$\boldsymbol{\theta}$-Vacuum from Functional Renormalization

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Yuepeng Guan

（玥鹏 关）

Center for Theoretical Physics and College of Physics, Jilin University

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Based on: **Y. G.**, Jan M. Pawlowski and Masatoshi Yamada, *Phys. Rev. D* 113, 025003 (2026), arXiv: 2507.22318 [hep-th].

Outline

- **Background & Motivation**
- **$U(1)$ Quantum Mechanics as Benchmark**
  - The model, θ-term, and winding sectors
  - Topological vacuum structure
  - Benchmarks of the relaxed $U(1)$ model
- **Functional Flow with Topology**
  - Regulator and topology freeze
  - Topological information at UV
  - Sector-wise RG flow and reconstruction
- **Results & Discussion**
- **Summary**
- **References**
- **Appendices**

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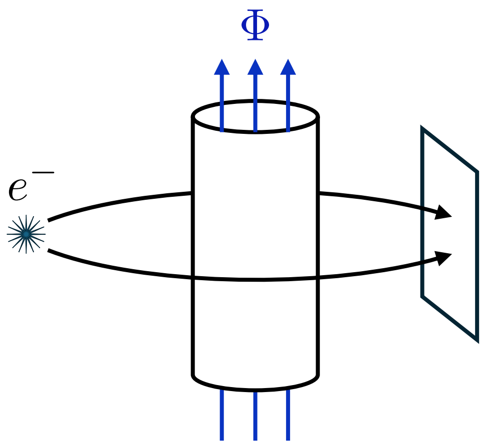
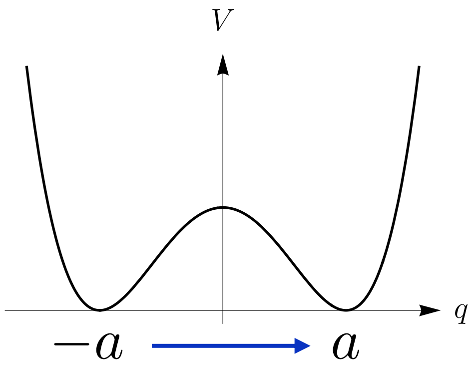
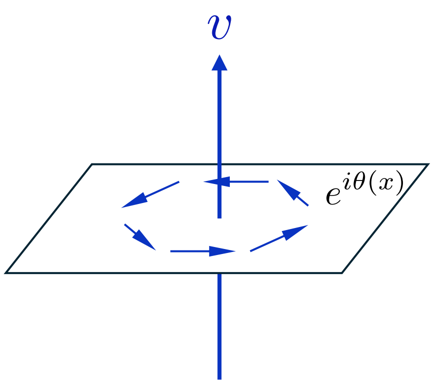
Background & Motivation

○ Topological properties in quantum field theories (QFTs):

- Governed by the global structure of the system rather than local details;
- Aharonov–Bohm effect, instanton-mediated tunneling effect, topological defects, quantum anomalies, etc.;







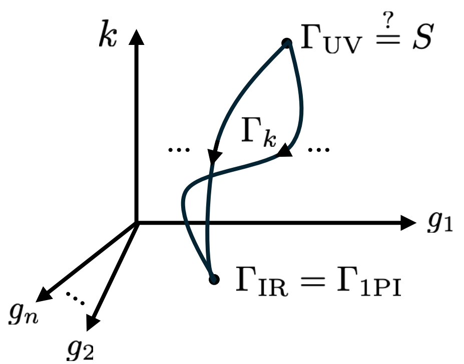
- Nonperturbative approaches should accommodate topological properties;

Background & Motivation

○ Functional renormalization group (fRG):

- A versatile nonperturbative functional approach with applications;
- How topological sectors are represented and carried through an RG flow is still not fully understood;

$$\partial_t \Gamma_k[\Phi] = \frac{1}{2}\,\mathrm{STr}\left[\left(\Gamma_k^{(2)} + \mathcal{R}_k\right)^{-1} \partial_t \mathcal{R}_k\right]$$



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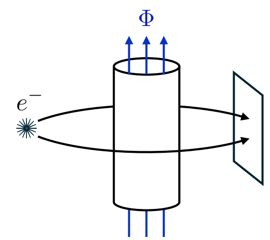

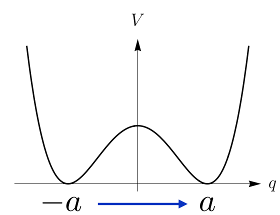

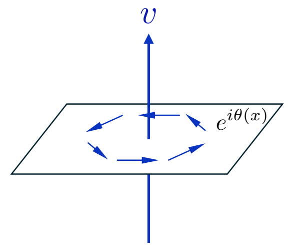

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○ Question:

<u>How can fRG flows faithfully encode topological vacuum structure?</u>

○ Strategy:

Controlled $U(1)$ quantum mechanics as a benchmark theory [1].

[1] K. Fukushima, T. Shimazaki, and Y. Tanizaki, JHEP 04 (2022) 040.

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$U(1)$ Quantum Mechanics as Benchmark

- The model, θ-term, and winding sectors

U(1)-symmetric Q.M.

$$S[\varphi] = \int_\tau \left[\frac{m}{2}\dot{\varphi}^*\dot{\varphi} - \frac{\theta}{4\pi}(\varphi^*\dot{\varphi} - \dot{\varphi}^*\varphi) + V(\varphi^*\varphi)\right]$$

$$V(\varphi^*\varphi) = \frac{g}{4}(\varphi^*\varphi - 1)^2$$
$$\varphi = r\, e^{i\vartheta}$$

$$S = \int_\tau \left[\frac{m}{2}(\dot{r}^2 + r^2\dot{\vartheta}^2) - i\frac{\theta}{2\pi}r^2\dot{\vartheta} + \frac{g}{4}(r^2 - 1)^2\right]$$

