Existence theorem on the UV limit of Wilsonian RG flows

*Class.Quant.Grav.***41**(2024)125009 and *J.Phys.***A59**(2026)035401

+ *new paper draft*

András LÁSZLÓ

laszlo.andras@wigner.hu

Wigner Research Centre for Physics, Budapest

joint work with Zsigmond Tarcsay and Jobst Ziebell

ERG2026, Brighton, 1-5 September 2026

Outline

I. Notations, recap on Feynman functional integration

II. Results on UV behavior of Wilsonian RG flows

Part I:

Notations, recap on Feynman functional integration

Recap on distribution theory

Will consider only scalar and bosonic fields for simplicity.

Will consider only flat (affine) spacetime manifold for simplicity.

- $\mathcal{E}$: space of all smooth fields over spacetime.
  They form a vector space with a topology: collection of “open” sets
  $\varphi_i \in \mathcal{E}$ $(i \in \mathbb{N}) \to 0$ iff all derivatives locally uniformly converge to zero.

- $\mathcal{S}$: space of rapidly decreasing smooth fields (Schwartz fields) over spacetime.
  They form a vector space with a topology:
  $\varphi_i \in \mathcal{S}$ $(i \in \mathbb{N}) \to 0$ iff all derivatives uniformly converge to zero faster than polynomial.

- $\mathcal{D}$: space of compactly supported smooth fields (test fields) over spacetime.
  They form a vector space with a topology:
  $\varphi_i \in \mathcal{D}$ $(i \in \mathbb{N}) \to 0$ iff they stay within a compact set and $\to 0$ in $\mathcal{E}$ sense.

only meaningful on flat spacetime

$$\underbrace{\mathcal{E}}_{\text{smooth fields}} \supset \underbrace{\mathcal{S}}_{\text{Schwartz fields}} \supset \underbrace{\mathcal{D}}_{\text{test fields}}$$

Distributions are continuous duals of $\mathcal{E}$, $\mathcal{S}$, $\mathcal{D}$.

- $\mathcal{E}'$: continuous $\mathcal{E} \to \mathbb{R}$ linear functionals.
  They are the compactly supported distributions.

- $\mathcal{S}'$: continuous $\mathcal{S} \to \mathbb{R}$ linear functionals.
  They are the tempered or Schwartz distributions.

- $\mathcal{D}'$: continuous $\mathcal{D} \to \mathbb{R}$ linear functionals.
  They are the space of all distributions.

They carry a corresponding natural topology (notion of "open" sets).

only meaningful on flat spacetime

$$\underbrace{\mathcal{E}'}_{\text{compactly supp distributions}} \subset \underbrace{\mathcal{S}'}_{\text{Schwartz distributions}} \subset \underbrace{\mathcal{D}'}_{\text{all distributions}}$$

[Of course, functions are also distributions, e.g. $\mathcal{D} \subset \mathcal{E}'$ and $\mathcal{E} \subset \mathcal{D}'$ etc.]

Recap on measure / integration / probability theory

- Let X be a set.
- Let Σ be a collection of subsets of X such that:
  - X is in Σ,
  - for all A in Σ, their complement is in Σ.
  - for all countably infinite system $A_i \in \Sigma$ $(i \in \mathbb{N})$, their union $\underset{i\in\mathbb{N}}{\cup} A_i$ is in Σ.

  Then, Σ is called a sigma-algebra (their elements called measurable sets of X).
  Typically, if X carries open sets (topology), the sigma-alg generated by them is used.
- Let $\mu : \Sigma \to \mathbb{R}_0^+$ be a weight assigning function to sets, such that:
  - $\mu(\emptyset) = 0$,
  - for all countably infinite disjoint system $A_i \in \Sigma$ $(i \in \mathbb{N})$: $\mu\big(\underset{i\in\mathbb{N}}{\cup} A_i\big) = \sum_{i\in\mathbb{N}} \mu(A_i)$.

