n)} (\bar{\psi}\psi)^n$ and extract flow equations for the derivatives $\partial_\Lambda \Gamma_\Lambda^{(n)}$ from the Wetterich equation
- ▶ Approximation set by **1-loop truncation** in practice ($\Gamma_\Lambda^{(n \geq 6)} = 0$)

Conventional fermionic fRG with interaction flow

Insert regulator in the bare propagator ($G_0 \rightarrow G_{0,\Lambda}$) and in the **bare interaction** ($\mathcal{U} \rightarrow \mathcal{U}_\Lambda$)
 for a fermionic theory with $S[c^\dagger, c] = - \sum_{\alpha_1, \alpha_2} c_{\alpha_1}^\dagger G_{0, \alpha_1 \alpha_2}^{-1} c_{\alpha_2} + \sum_{\alpha_1, \alpha_2, \alpha_3, \alpha_4} \mathcal{U}_{\alpha_1 \alpha_2 \alpha_3 \alpha_4} c_{\alpha_1}^\dagger c_{\alpha_2}^\dagger c_{\alpha_3} c_{\alpha_4}$:

$$\begin{aligned} \partial_\Lambda \Gamma_\Lambda^{(2)} &= \text{diagram 1} + \text{diagram 2} + \text{diagram 3} + \text{diagram 4} + \text{diagram 5} + \text{diagram 6} \\ \partial_\Lambda \Gamma_\Lambda^{(4)} &= \text{diagram 7} + \text{diagram 8} + \text{diagram 9} + \text{diagram 10} + \text{diagram 11} \\ &+ \text{diagram 12} + \text{diagram 13} + \text{diagram 14} + \text{diagram 15} \end{aligned}$$

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Insert regulator in the bare propagator ($G_0 \rightarrow G_{0,\Lambda}$)

$$\partial_\Lambda \Gamma_\Lambda^{(2)} = \text{Diagram: A square loop with a blue box labeled } \Gamma_\Lambda^{(4)} \text{ in the center. A diagonal slash is on the top edge.}$$

$$\partial_\Lambda \Gamma_\Lambda^{(4)} = \text{Diagram: Two blue boxes labeled } \Gamma_\Lambda^{(4)} \text{ connected by a horizontal line. A diagonal slash is on the top edge of the right box.}$$

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⚠ Several diagrams with nested loops $\Rightarrow$ Highly **inefficient** to formulate interaction flow in this language!

Multiloop fRG and its bosonization with the single-boson exchange approach

Conventional fRG vs Multiloop fRG

Conventional fRG

Expand Wetterich equation

[Morris, IJMP A09, 2411 (1994)]

$$\Gamma[\psi] = -\ln \left(\int \mathcal{D}c \, e^{-S[c] + Jc} \right) + J\psi$$

$$\Downarrow \quad G_0 \rightarrow G_{0,\Lambda}$$

$$\partial_\Lambda \Gamma_\Lambda[\psi] = \frac{1}{2} \text{Tr} \left[\left(\Gamma_\Lambda^{(2)}[\psi] + R_\Lambda \right)^{-1} \partial_\Lambda R_\Lambda \right]$$

$$\Downarrow$$

$$\partial_\Lambda \Sigma_\Lambda = \text{Diagram: A box labeled } \Gamma_\Lambda^{(4)} \text{ with a self-energy loop on top.}$$

$$\partial_\Lambda \Gamma_\Lambda^{(4)} = \text{Diagram: Two } \Gamma_\Lambda^{(4)} \text{ boxes connected by a loop labeled } \partial_\Lambda G_{0,\Lambda}^{-1} \text{ plus a diagram with a } \Gamma_\Lambda^{(6)} \text{ box and a self-energy loop.}$$

$$\partial_\Lambda \Gamma_\Lambda^{(6)} = \dots$$

$\vdots$

Conventional fRG vs Multiloop fRG

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$$\partial_\Lambda \Gamma_\Lambda^{(6)} = \dots$$

⋮

Multiloop fRG

Differentiate Schwinger-Dyson, Bethe-Salpeter equations

[Blaizot, Pawłowski, Reinos, PLB 696, 523 (2011)]

[Kugler, von Delft, PRL 120, 057403 (2018)]

Conventional fRG vs Multiloop fRG

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$$\partial_\Lambda \Gamma_\Lambda^{(6)} = \dots$$