$g \to \infty$

Radial mean field limit $r^2 = 1$

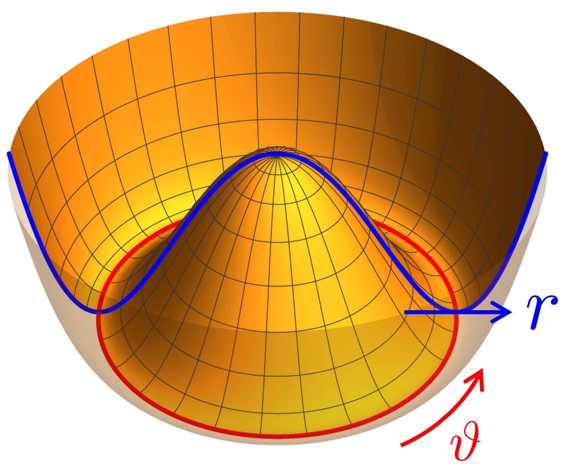
Quantum rotor

$$S = \int_\tau \left(\frac{m}{2}\dot{\vartheta}^2 - i\frac{\theta}{2\pi}\dot{\vartheta}\right)$$

θ-term

$$\frac{\theta}{4\pi}\int_0^\beta \mathrm{d}\tau\,(\varphi^*\dot{\varphi} - \dot{\varphi}^*\varphi) = \frac{\theta}{2\pi}\int_{\vartheta(0)}^{\vartheta(\beta)} \mathrm{d}\vartheta = i\theta\,\nu$$

Doesn't contribute to classical E.o.M.



Topological field configurations

$$\vartheta_\nu = \underbrace{\boxed{\vartheta}}_{\text{Angular Fluctuation}} + \underbrace{\boxed{2\pi\frac{\tau}{\beta}\nu}}_{\text{Winding Background}} \qquad \nu \in \mathbb{Z}$$

$U(1)$ Quantum Mechanics as Benchmark

- Topological vacuum structure

Partition function

$$\mathcal{Z} = \sum_{\nu\in\mathbb{Z}} \int [\mathcal{D}\vartheta_\nu]\, e^{-S[\vartheta_\nu]}$$

kinetic fluctuations

$$= \boxed{\mathcal{N}(\beta)} \boxed{\sum_{\nu\in\mathbb{Z}} e^{-\frac{2\pi^2 m}{\beta}\nu^2 + i\theta\nu},}$$

winding sector

Poisson resummation

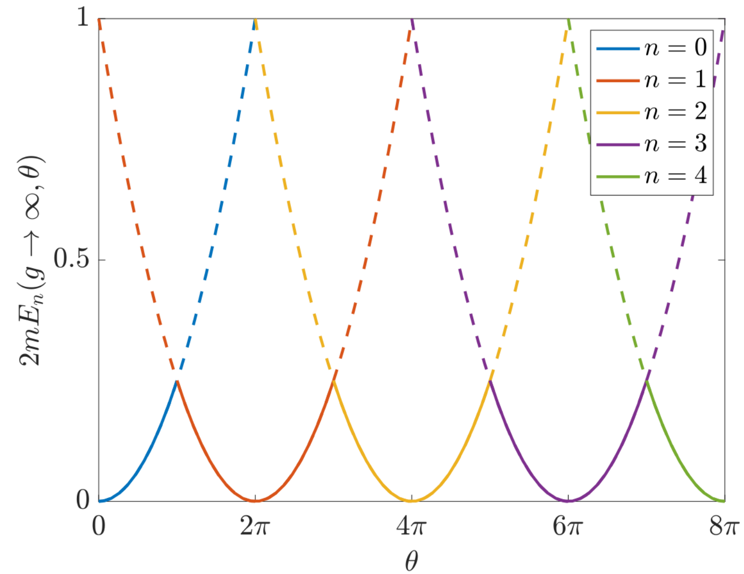
$$Z = \mathcal{N}'(\beta) \sum_{n\in\mathbb{Z}} e^{-\frac{\beta}{2m}(n-\frac{\theta}{2pi})^2} \sim \sum_{n\in\mathbb{Z}} e^{-\beta E_n(\theta)}$$

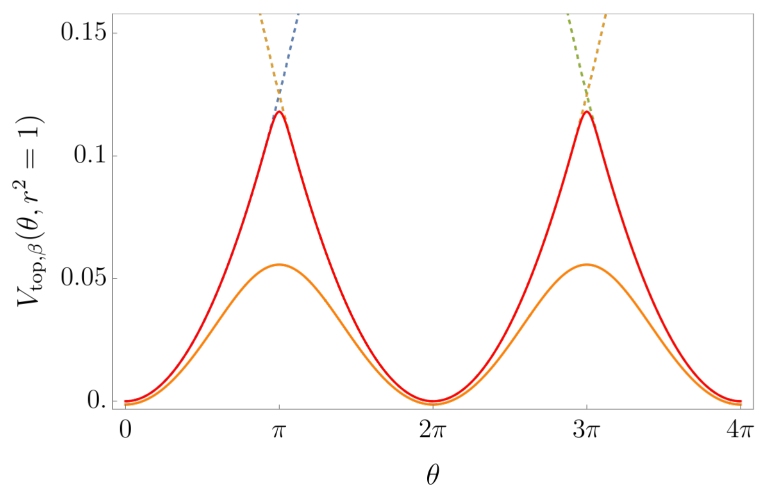
$$E_n(\theta) = \frac{1}{2m}\left(n - \frac{\theta}{2\pi}\right)^2$$

Free energy

$$E(\theta) = -\frac{1}{\beta}\log\left(\sum_n e^{-\beta E_n}\right)$$

$$\rightarrow \; -\frac{1}{\beta}\log\left\{\sqrt{\frac{2\pi m}{\beta}}\vartheta_3\left(-\frac{\theta}{2}, e^{\frac{2m\pi^2}{\beta}}\right)\right\}$$





$U(1)$ Quantum Mechanics as Benchmark

- Benchmarks of the relaxed $U(1)$ model

Back into the $U(1)$-invariant theory:

$$S = \int_\tau \left[\frac{m}{2}\left(\dot{r}^2 + r^2\dot{\vartheta}^2\right) + \frac{\theta}{2\pi}r^2\dot{\vartheta} + \frac{g}{4}\left(r^2-1\right)^2\right].$$

Benchmark: numerical solution of the Schrödinger equation.
The Hamiltonian in (r, ϑ) basis reads

$$H = \frac{1}{2m}\Pi_r^2 + \frac{1}{2mr^2}\left(\Pi_\vartheta - \frac{\theta}{2\pi}r^2\right)^2 + \frac{g}{4}(r^2-1)^2$$