  Then, μ is called a probability measure. [One may always choose $\mu(X) = 1$.]

(X, Σ, μ) is called a probability measure space.
Will study probability measures on $X = \mathcal{E}, \mathcal{S}, \mathcal{D}, \mathcal{E}', \mathcal{S}', \mathcal{D}'$.

- Function $f: X \to \mathbb{C}$ is called measurable iff preimage of measurable set is measurable.
  Theorem: f is measurable iff approximable pointwise by “histograms” with bins from Σ.

- The integral $\int\limits_{\phi\in X} f(\phi)\,\mathrm{d}\mu(\phi)$ is defined via the histogram “area” approximations.
  Theorem: this is well-defined.

- Let (X,Σ,μ) be a measure space and (Y,Δ) an other one, with unspecified measure.
  Let $C: X \to Y$ be a measurable mapping.
  Then, one can define the pushforward (or marginal) measure $C_*\mu$ on Y.
  [For all $B\in\Delta$ one defines $(C_*\mu)(B)$ as μ measure of the preimage of B by C.]

- Pushforward (marginal) measure means simply transformation of integration variable.
  If forgetful transformation, “forgotten” d.o.f. are “integrated out”.

- If μ is a probability measure, $Z(j) := \int\limits_{\phi\in X} \mathrm{e}^{\mathrm{i}(j|\phi)}\,\mathrm{d}\mu(\phi)$ is its Fourier transform.

Recap on Euclidean Feynman functional integral

- Take an Euclidean action $S = T + V$, with kinetic + potential term splitting. Say, $T(\varphi) = \int_{\mathcal{M}} \varphi\,(-\Delta + m^2)\varphi$, and $V(\varphi) = g \int_{\Omega} \varphi^4$.
- Then T, i.e. $(-\Delta + m^2)$ has a propagator $K(\cdot,\cdot)$ which is positive definite:
  - $(-\Delta + m^2)_x\, K(x, y) = \delta_y(x)$,
  - for all $j \in \mathcal{D}$ sources: $(K|j \otimes j) \geq 0$.
- Due to above, the function $Z_T(j) := \mathrm{e}^{-(K|j\otimes j)}$ $(j \in \mathcal{D})$ has "quite nice" properties.
- Bochner-Minlos theorem: because of
  - "quite nice" properties of Z_T,
  - "quite nice" properties of the space $\mathcal{D}$,

  $\exists|$ probability measure γ on $\mathcal{D}'$, whose Fourier transform is Z_T.
  It is the Feynman measure for free theory: $\int\limits_{\phi\in\mathcal{D}'} (\ldots)\,\mathrm{d}\gamma(\phi) = \int\limits_{\phi\in\mathcal{D}'} (\ldots)\,\mathrm{e}^{-T(\phi)}$ "$\mathrm{d}\phi$".
- Tempting definition for Feynman measure of interacting theory:

$$\int\limits_{\phi\in\mathcal{D}'} (\ldots)\ \mathrm{e}^{-V(\phi)}\ \mathrm{d}\gamma(\phi) \qquad \left[= \int\limits_{\phi\in\mathcal{D}'} (\ldots)\ \underbrace{\mathrm{e}^{-(T(\phi)+V(\phi))}}_{=\mathrm{e}^{-S(\phi)}}\ \text{“}\mathrm{d}\phi\text{”} \right]$$

Recap on Wilsonian regularization

- Problem, the interacting Feynman measure $\mu := \mathrm{e}^{-V} \cdot \gamma$ is undefined:

$$\int\limits_{\phi\in\mathcal{D}'} (\ldots) \underbrace{\mathrm{d}\mu(\phi)}_{\text{wannabe Feynman measure}} := \int\limits_{\phi\in\mathcal{D}'} (\ldots) \underbrace{\mathrm{e}^{-V(\phi)}}_{\text{lives on function sense fields}} \underbrace{\mathrm{d}\gamma(\phi)}_{\text{lives on distribution sense fields}}$$

  Because V is spacetime integral of pointwise product of fields, e.g. $V(\varphi) = g\int_\Omega \varphi^4$.
  How to bring e^{-V} and γ to common grounds?