⋮

Multiloop fRG

Differentiate Schwinger-Dyson, Bethe-Salpeter equations

[Blaizot, Pawłowski, Reinos, PLB 696, 523 (2011)]

[Kugler, von Delft, PRL 120, 057403 (2018)]

$$\Sigma = \text{Diagram: A box labeled } \Gamma^{(4)} \text{ with a loop on top.}, \quad \phi^X = \text{Diagram: Two boxes labeled } \Gamma^{(4)} \text{ connected by a loop, with } \Gamma^{(4)} - \phi^X \text{ above the loop.}$$

Conventional fRG vs Multiloop fRG

Conventional fRG

Expand Wetterich equation

[Morris, IJMP A09, 2411 (1994)]

$$\Gamma[\psi] = -\ln \left(\int \mathcal{D}c \, e^{-S[c] + Jc} \right) + J\psi$$

$$\Downarrow G_0 \rightarrow G_{0,\Lambda}$$

$$\partial_\Lambda \Gamma_\Lambda[\psi] = \frac{1}{2} \text{Tr} \left[\left(\Gamma_\Lambda^{(2)}[\psi] + R_\Lambda \right)^{-1} \partial_\Lambda R_\Lambda \right]$$

$$\Downarrow$$

$$\partial_\Lambda \Sigma_\Lambda = \text{Diagram: A box labeled } \Gamma_\Lambda^{(4)} \text{ with a self-energy loop on top.}$$

$$\partial_\Lambda \Gamma_\Lambda^{(4)} = \text{Diagram: Two boxes labeled } \Gamma_\Lambda^{(4)} \text{ connected by a loop, with a label } \partial_\Lambda G_{0,\Lambda}^{-1} \text{ above the loop.} + \text{Diagram: A box labeled } \Gamma_\Lambda^{(6)} \text{ with a self-energy loop on top.}$$

$$\partial_\Lambda \Gamma_\Lambda^{(6)} = \dots$$

⋮

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$$\Downarrow G_0 \rightarrow G_{0,\Lambda}$$

$$\partial_\Lambda \Sigma_\Lambda = \partial_\Lambda \left(\text{Diagram: A box labeled } \Gamma_\Lambda^{(4)} \text{ with a self-energy loop on top.} \right), \quad \partial_\Lambda \phi_\Lambda^X = \partial_\Lambda \left(\text{Diagram: Two boxes labeled } \Gamma_\Lambda^{(4)} \text{ connected by a loop, with a label } \Gamma_\Lambda^{(4)} - \phi_\Lambda^X \text{ above the loop.} \right)$$

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$$\partial_\Lambda \Gamma_\Lambda^{(4)} = \text{Diagram: two boxes labeled } \Gamma_\Lambda^{(4)} \text{ connected by a loop} \xrightarrow{\partial_\Lambda G_{0,\Lambda}^{-1}} \text{Diagram: a box labeled } \Gamma_\Lambda^{(6)}$$

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$$\Downarrow \partial_\Lambda \phi^X = \sum_{\ell=1}^{\infty} \partial_\Lambda \phi^{X(\ell)}$$

$$\partial_\Lambda \Sigma_\Lambda = \partial_\Lambda \left(\text{Diagram: a box labeled } \Gamma_\Lambda^{(4)} \text{ with a loop on top} \right)$$

$$\partial_\Lambda \phi_\Lambda^{(1)X} = \text{Diagram: two boxes labeled } \Gamma_\Lambda^{(4)} \text{ connected by a loop, with a cross on the loop labeled } \partial_\Lambda G_\Lambda^{-1}$$

$$\partial_\Lambda \phi_\Lambda^{(2)X} = \dots$$

⋮

Conventional fRG vs Multiloop fRG

Conventional fRG

Expand Wetterich equation

[Morris, IJMP A09, 2411 (1994)]

$$\begin{aligned}
 \partial_\Lambda \Sigma_\Lambda &= \text{Diagram 1: A square box labeled } \Gamma_\Lambda^{(4)} \text{ with a loop on top.} \\
 \partial_\Lambda \Gamma_\Lambda^{(4)} &= \text{Diagram 2: Two square boxes labeled } \Gamma_\Lambda^{(4)} \text{ connected by a horizontal line, with a loop on top labeled } \partial_\Lambda G_{0,\Lambda}^{-1}. \quad + \quad \text{Diagram 3: A hexagon labeled } \Gamma_\Lambda^{(6)} \text{ with a loop on top.} \\
 \partial_\Lambda \Gamma_\Lambda^{(6)} &= \dots \\
 &\vdots
 \end{aligned}$$