Schrödinger equation in r-direction

$$\left[-\frac{1}{2mr}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{2mr^2}\left(\boxed{n} - \frac{\theta}{2\pi}r^2\right)^2 + \frac{g}{4}(r^2-1)^2\right]\psi_{l,n}(r) = \boxed{E_{l,n}(g,\theta)}\,\psi_{l,n}(r)$$

Quantum number of Π_ϑ

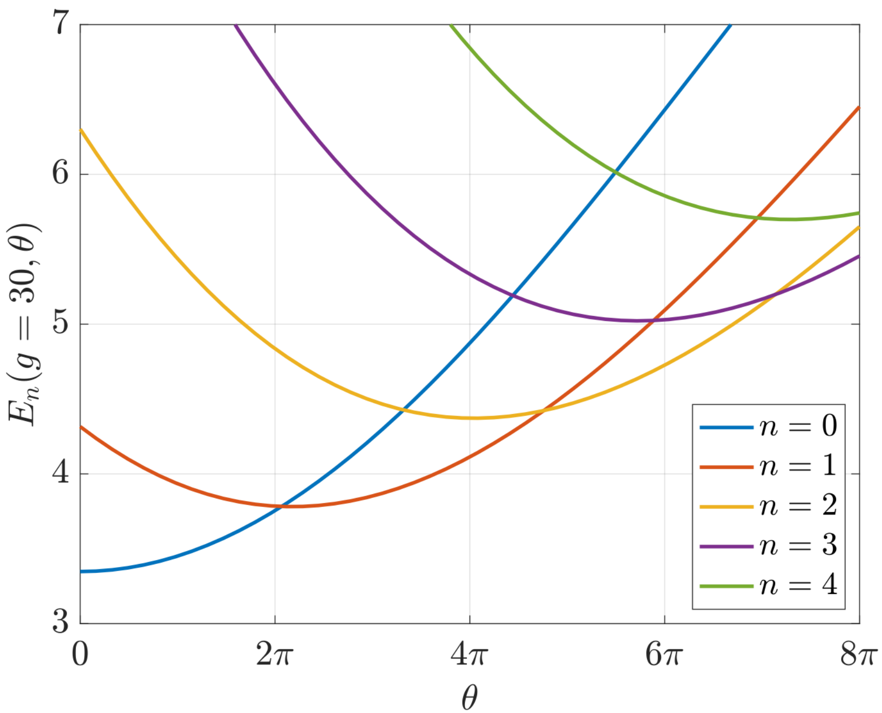
Energy eigenvalues we are looking for

$U(1)$ Quantum Mechanics as Benchmark

- Benchmarks of the relaxed $U(1)$ model

Energy levels

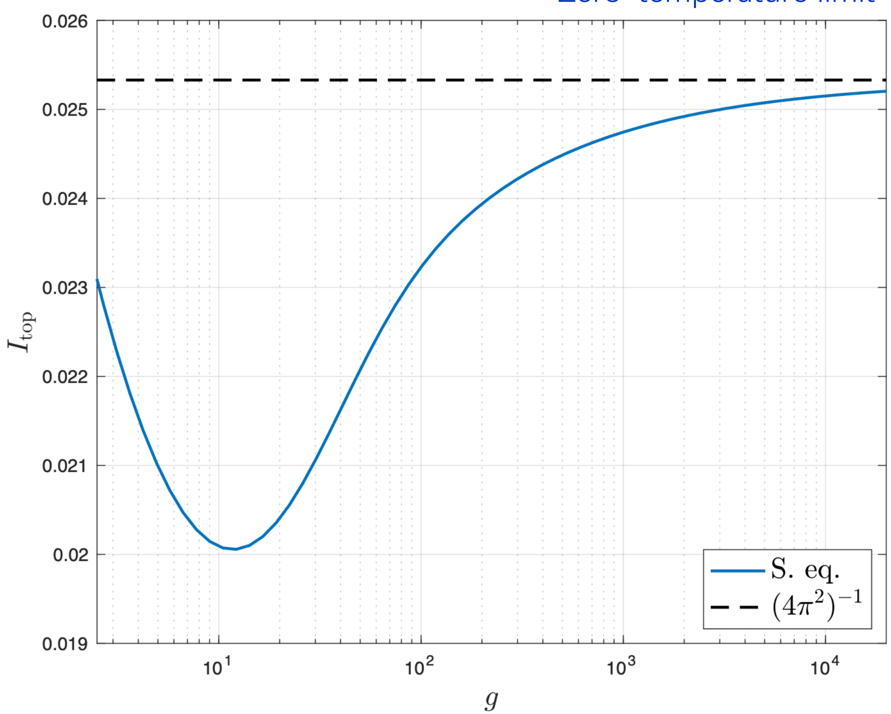
$$E_{l,n}(g,\theta)$$



Topological susceptibility

$$I_{\text{top}}(g) = \frac{\partial^2}{\partial\theta^2}E_0(\theta;g)$$

"Zero-temperature limit"



(*Relaxing the radial constraint breaks the exact θ-periodicity.*)

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Functional Flow with Topology

○ Regulator and topology freeze

Standard fRG-LPA setup [2-4]

1PIEA ansatz:

$$\Gamma_k = \int \frac{\mathrm{d}p}{2\pi}\left[\frac{m}{2}u^*\left(p + i\,\frac{\theta}{2m\pi}\right)^2 u + V_k(u^*, u)\right]$$