- Physicist workaround: brute force, Wilsonian regularization.
  Take a continuous linear mapping C: (distributional fields) $\to$ (function sense fields).
  Take the pushforward Gaussian measure $\gamma_C := C_*\gamma$ lives on $\mathrm{Ran}(C)$
  Those are functions, so safe to integrate e^{-V} there:

$$\int\limits_{\varphi\in\mathrm{Ran}(C)} (\ldots)\ \mathrm{e}^{-V(\varphi)}\ \mathrm{d}\gamma_C(\varphi) \qquad \left[= \int\limits_{\varphi\in\mathrm{Ran}(C)} (\ldots)\ \mathrm{e}^{-(T(\varphi)+V(\varphi))}\ \text{“}\mathrm{d}\varphi\text{”}\right]$$

  a space of UV regularized fields

  [Schwartz kernel theorem: C is convolution by a test function, if translationally invariant.
  I.e., it is a momentum space damping, or coarse-graining of fields.]

Recap on Wilsonian RG flows

- Issue, the so defined $\mathrm{e}^{-V} \cdot \gamma_C$ is C-dependent. How to get rid of C-dependence?

- Take a family V_C $(C \in \{$coarse-grainings$\})$ of interaction terms. $\leftrightarrow \mu_C := \mathrm{e}^{-V_C} \cdot \gamma_C$
We say that it is a nonterminating Wilsonian renormalization group flow iff:
$\exists$ some continuous functional z : $\{$coarse-grainings$\} \to \mathbb{R}$, such that
$\forall$ coarse-grainings C, C', C'' with $C'' = C'\,C$:
$$z(C'')_*\, \mu_{C''} \quad = \quad z(C)_*\, C'_*\, \mu_C\,.$$
[z is called the running field renormalization factor.]

- Physics idea behind:
expectation value of a smoothed observable can be calculated at least C-consistently.

- Expressed on moments: let $\mathcal{G}_C = (\mathcal{G}_C^{(0)}, \mathcal{G}_C^{(1)}, \mathcal{G}_C^{(2)}, \dots)$ be the moments of μ_C, then
$\exists$ some continuous functional z : $\{$coarse-grainings$\} \to \mathbb{R}$, such that
$\forall$ coarse-grainings C, C', C'' with $C'' = C'\,C$:
$$z(C'')^n\, \mathcal{G}_{C''}^{(n)} \quad = \quad z(C)^n \otimes^n C'\, \mathcal{G}_C^{(n)} \qquad \text{for all } n = 0, 1, 2, \dots$$
[Valid also in Lorentz signature and on manifolds, for formal moments (correlators).]

[We can always set $z(C) = 1$, by rescaling fields: $\tilde{\mu}_C := z(C)_*\, \mu_C$ or $\tilde{\mathcal{G}}_C^{(n)} := z(C)^n\, \mathcal{G}_C^{(n)}$.]

Part II:

Results on UV behavior of Wilsonian RG flows

UV theorems on nonterminating Wilsonian RG flows

[A.László, Z.Tarcsay, J.Ziebell *J.Phys.***A59**(2026)035401 + *Class.Quant.Grav.***41**(2024)125009 + *new*]:

- Flow of correlators form "nice" gen.funct.space (manifolds, any signature, any fields): similar to ordinary distributions (loc.conv, complete, nuclear, semi-Montel, Schwartz).
- Existence of UV limit of correlators (flat spacetime, any signature, bosonic fields): $\exists|$ C-independent distributional correlator $G = (G^{(0)}, G^{(1)}, G^{(2)}, \dots)$, such that $\mathcal{G}_C^{(n)} = \otimes^n C\, G^{(n)}$ holds.
- Existence of UV limit of Feynman measure (flat, Euclidean signature): $\exists|$ C-independent probability measure μ on the distribution sense fields, such that $\mu_C = C_* \mu$ holds.
- Existence of UV limit of interaction potential (flat, Euclidean): Parameters of the reference γ_C cannot be C-dependent. ($\leftarrow$ not even for eff.theories) $\exists|$ C-independent interaction potential V on the distribution sense fields ϕ, such that $V_C(C\,\phi) = V(\phi)$ holds γ-a.e., for positive Fourier spectrum coarse-grainings C.
- Preservation of lower bound (flat, Euclidean): if V_C were bounded from below for a positive Fourier spectrum C, then V and all V_C are bounded from below, and with the same bound. $\rightarrow$ This leads to a new kind of renormalizability test.

Summary

- Wilsonian RG flows canonically defined on manifolds.
- Nonterminating WRG flows have UV limit in terms
  - of correlators (any signature, bosonic, flat spctm)
  - of Feynman measures (Euclidean, bosonic, flat spctm)
  - of interaction potentials
    (Euclidean, if we allow Gaussian tailed regulators)
- Kinetic Gaussian params cannot run.
  (Euclidean, also for effective theories!)
- Flow preserves lower bound of interaction potentials.
  (Euclidean, if we allow Gaussian tailed regulators)
  $\rightarrow$ Can be used for a renormalizability test.

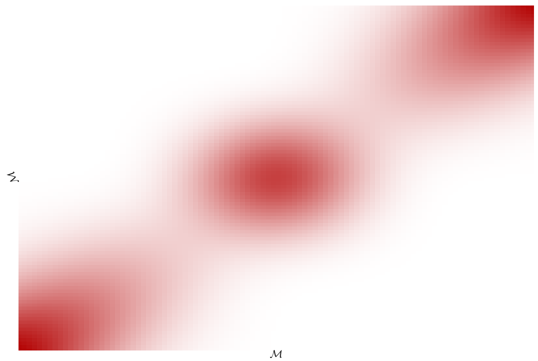
Backup slides

On coarse-grainings

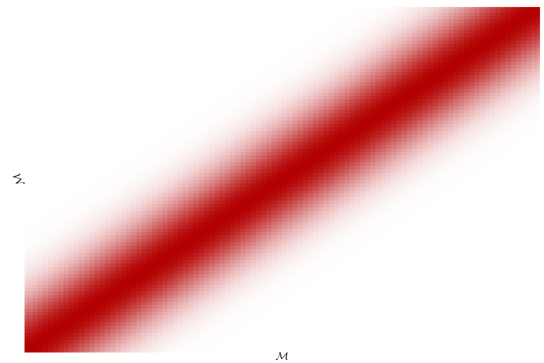
Original problematics: need to make functions of distributions, so that $\int_\Omega \varphi^4$ is meaningful.

Let $C:\ \mathcal{D}' \to \mathcal{E}$ continuous linear mapping.

Schwartz kernel theorem:
$\exists|\ \kappa(x,y)$ smooth section on $\mathcal{M}\times\mathcal{M}$:
$\quad\forall\, T\in\mathcal{D}'$: $\ (C\,T)\big|_y = (T\,|\,\kappa(\cdot,y))$ holds.



If $\mathcal{M}\equiv\mathbb{R}^N$, let C_κ be translation invariant.
Then, $\kappa(x,y)=\eta(x-y)$ for some $\eta\in\mathcal{D}\setminus\{0\}$.



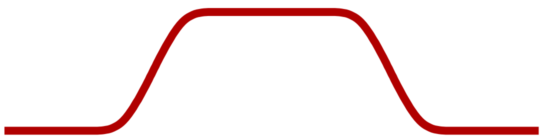
That is, $C_\kappa = C_\eta = \eta\star(\cdot)$ for some $\eta\in\mathcal{D}\setminus\{0\}$.