Multiloop fRG

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 \partial_\Lambda \phi_\Lambda^{(2)X} &= \dots \\
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Conventional fRG vs Multiloop fRG

Conventional fRG

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 \partial_\Lambda \Gamma_\Lambda^{(6)} &= \dots \\
 &\vdots
 \end{aligned}$$

Multiloop fRG

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 \partial_\Lambda \phi_\Lambda^{(1)X} &= \text{Diagram 2: Two square boxes labeled } \Gamma_\Lambda^{(4)} \text{ connected by a horizontal line, with a loop on top labeled } \partial_\Lambda G_\Lambda^{-1}. \\
 \partial_\Lambda \phi_\Lambda^{(2)X} &= \dots \\
 &\vdots
 \end{aligned}$$

Conventional fRG vs Multiloop fRG

Conventional fRG

Expand Wetterich equation

[Morris, IJMP A09, 2411 (1994)]

$$\partial_\Lambda \Sigma_\Lambda = \text{Diagram 1}$$

$$\partial_\Lambda \Gamma_\Lambda^{(4)} = \text{Diagram 2} + \text{Diagram 3}$$

$$\cancel{\partial_\Lambda \Gamma_\Lambda^{(6)} = \dots}$$

×

Multiloop fRG

Differentiate Schwinger-Dyson, Bethe-Salpeter equations

[Blaizot, Pawłowski, Reinosa, PLB 696, 523 (2011)]

[Kugler, von Delft, PRL 120, 057403 (2018)]

$$\partial_\Lambda \Sigma_\Lambda = \partial_\Lambda \left(\text{Diagram 4} \right)$$

$$\partial_\Lambda \phi_\Lambda^{(1)X} = \text{Diagram 5}$$

$$\cancel{\partial_\Lambda \phi_\Lambda^{(2)X} = \dots}$$

$$\times \quad \Gamma_\Lambda^{(4)} = \sum_X \phi_\Lambda^X + \underbrace{\cancel{\gamma_\Lambda^{2PI}}}_{\mathcal{U}}$$

Conventional fRG vs Multiloop fRG

Conventional fRG

Expand Wetterich equation

[Morris, IJMP A09, 2411 (1994)]

$$\begin{aligned}\partial_\Lambda \Sigma_\Lambda &= \text{Diagram: a box labeled } \Gamma_\Lambda^{(4)} \text{ with a self-energy loop on top} \\ \partial_\Lambda \Gamma_\Lambda^{(4)} &= \text{Diagram: two boxes labeled } \Gamma_\Lambda^{(4)} \text{ connected by a line labeled } \partial_\Lambda G_{0,\Lambda}^{-1} \text{ plus a crossed-out diagram} \\ \cancel{\partial_\Lambda \Gamma_\Lambda^{(6)} = \dots} &\quad \times\end{aligned}$$

Multiloop fRG

Differentiate Schwinger-Dyson, Bethe-Salpeter equations

[Blaizot, Pawłowski, Reinos, PLB 696, 523 (2011)]

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► **1-loop truncation** yields same flow equations for the 4-point vertex, with $\partial_\Lambda G_{0,\Lambda}^{-1} \leftrightarrow \partial_\Lambda G_\Lambda^{-1}$ (Katanin substitution)

Conventional fRG vs Multiloop fRG

Conventional fRG

Expand Wetterich equation

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$$\begin{aligned}\partial_\Lambda \Sigma_\Lambda &= \text{Diagram: a box labeled } \Gamma_\Lambda^{(4)} \text{ with a self-energy loop on top} \\ \partial_\Lambda \Gamma_\Lambda^{(4)} &= \text{Diagram: two boxes labeled } \Gamma_\Lambda^{(4)} \text{ connected by a line with } \partial_\Lambda G_{0,\Lambda}^{-1} \text{ above it} + \text{Diagram: a box labeled } \Gamma_\Lambda^{(4)} \text{ with a self-energy loop on top, crossed out with a red X} \\ \cancel{\partial_\Lambda \Gamma_\Lambda^{(6)}} &= \dots \\ &\quad \times\end{aligned}$$