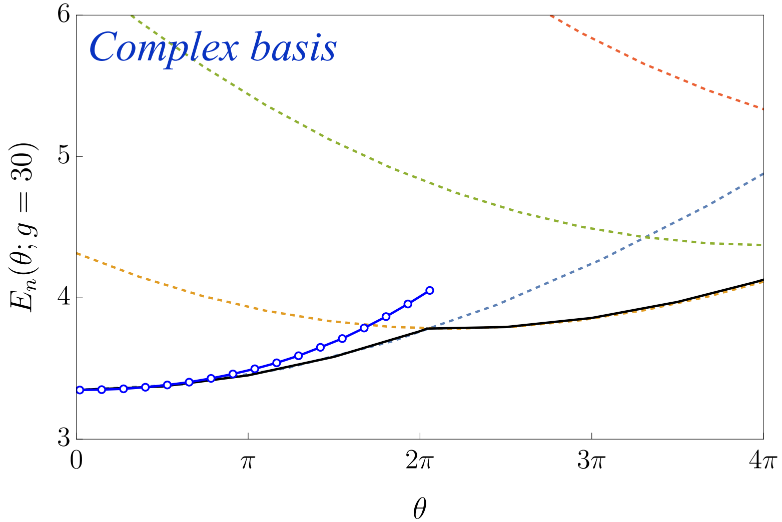
Litim-type regulator (complex variable):

$$R_k^{\mathrm{Litim}}(p_\theta) = \mathrm{Re}(k^2 - p_\theta^2)\Theta(k^2 - p_\theta^2)$$

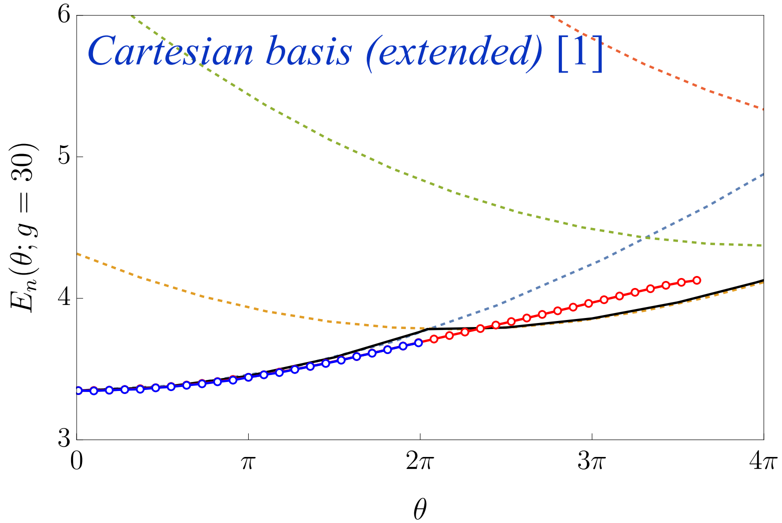
Flow equation (complex basis):

$$\partial_t V_k = \frac{k^2}{2\pi|\tilde{\theta}|}\left[\arctan\left(\frac{2|\tilde{\theta}|\sqrt{\tilde{\theta}^2 + k^2}}{c_k^{(1)}}\right) + \arctan\left(\frac{2|\tilde{\theta}|\sqrt{\tilde{\theta}^2 + k^2}}{c_k^{(2)}}\right)\right]$$

$$E(\theta; g) = \tilde{V}_{k=0}(r^2 = \langle r\rangle^2, \theta)$$



Level crossing is not recovered



Functional Flow with Topology

- Regulator and topology freeze

Regulated generating functional

$$Z_\phi[J_\phi] = \int \mathcal{D}\hat{\varphi}\, e^{-S[\hat{\varphi}]+\boxed{\Delta S_\phi[\hat{\phi}]}+\int_\tau J^a_\phi \hat{\phi}^a}$$

Cutoff terms

$$\Delta S_{\rm pol}[r,\vartheta_\nu] = \Delta S_r[r] + \Delta S_\vartheta[\vartheta_\nu]$$

Radial:

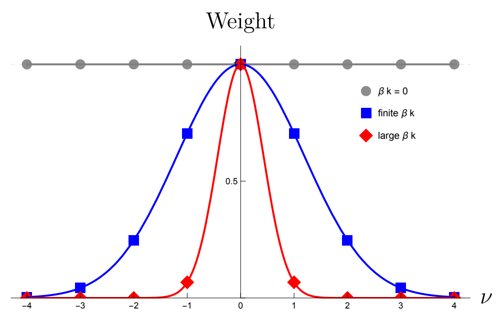
$$\Delta S_r[r] = \frac{1}{2}\int_\tau r(\tau) R_r(\partial_\tau^2) r(\tau)$$

Angular:

$$\boxed{\Delta S_\vartheta[\vartheta_\nu] = \frac{1}{2}\int_\tau \vartheta_\nu(\tau) R_\vartheta(\partial_\tau^2)\vartheta_\nu(\tau)}$$

$$\Delta S_\vartheta[\vartheta_\nu] = \Delta S_\vartheta[\vartheta] + \boxed{\frac{2\pi^2}{3}\nu^2 \beta\, k}$$

Suppressing the nontrivial winding configurations in $\beta \to \infty$



Functional Flow with Topology

○ Topological information at UV

Path-integral form of 1PIEA

$$e^{-\Gamma_k[\varphi]} = \lim_{\beta\to\infty} \sum_\nu \int \mathcal{D}\hat{\varphi}\, e^{-S\left[\hat{\varphi}\, e^{\frac{2\pi i\nu}{\beta}\tau}\right] + \Delta S_k[\hat{\varphi}-\varphi]}$$

$$\times\, e^{\int_\tau (\hat{\varphi}-\varphi)\frac{\delta\Gamma[\varphi]}{\delta\varphi}}$$

$\hat{\varphi},\ \varphi$: trivial winding

UV limit: local fluctuations suppressed

$$\lim_{k\to\infty} e^{-\Gamma_k[\varphi]} \propto \sum_\nu e^{-S\left[\varphi\, e^{\frac{2\pi i\nu}{\beta}\tau}\right]}$$

$$S\left[\varphi\, e^{\frac{2\pi i\nu}{\beta}\tau}\right] = S(\varphi) + \left[\frac{m}{2}\frac{(2\pi\nu)^2}{\beta} - i\nu\theta\right] r^2$$

$$\Gamma_\Lambda[\varphi] \simeq S[\varphi] + \beta\, V_{\mathrm{top},\beta}(r^2;\theta)$$