On flat spacetime we can also play this on Schwartz distributions:
let $C: \ \mathcal{S}' \to \mathcal{E}$ continuous linear mapping, which is translationally invariant.
Then: $C_\kappa = C_\eta = \eta \star (\cdot)$ for some $\eta \in \mathcal{S} \setminus \{0\}$.

Customary to make further restrictions on $F(\eta)$.
E.g. flat unity top + Schwartz tail.

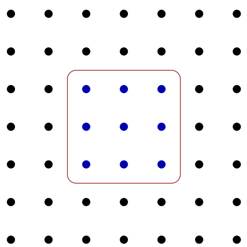
These admit ones e.g. with Gaussian tail.

Sometimes, smooth bandlimited tail is required.



Extreme case: not trivial, but sharp cutoff is γ-a.e. meaningful.

On manifolds, bandlimiting is not meaningful.
Not natural even on flat spacetime:



Lattice averaging corresponds to C_η with $\eta \in \mathcal{D}\setminus\{0\}$.
But Paley-Wiener-Schwartz theorem:
then $F(\eta)$ is not bandlimited.

What stays true with strict bandlimited momentum cutoff

Some QFT literatures postulate that Fourier profile of regulators are strictly bandlimited:

Sharp bandlimited momentum cutoff (in a tricky way) can also be defined, γ -a.e.:

What stays true from our theorems?

1. On flat spacetime for bosonic fields, if μ_C -s have second moment, $\exists$ UV measure μ.
2. Parameters of kinetic $(-d^2\Delta + m^2)$, to which γ is associated, cannot run.
   (Not even for terminating flows! That is, also for effective field theories.)

   Don't yet know: if existence of UV limit potential stays true.

Info: bandlimiting not meaningful on manifolds, "not natural".

Higher derivative theories?

Take an Euclidean spacetime $\sim \mathbb{R}^N$.

Kinetic term may include e.g. $\int_{\mathcal{M}} \varphi(-\Delta)^M \varphi$ type higher derivative terms $(M \geq 1)$.

Then, in momentum space the free propagator decays like $\sim 1/p^{2M}$.

This is equivalent to an unremoved Wilsonian regularization with $\sim 1/p^{M-1}$ momentum tail!

For $M \geq N/2$, corresponding kinetic Gaussian γ is supported on better distributions.

In that case, the theory becomes renormalizable *a la* φ^4 on $\mathbb{R}^2$ (Wick ordering).

For $M > N/2$, corresponding kinetic Gaussian γ is supported on functions.

In that case, the theory becomes finite.

Relation to usual RG theory

Fix some $\eta \in \mathcal{S}$ such that $\int \eta = 1$ and $F(\eta) > 0$.
Introduce scaled η, that is $\eta_\Lambda(x) := \Lambda^N \eta(\Lambda\, x)$ (for all $x \in \mathbb{R}^N$ and scaling $1 \leq \Lambda < \infty$).
One has $\eta_\Lambda \xrightarrow{\mathcal{S}'} \delta$ as $\Lambda \longrightarrow \infty$.

By our theorem, for all Λ, one has $V_{C_{\eta_\Lambda}}(C_{\eta_\Lambda}\,\phi) = V(\phi)$ for γ-a.e. $\phi \in \mathcal{S}'$.

$\Downarrow$

ODE for $V_{C_{\eta_\Lambda}}$, namely $\frac{\mathrm{d}}{\mathrm{d}\Lambda}\left(\mathrm{e}^{-V_{C_{\eta_\Lambda}} \circ C_{\eta_\Lambda}} \cdot \gamma\right) = 0$ for $1 \leq \Lambda < \infty$.

In QFT, trying to solve such flow equation, given initial data $V_{C_\Lambda}\big|_{\Lambda=1}$.