Multiloop fRG

Differentiate Schwinger-Dyson, Bethe-Salpeter equations

[Blaizot, Pawłowski, Reinosa, PLB 696, 523 (2011)]

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$$\begin{aligned}\partial_\Lambda \Sigma_\Lambda &= \partial_\Lambda \left(\text{Diagram: a box labeled } \Gamma_\Lambda^{(4)} \text{ with a self-energy loop on top} \right) \\ \partial_\Lambda \phi_\Lambda^{(1)X} &= \text{Diagram: two boxes labeled } \Gamma_\Lambda^{(4)} \text{ connected by a line with } \partial_\Lambda G_\Lambda^{-1} \text{ above it} \\ \cancel{\partial_\Lambda \phi_\Lambda^{(2)X}} &= \dots \\ &\quad \times \quad \Gamma_\Lambda^{(4)} = \sum_X \phi_\Lambda^X + \underbrace{\cancel{\tau_\Lambda^{2PI}}}_{\mathcal{U}}\end{aligned}$$

- **1-loop truncation** yields same flow equations for the 4-point vertex, with $\partial_\Lambda G_{0,\Lambda}^{-1} \leftrightarrow \partial_\Lambda G_\Lambda^{-1}$ (Katanin substitution)
- Much more affordable to go **beyond 1-loop truncation** with multiloop fRG than with conventional fRG

$$\partial_\Lambda \phi_\Lambda^{(2)X} = \sum_{X' \neq X} \left(\text{Diagram: a box labeled } \partial_\Lambda \phi_\Lambda^{X'(1)} \text{ connected to a box labeled } \Gamma_\Lambda^{(4)} \text{ with a self-energy loop on top} + \text{Diagram: a box labeled } \Gamma_\Lambda^{(4)} \text{ connected to a box labeled } \partial_\Lambda \phi_\Lambda^{X'(1)} \text{ with a self-energy loop on top} \right)$$

Bosonization with the single-boson exchange approach

Single-boson exchange (SBE) decomposition of the 4-point vertex:

[Krien, Valli, Capone, PRB 100, 155149 (2019)]

$$\Gamma^{(4)} = \sum_X \phi^X + \mathcal{I}^{2\text{PI}}, \quad \text{with} \quad \mathcal{I}^{2\text{PI}} = \underbrace{\text{X} + \text{[Diagram]} + \dots}_{\text{only 2PI diagrams}}$$


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► Heart of the SBE decomposition:

$$\phi^X(Q, k, k') = \underbrace{\text{[Diagram: triangle with vertices and internal lines]}}_{\bar{\lambda}^X(Q, k) w^X(Q) \lambda^X(Q, k')} + M(Q, k, k') - \mathcal{U}$$

w^X : bosonic propagator

λ^X : Yukawa coupling

M^X : rest function

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► SBE part of ϕ^X only contains **$\mathcal{U}$ -reducible diagrams**:

$$\text{Diagram} = \text{Diagram 1} + \text{Diagram 2} + \dots$$

Bosonization with the single-boson exchange approach

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► Heart of the SBE decomposition:

$$\phi^X(Q, k, k') = \underbrace{\text{[Diagram: triangle with w^X and \lambda^X]}}_{\bar{\lambda}^X(Q, k) w^X(Q) \lambda^X(Q, k')} + M(Q, k, k') - \mathcal{U}$$

w^X : bosonic propagator
 λ^X : Yukawa coupling
 M^X : rest function

► SBE part of ϕ^X only contains **$\mathcal{U}$ -reducible diagrams**:

$$\text{[Diagram: triangle with w^X and \lambda^X]} = \text{[Diagram: bubble with w^X]} + \text{[Diagram: bubble with w^X]} + \dots$$

Bosonization with the single-boson exchange approach

Single-boson exchange (SBE) decomposition of the 4-point vertex:

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$$\Gamma^{(4)} = \sum_X \phi^X + \mathcal{I}^{2\text{PI}}, \quad \text{with} \quad \mathcal{I}^{2\text{PI}} = \underbrace{\text{X} + \text{[Diagram: Box with four external lines and two internal lines forming a square]} + \dots}_{\text{only 2PI diagrams}}$$