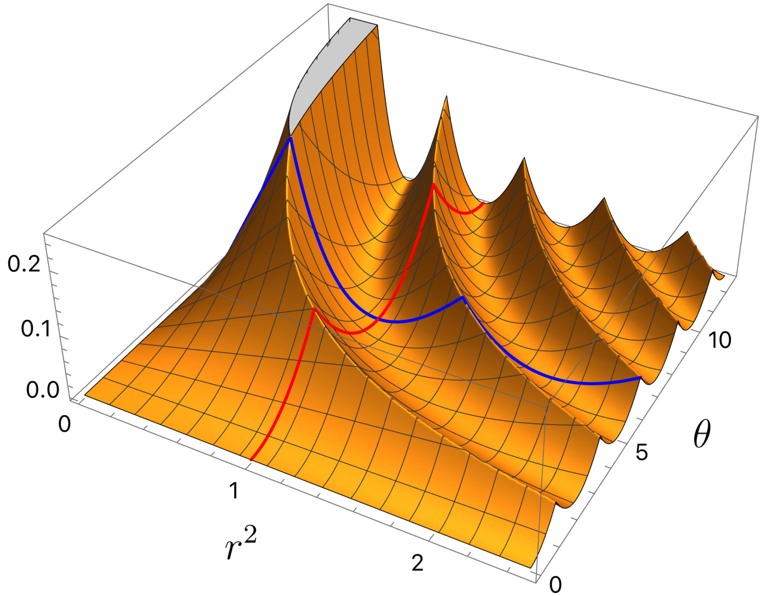
"Topological" potential

$$V_{\mathrm{top},\beta}(r^2;\theta) = -\frac{1}{\beta}\log\left[\sqrt{\frac{2\pi m r^2}{\beta}}\,\vartheta_3\left(-\frac{\theta}{2}r^2, e^{\frac{2m\pi^2}{\beta}r^2}\right)\right]$$

$$V_{\mathrm{top}}(r^2;\theta) = \lim_{\beta\to\infty} V_{\mathrm{top},\beta}(r^2;\theta)$$

$$= \frac{1}{2m}\left(\frac{\theta}{2\pi}r^2 - \frac{n(\theta,r)}{r}\right)^2$$

Indicates the energy levels



Functional Flow with Topology

○ Sector-wise RG flow and reconstruction

1PIEA ansatz:

$$\Gamma_k[\varphi] = \int \frac{\mathrm{d}p}{2\pi} \left[\frac{m}{2}\varphi^*(-p)p^2\varphi(p)\right] + \int_\tau V_k(\varphi^*\varphi)$$

Flow equation:

$$\partial_t V_k = \int \frac{\mathrm{d}p}{2\pi} \frac{\frac{m}{2}\partial_t R_k(p)\left[\frac{m}{2}P_k(p) + V' + r^2 V''\right]}{\left[\frac{m}{2}P_k(p) + V' + r^2 V''\right]^2 - \left[r^2 V''\right]^2}$$

Boundary condition:

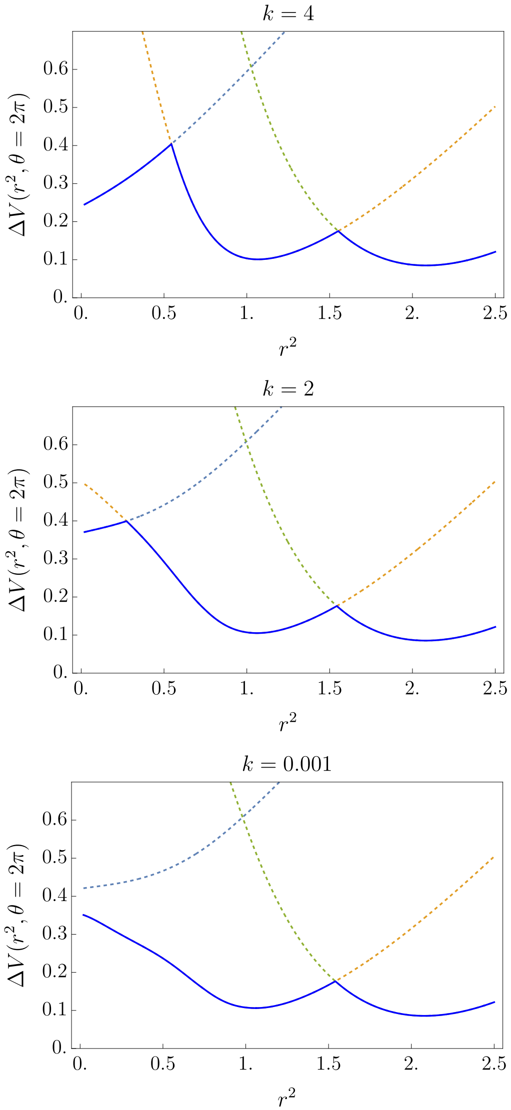
$$V_{k=\Lambda}(r^2) = V_{\mathrm{top}}(r^2;\theta) + \frac{g}{4}(r^2-1)^2$$

Reconstruct $V_k(r^2;\theta) \sim \min_n V_{n,k}(r^2;\theta)$

"Sector-wise" boundary condition:

$$V_{n,k=\Lambda}(r^2) = \frac{1}{2mr^2}\left(n - \frac{\theta}{2\pi}r^2\right)^2 + \frac{g}{4}(r^2-1)^2$$

"Moving cusps"