But why bother? By our theorem, all RG flows of such kind has some V at the UV end.
Look directly for V?

What really the game is about?

Original problem:

- We had $\mathcal{V}:\ \{\text{function sense fields}\} \to \mathbb{R}\cup\{\pm\infty\}$, say $\mathcal{V}(\varphi) = g\int\varphi^4$.
- We would need to integrate it against γ, but that lives on $\mathcal{S}'$ fields.
- γ known to be supported "sparsely", i.e. not on function fields, but really on $\mathcal{S}'$.
- So, we really need to extend $\mathcal{V}$ at least γ-a.e. to make sense of $\mu := \mathrm{e}^{-V}\cdot\gamma$.

Concern of physicists: this may be impossible.

- We are afraid that V on $\mathcal{S}'$ might not exist.
- Instead, let us push γ to smooth fields by C, do there $\mu_C := \mathrm{e}^{-V_C}\cdot\gamma_C$.
- Then, get rid of C-dependence of μ_C by concept of Wilsonian RG flow.
  Maybe even $\mu_C \to \mu$ could exist as $C \to \delta$ if we are lucky...

Our result: we are back to the start.

- The UV limit Feynman measure μ then indeed exists.
- But we just proved that then there must exist some extension V of $\mathcal{V}$ to $\mathcal{S}'$, γ-a.e.
- So, we'd better look for that ominous extension V.
- For bounded from below $\mathcal{V}$, bounded from below measurable V needed.
  If we find one, $\mu := \mathrm{e}^{-V}\cdot\gamma$ is then finite measure automatically.
  Only pathology: overlap integral of e^{-V} and γ expected small, maybe zero.
  We only need to make sure that $\int_{\phi\in\mathcal{S}'}\mathrm{e}^{-V(\phi)}\,\mathrm{d}\gamma(\phi) > 0$!

A natural extension[A.László, Z.Tarcsay, J.Ziebell *J.Phys.*__A59__(2026)035401]:

If $\mathcal{V}$ is bounded from below, there is an optimal extension, the "greedy" extension.

$V(\cdot) \;:=\; \underset{\{\eta_n\to\delta\}}{(\gamma)\mathrm{inf}}\;\liminf_{\eta_n\to\delta}\;\mathcal{V}(\eta_n\star\;\cdot\;)$

This is the lower envelope of extensions, i.e. overlap of e^{-V} and γ largest.

We used $(\gamma)\mathrm{inf}$ trick to make V measurable.

Theorem[A.László, Z.Tarcsay, J.Ziebell *J.Phys.*__A59__(2026)035401]:

The interacting Feynman measure $\mu:=\mathrm{e}^{-V}\cdot\gamma$ by greedy extension is nonzero iff

$$\exists\;\eta_n\to\delta\;:\qquad \int\limits_{\phi\in\mathcal{S}'}\limsup_{n\to\infty}\;\mathrm{e}^{-\mathcal{V}(\eta_n\star\phi)}\;\mathrm{d}\gamma(\phi)\quad>\quad 0.$$

Sufficient condition:

$$\exists\;\eta_n\to\delta\;:\qquad \lim_{n\to\infty}\int\limits_{\phi\in\mathcal{S}'}\mathrm{e}^{-\mathcal{V}(\eta_n\star\phi)}\;\mathrm{d}\gamma(\phi)\quad>\quad 0.$$

Makes μ for all bounded $\mathcal{V}$ meaningful. [On triviality, see *CMP*__406__(2025)211.]

[Weidling, sine-Gordon yes. φ^4 no. $(\varphi^2-\psi^2)^2$ maybe.]

Sketch of proofs for existence of UV correlators

1. On manifolds, Wilsonian RG flows form a "nice" top.vector space, similar to distributions.
[It is Hausdorff, locally convex, complete, nuclear, semi-Montel, Schwartz.]