► Heart of the SBE decomposition:

$$\phi^X(Q, k, k') = \underbrace{\text{[Diagram: Triangle with external lines and internal lines]}_{\bar{\lambda}^X(Q, k) w^X(Q) \lambda^X(Q, k')} + M(Q, k, k') - \mathcal{U}$$

w^X : bosonic propagator
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► SBE part of ϕ^X only contains **$\mathcal{U}$ -reducible diagrams**:

$$\text{[Diagram: Triangle with external lines and internal lines]} = \text{[Diagram: Triangle with external lines and internal lines]} + \text{[Diagram: Triangle with external lines and internal lines]} + \dots$$

► (Multiloop) fRG flow equations can be formulated for w^X , λ^X and M^X :

[Gievers, Walter, Ge, von Delft, Kugler, EPJB 95, 108 (2022)]

$$\partial_\Lambda w_\Lambda^{X(1)} = \text{[Diagram: Triangle with external lines and internal lines]}, \quad \partial_\Lambda \lambda_\Lambda^{X(1)} = \text{[Diagram: Triangle with external lines and internal lines]}, \quad \partial_\Lambda M_\Lambda^{X(1)} = \text{[Diagram: Triangle with external lines and internal lines]}$$

Bosonization with the single-boson exchange approach

Single-boson exchange (SBE) decomposition of the 4-point vertex:

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► Heart of the SBE decomposition:

$$\phi^X(Q, k, k') = \underbrace{\text{[Diagram: Triangle with w^X and \lambda^X]}}_{\bar{\lambda}^X(Q, k) w^X(Q) \lambda^X(Q, k')} + M(Q, k, k') - \mathcal{U}$$

w^X : bosonic propagator
 λ^X : Yukawa coupling
 M^X : rest function

► SBE part of ϕ^X only contains **$\mathcal{U}$ -reducible diagrams**:

$$\text{[Diagram: Triangle with w^X and \lambda^X]} = \text{[Diagram: Triangle with w^X and \lambda^X, U-reducible]} + \text{[Diagram: Triangle with w^X and \lambda^X, U-reducible]} + \dots$$

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⚠ $w^X(Q)$ and $\lambda^X(Q, k)$ much simpler objects than $\phi^X(Q, k, k')$ $\Rightarrow$ Room for **efficient approximations**

Multiloop SBE fRG results for the 2D Hubbard model

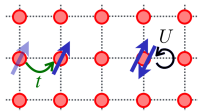
2D Hubbard model Hamiltonian:

[Hubbard, PRSL A 276, 238 (1963)]

[Gutzwiller, PRL 10, 159 (1963)]

[Kanamori, PTP 30, 275 (1963)]

$$\mathcal{H} = -t \sum_{\langle ij \rangle \sigma} c_{i\sigma}^\dagger c_{j\sigma} + U \sum_i c_{i\uparrow}^\dagger c_{i\uparrow} c_{i\downarrow}^\dagger c_{i\downarrow}$$



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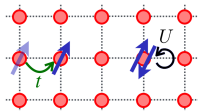
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Multiloop SBE fRG results without rest functions ($M^X(Q, k, k') = 0$):

[Fraboulet, Al-Eryani, Heinzelmann, Kauch, Andergassen, arXiv:2512.11190]

Multiloop SBE fRG results for the 2D Hubbard model

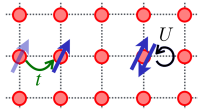
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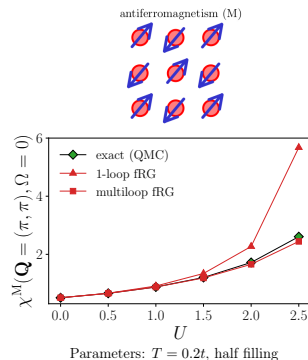
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- Possible to reach **quantitative accuracy** at loop convergence



Multiloop SBE fRG results for the 2D Hubbard model

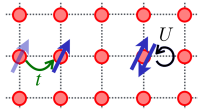
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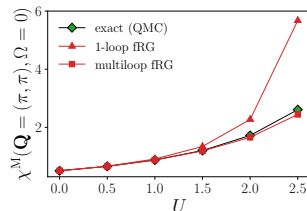
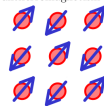


Multiloop SBE fRG results without rest functions ($M^X(Q, k, k') = 0$):

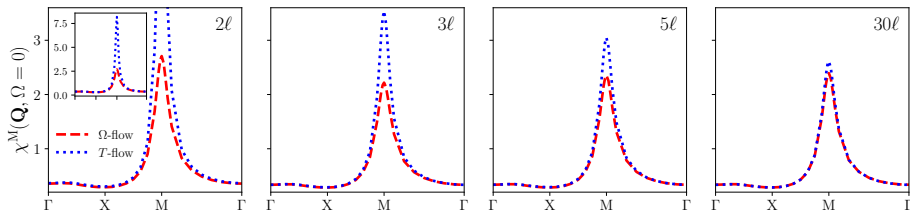
[Fraboulet, Al-Eryani, Heinzlmann, Kauch, Andergassen, arXiv:2512.11190]

- Possible to reach **quantitative accuracy** at loop convergence
- **Weak regulator dependence** observed at loop convergence

antiferromagnetism (M)



Parameters: $T = 0.2t$, half filling



Parameters: $U = 2.5t$, $T = 0.2t$, half filling

Interaction flow with multiloop SBE fRG flow equations

Multiloop SBE fRG equations **scarcely** modified when adding a regulator also in the bare interaction ($\mathcal{U} \rightarrow \mathcal{U}_\Lambda$):

[Al-Eryani, Gievers, Fraboulet, PRR 8, 023029 (2026)]

$$\partial_\Lambda w_\Lambda^{X(1)} = \underbrace{\text{Diagram}}_{w_\Lambda^X \lambda_\Lambda^X (\partial_\Lambda \Pi_\Lambda^X) \bar{\lambda}_\Lambda^X w_\Lambda^X} - w_\Lambda^X (\partial_\Lambda \mathcal{U}_\Lambda^{-1}) w_\Lambda^X$$

- New terms involving $\partial_\Lambda \mathcal{U}_\Lambda$ essentially in $\partial_\Lambda w_\Lambda^{X(1)}$
 $\triangle w_\Lambda^X(Q)$ much simpler than $\lambda_\Lambda^X(Q, k)$ or $M_\Lambda^X(Q, k, k')$

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- New terms involving $\partial_\Lambda \mathcal{U}_\Lambda$ essentially in $\partial_\Lambda w_\Lambda^{X(1)}$
 - ⚠ $w_\Lambda^X(Q)$ much simpler than $\lambda_\Lambda^X(Q, k)$ or $M_\Lambda^X(Q, k, k')$
- New flow equations for w_Λ^X directly follows from Dyson equation:

$$w^X = \mathcal{U}^{-1} - P^X \Rightarrow \partial_\Lambda w_\Lambda^X = \partial_\Lambda \mathcal{U}_\Lambda^{-1} - P_\Lambda^X$$

- ⚠ Polarization: $P^X = \lambda^X \Pi^X$

Interaction flow with multiloop SBE fRG flow equations

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- New terms involving $\partial_\Lambda \mathcal{U}_\Lambda$ essentially in $\partial_\Lambda w_\Lambda^{X(1)}$
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- New flow equations for w_Λ^X directly follows from Dyson equation:

$$w^X = \mathcal{U}^{-1} - P^X \Rightarrow \partial_\Lambda w_\Lambda^X = \partial_\Lambda \mathcal{U}_\Lambda^{-1} - P_\Lambda^X$$

$$\triangle \text{Polarization: } P^X = \lambda^X \Pi^X$$

- Possible to avoid dealing with the inverse $\mathcal{U}_\Lambda^{-1}$ using

$$w_\Lambda^X (\partial_\Lambda \mathcal{U}_\Lambda^{-1}) w_\Lambda^X = \partial_\Lambda \mathcal{U}_\Lambda + w_\Lambda^X \lambda_\Lambda^X \Pi_\Lambda^X (\partial_\Lambda \mathcal{U}_\Lambda) + (\partial_\Lambda \mathcal{U}_\Lambda) \Pi_\Lambda^X \lambda_\Lambda^X w_\Lambda^X + w_\Lambda^X \lambda_\Lambda^X \Pi_\Lambda^X (\partial_\Lambda \mathcal{U}_\Lambda) \Pi_\Lambda^X \lambda_\Lambda^X w_\Lambda^X$$