- Coarse-grainings have a natural ordering of being less UV than an other:
$C'' \preceq C$ iff $C'' = C$ or $\exists C' :\ C'' = C'\, C$.
- With this, the space of Wilsonian RG flows is seen to be projective limit of copies of $\mathcal{T}(\mathcal{E})$.
- Check known properties of $\mathcal{T}(\mathcal{E})$, some of them are preserved by projective limit.

2. On flat spacetime for bosonic fields, all Wilsonian RG flows are $\mathcal{G}^{(n)}_{C} = \otimes^n C\, G^{(n)}$.

- On flat spacetime, convolution ops by test functions $C_\eta := \eta \star (\cdot)$ exist and commute.
- Due to RGE, commutativity of convolution ops, and polarization formula for n-forms, for bosonic fields $\mathcal{G}^{(n)}_{C_\eta}$ is n-order homogeneous polynomial in η.
That is, $\exists|\ \mathcal{G}^{(n)}_{\eta_1,\dots,\eta_n}$ symmetric n-linear map in $\eta_1, \dots, \eta_n$, such that $\mathcal{G}^{(n)}_{C_\eta} = \mathcal{G}^{(n)}_{\eta,\dots,\eta}$.
- Due to RGE, commutativity of convolution ops, and a Banach-Steinhaus thm variant, $\mathcal{G}^{(n)}_{\eta_1^t,\dots,\eta_n^t}\Big|_0$ extends to an n-variate distribution, it will do the job as $(G^{(n)}\,|\,\eta_1 \otimes \dots \otimes \eta_n)$.

[A Banach-Steinhaus theorem variant (the key lemma – A.László, Z.Tarcsay):
If a sequence of n-variate distributions pointwise converge on $\otimes^n \mathcal{D}$, then also on full $\mathcal{D}_n$.]

Sketch of proofs for existence of UV measures

1. On flat spacetime for bosonic fields, all Wilsonian RG flows are of the form $\mu_C = C_* \, \mu$.

- We prove it for Fourier transforms (partition functions), and then use Bochner-Minlos.
  We use that $\mathcal{S} \star \mathcal{S} = \mathcal{S}$, moreover
  that for all $\mathcal{J} \subset \mathcal{S}$ compact $\exists\, \eta \in \mathcal{S}$ and $\mathcal{L} \subset \mathcal{S}$ compact such that $\mathcal{J} = \eta \star \mathcal{L}$.

2. Parameters of the reference kinetic Gaussian γ cannot run (also for eff.theories).

- Pushforward preserves abs.continuity, plus a rigidity property of Gaussian measures.

3. There exists some measurable potential $V : \mathcal{S}' \to \overline{\mathbb{R}}$, such that $\mu = \mathrm{e}^{-V} \cdot \gamma$.

- We apply Radon-Nikodym theorem, the fact that $\mathcal{S}'$ is so-called Souslin space,
  and that for $\eta \in \mathcal{S}$ with $F(\eta) > 0$ the coarse-graining $C_\eta := \eta \star (\cdot)$ is injective.

4. For C, with nowhere vanishing Fourier spectrum, one has $\mathrm{e}^{-V_C \circ C} \cdot \gamma = \mathrm{e}^{-V} \cdot \gamma$.

- Fundamental formula of integration variable substitution vs pusforward, Souslin-ness of $\mathcal{S}'$,
  injectivity of coarse-graining $C_\eta := \eta \star (\cdot)$ with $\eta \in \mathcal{S}$, $F(\eta) > 0$.

What is QFT in arbitrary signature?

So, it turns out that Wilsonian RG flow of correlators $\leftrightarrow$ distributional correlators.
(under mild conditions)

Executive summary:
- In QFT, the fundamental objects of interest are distributional field correlators.
- Physical ones selected by a “field equation”, the master Dyson-Schwinger equation.
  Through their smoothed (Wilsonian regularized) instances [*CQG* **39**(2022)185004].