Interaction flow with multiloop SBE fRG flow equations

Multiloop SBE fRG equations **scarcely** modified when adding a regulator also in the bare interaction ($\mathcal{U} \rightarrow \mathcal{U}_\Lambda$):

[Al-Eryani, Gievers, Fraboulet, PRR 8, 023029 (2026)]

$$\partial_\Lambda w_\Lambda^{X(1)} = \underbrace{\text{Diagram}}_{w_\Lambda^X \lambda_\Lambda^X (\partial_\Lambda \Pi_\Lambda^X) \bar{\lambda}_\Lambda^X w_\Lambda^X} - w_\Lambda^X (\partial_\Lambda \mathcal{U}_\Lambda^{-1}) w_\Lambda^X$$

- New terms involving $\partial_\Lambda \mathcal{U}_\Lambda$ essentially in $\partial_\Lambda w_\Lambda^{X(1)}$
 $\triangle w_\Lambda^X(Q)$ much simpler than $\lambda_\Lambda^X(Q, k)$ or $M_\Lambda^X(Q, k, k')$
- New flow equations for w_Λ^X directly follows from Dyson equation:

$$w^X = \mathcal{U}^{-1} - P^X \Rightarrow \partial_\Lambda w_\Lambda^X = \partial_\Lambda \mathcal{U}_\Lambda^{-1} - P_\Lambda^X$$

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Use this generalized fRG approach to formulate a **T-flow** for a model with **retarded interaction** (Anderson-Holstein impurity)

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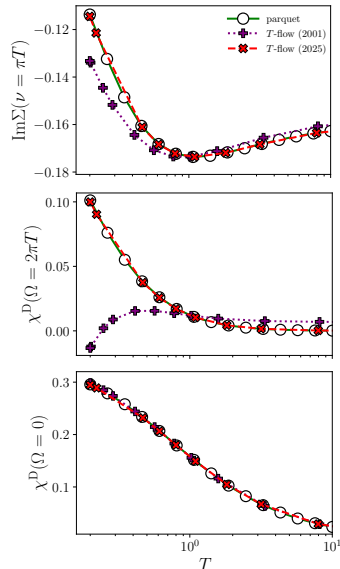
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Use this generalized fRG approach to formulate a **T-flow** for a model with **retarded interaction** (Anderson-Holstein impurity)

$\Rightarrow$ Get back the parquet approximation ($\mathcal{I}^{2PI} = \mathcal{U}$) at each step of the flow when loop convergence is reached



Parameters: $U = 1.4$, $\omega_0 = 1$, $V_H = 0.5$, half filling

Conclusion

- Interaction flows with both $G_0 \rightarrow G_{0,\Lambda}$ and $\mathcal{U} \rightarrow \mathcal{U}_\Lambda$ very efficiently formulated within multiloop fRG combined with SBE approach

Conclusion

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- ▶ Introducing regulators via both $G_0 \rightarrow G_{0,\Lambda}$ and $\mathcal{U} \rightarrow \mathcal{U}_\Lambda$ not only useful to extend the T -flow to electron-phonon systems:

Conclusion

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- Introducing regulators via both $G_0 \rightarrow G_{0,\Lambda}$ and $\mathcal{U} \rightarrow \mathcal{U}_\Lambda$ not only useful to extend the T -flow to electron-phonon systems:
 - Use $\mathcal{U}_\Lambda$ to fix useful constraints during the flow (Ward identities, ...)
 - Extended DMFT + fRG for non-local interactions

[Taranto, Andergassen, Bauer, Held, Katanin, Metzner, Rohringer, Toschi, PRL 112, 196402 (2014)]

[Vilardi, Taranto, Metzner, PRB 99, 104501 (2019)]

[Katanin, PRB 99, 115112 (2019)]

[Bonetti, Toschi, Hille, Andergassen, Vilardi, PRR 4, 013034 (2022)]

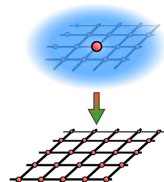


Fig. from Bonetti *et al.*, PRR 4, 013034 (2022)

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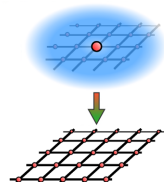


Fig. from Bonetti *et al.*, PRR 4, 013034 (2022)

- fRG solver publicly available at [Al-Eryani, Patricolo, Fraboulet, arXiv:2604.04232](#)

Backup slides

Multiloop SBE fRG equations:

[Gievers, Walter, Ge, von Delft, Kugler, EPJB 95, 108 (2022)]

$$\partial_\Lambda w_\Lambda^{X(1)} = \text{Diagram 1}, \quad \partial_\Lambda \lambda_\Lambda^{X(1)} = \text{Diagram 2}, \quad \partial_\Lambda M_\Lambda^{X(1)} = \text{Diagram 3}$$

Diagram 1: A loop with two external wavy lines. The top vertex is labeled $\partial_\Lambda G_\Lambda^{-1}$ with a cross. The left and right internal vertices are labeled λ_Λ^X .
 Diagram 2: A loop with two external lines. The left internal vertex is labeled $T_{U\text{irr},\Lambda}^X$ and the right internal vertex is labeled λ_Λ^X .
 Diagram 3: A loop with two external lines. Both the left and right internal vertices are labeled $T_{U\text{irr},\Lambda}^X$.

$$\partial_\Lambda w_\Lambda^{X(n)} = \sum_{X' \neq X} \left(\text{Diagram 4} \right), \quad \partial_\Lambda \lambda_\Lambda^{X(n)} = \sum_{X' \neq X} \left(\text{Diagram 5} + \text{Diagram 6} \right)$$

Diagram 4: A loop with two external wavy lines. The left internal vertex is labeled λ_Λ^X , the top internal vertex is labeled $\partial_\Lambda \phi_\Lambda^{X'(n-1)}$, and the right internal vertex is labeled λ_Λ^X .
 Diagram 5: A loop with two external lines. The left internal vertex is labeled $\partial_\Lambda \phi_\Lambda^{X'(n-1)}$ and the right internal vertex is labeled λ_Λ^X .
 Diagram 6: A loop with two external lines. Both the left and right internal vertices are labeled $T_{U\text{irr},\Lambda}^X$.

$$\partial_\Lambda M_\Lambda^{X(n)} = \sum_{X' \neq X} \left(\text{Diagram 7} + \text{Diagram 8} + \text{Diagram 9} \right)$$

Diagram 7: A loop with two external lines. The left internal vertex is labeled $\partial_\Lambda \phi_\Lambda^{X'(n-1)}$ and the right internal vertex is labeled $T_{U\text{irr},\Lambda}^X$.
 Diagram 8: A loop with two external lines. The left internal vertex is labeled $T_{U\text{irr},\Lambda}^X$ and the right internal vertex is labeled $\partial_\Lambda \phi_\Lambda^{X'(n-1)}$.
 Diagram 9: A loop with two external lines. Both the left and right internal vertices are labeled $T_{U\text{irr},\Lambda}^X$.

$$\partial_\Lambda w_\Lambda^X = \sum_{\ell=1}^{\infty} \partial_\Lambda w_\Lambda^{X(\ell)}, \quad \partial_\Lambda \lambda_\Lambda^X = \sum_{\ell=1}^{\infty} \partial_\Lambda \lambda_\Lambda^{X(\ell)}, \quad \partial_\Lambda M_\Lambda^X = \sum_{\ell=1}^{\infty} \partial_\Lambda M_\Lambda^{X(\ell)}$$

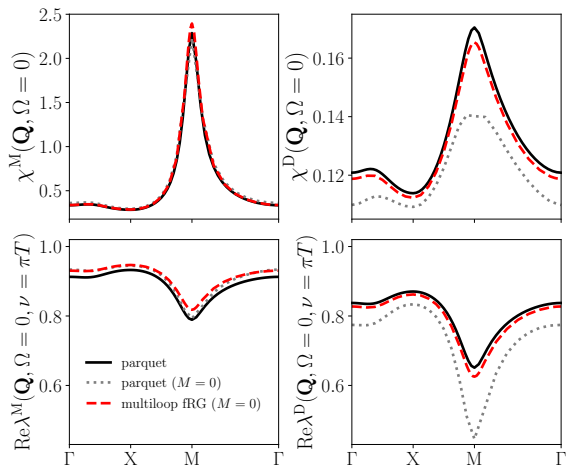


Fig. from [Fraboulet, Al-Eryani, Heinzlmann, Kauch, Andergassen, arXiv:2512.11190](#)

2D Hubbard model, $U = 2.5t$, $T = 0.2t$, half filling