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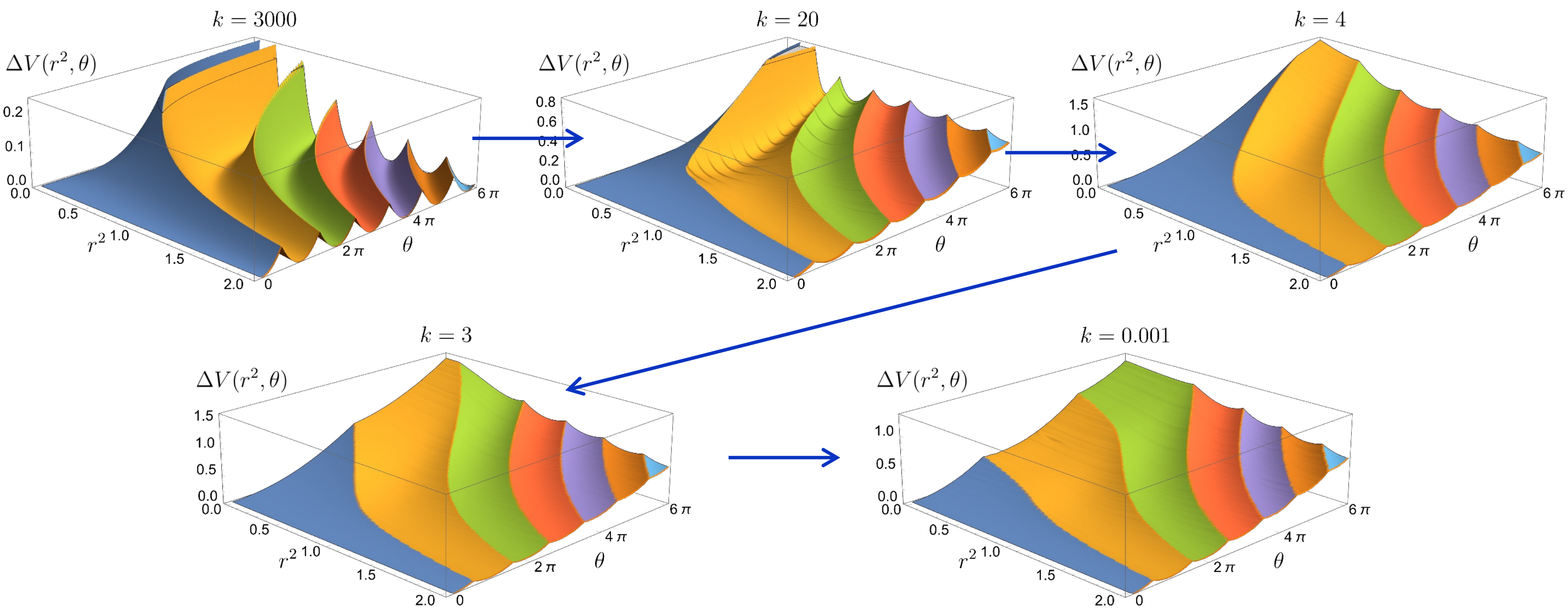
Results & Discussion

- Effective potential flow with scale (Litim-type regulator)

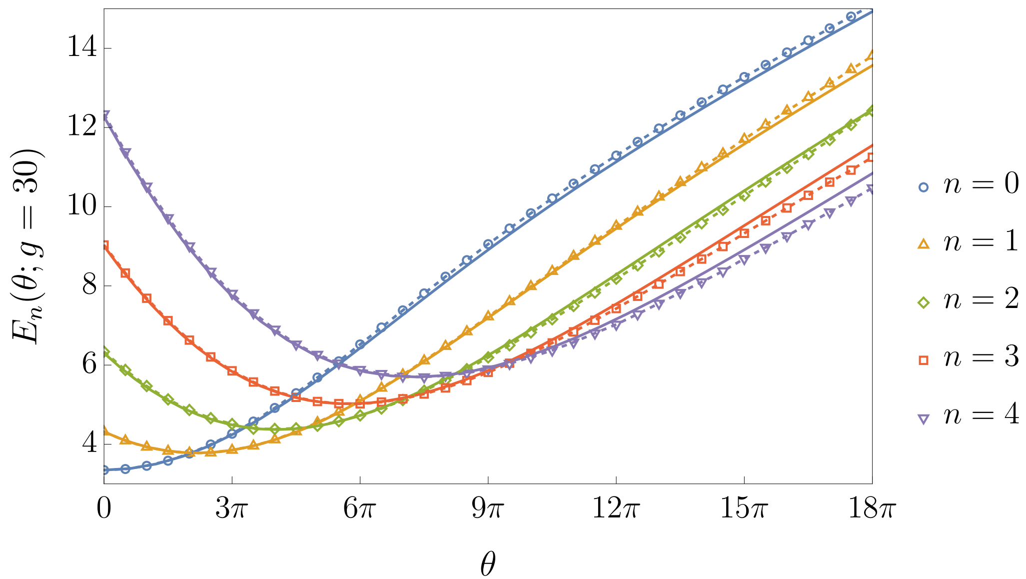
$$\Delta V(r^2, \theta) = V(r^2, \theta) - V(r^2, \theta = 0)$$



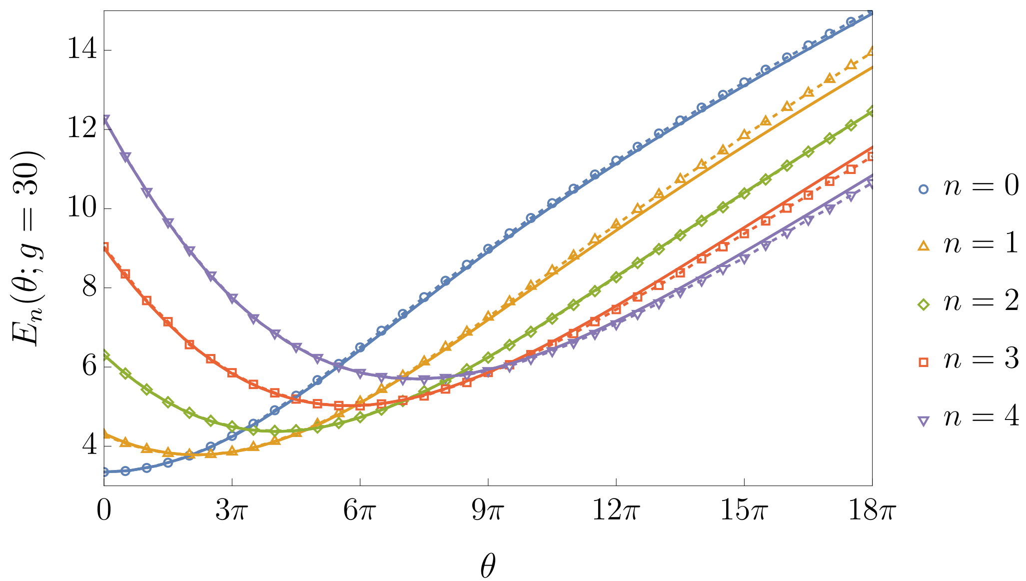
Results & Discussion

○ Energy levels

Litim-type:

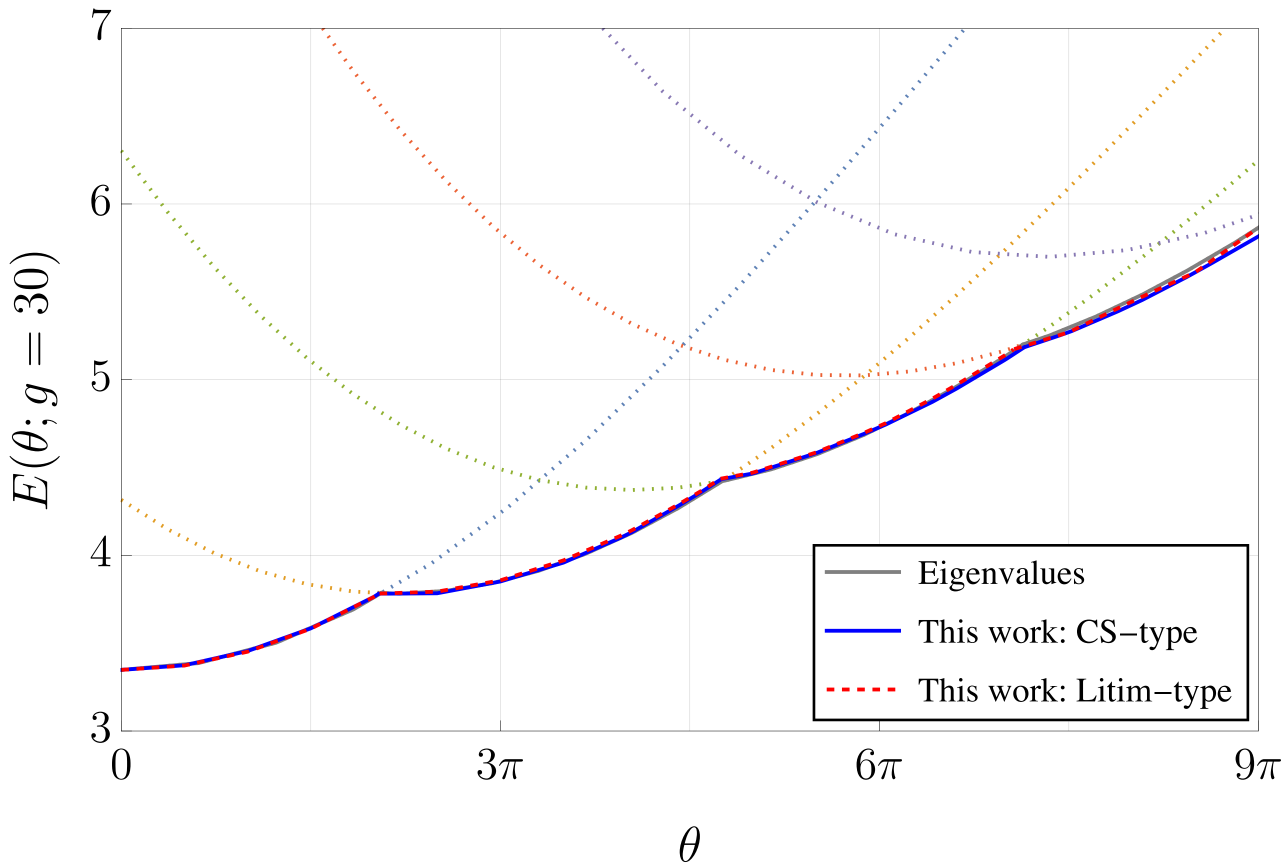


CS-type:



Results & Discussion

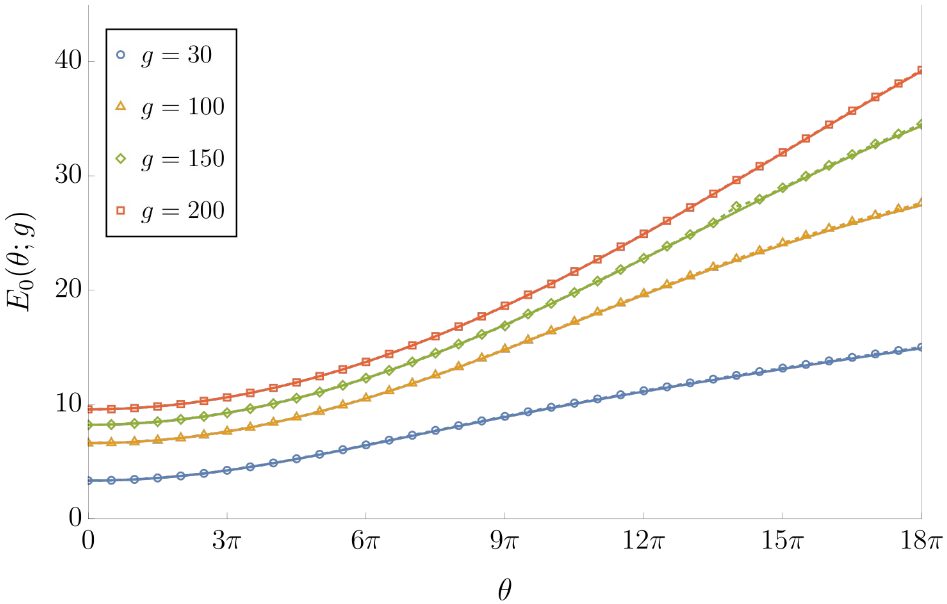
- Vacuum energy



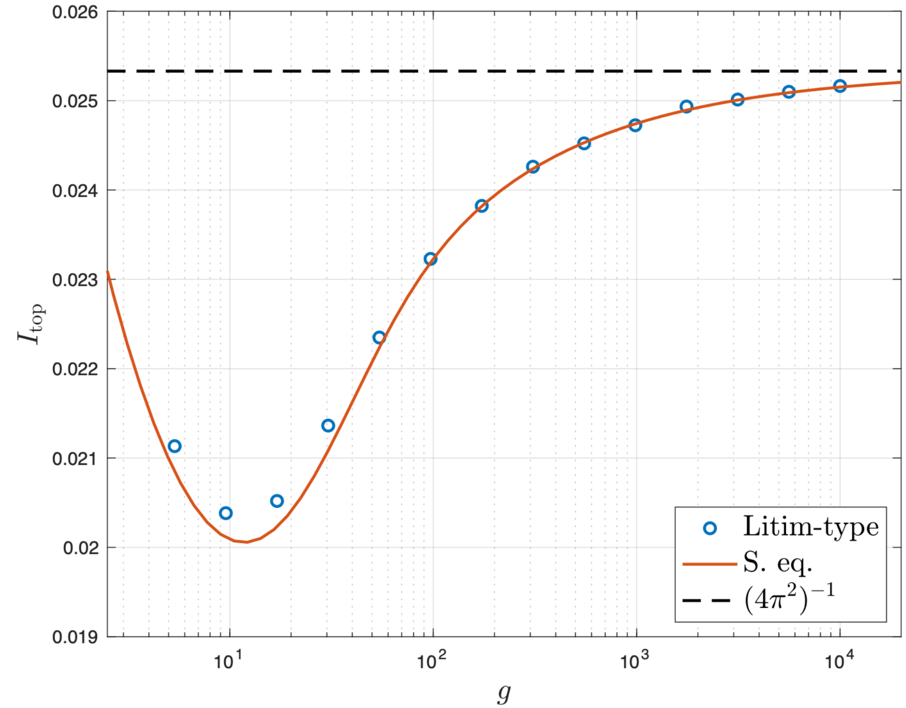
Results & Discussion

- Varying with g and topological susceptibility

Litim-type:



$$I_{\text{top}}(g) = \frac{\partial^2}{\partial\theta^2} E_0(\theta; g)$$



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Summary

- In the zero-temperature limit, conventional regulators suppress nontrivial winding sectors, so a naïve LPA flow does not recover the expected level-crossing structure.
- In the current work, we demonstrated that summing over winding sectors before the RG evolution generates the additional topological potential, which contains the information of the different energy branches.
- Each n-sector evolves smoothly, while the physical potential is obtained from the lower envelope of the sector potentials; the intersections generate the scale-dependent cusps .
- Both Litim- and CS-type regulators reproduce the Schrödinger energy levels and vacuum-energy structure, while the topological susceptibility approaches the quantum-rotor limit as $g \to \infty$.
- Outlook: extend the idea to genuine gauge theories and higher dimensions?
- Outlook: topologically non-trivial configurations emerge along the RG flow?

Thanks!

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[1] Kenji Fukushima, Takuya Shimazaki, and Yuya Tanizaki. Exploring the θ-vacuum structure in the functional renormalization group approach. JHEP, 04:040, 2022.

[2] C. Wetterich, Exact evolution equation for the effective potential, Phys. Lett. B 301 (1993) 90 [1710.05815].

[3] D. F. Litim, Optimized renormalization group flows, Phys. Rev. D 64, 105007 (2001).

[4] H. Gies, Introduction to the functional RG and applications to gauge theories, Lect. Notes Phys. 852 (2012) 287 [hep-ph/0611146].

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Appendices

○ Poisson resummation in quantum rotor

$$\mathcal{N}(\beta) \equiv \int [\mathcal{D}z_0]\, e^{-\int_0^\beta \mathrm{d}\tau\, L} = \sqrt{\frac{\beta}{2\pi m}}$$

Quantum rotor

$$S[z] = \int_\tau \left[\frac{m}{2}\dot{z}^*\dot{z} - \frac{\theta}{4\pi}(z^*\dot{z} - \dot{z}^* z)\right]$$

$$z = e^{i\vartheta}$$

Generating functional

$$Z_z[J_z] = \int \mathcal{D}\hat{z}\, e^{-S_\theta[z] + \int_\tau J\hat{z}}$$

Winding configurations

$$z_\nu(\tau) = e^{i\frac{2\pi\nu}{\beta}\tau} z_0(\tau)\,,$$

with $\quad z_0(\tau+\beta) = z_0(\tau)$

Partition function

$$\mathcal{Z} = \sum_{\nu\in\mathbb{Z}} \int [\mathcal{D}z_\nu]\, e^{-S[z_\nu(\tau), z_\nu^*(\tau)]}$$

$$= \mathcal{N}(\beta) \sum_{\nu\in\mathbb{Z}} e^{-\frac{2\pi^2 m}{\beta}\nu^2 + i\theta\nu}$$

Poisson resummation formula

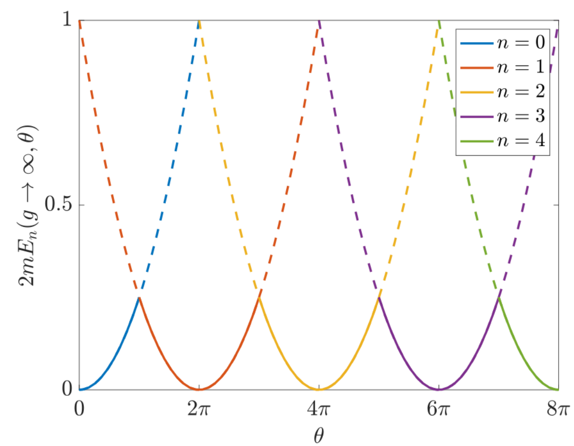
$$\sum_{\nu\in\mathbb{Z}} s(\nu) = \sum_{n\in\mathbb{Z}} \int_{-\infty}^{+\infty} \mathrm{d}x\, s(x) e^{i2\pi n x} \equiv \sum_{n\in\mathbb{Z}} S(n)$$

Spectral basis

$$\mathcal{Z} = \mathcal{N}'(\beta) \sum_{n\in\mathbb{Z}} e^{-\underline{\frac{\beta}{2m}\left(n - \frac{\theta}{2\pi}\right)^2}}$$

Canonical ensemble

$$E_n = \frac{1}{2m}\left(n - \frac{\theta}{2\pi}\right)^2$$



Appendices

○ Derivation of V_{top}

$$S = \int_\tau \left[\frac{m}{2}(\dot r^2 + r^2\dot\vartheta^2) - i\frac{\theta}{2\pi}r^2\dot\vartheta + \frac{g}{4}(r^2-1)^2\right]$$

UV limit with unsuppressed windings

$$\lim_{k\to\infty} e^{-\Gamma_k[\varphi]} \propto \sum_\nu e^{-S\left[\varphi\, e^{\frac{2\pi i\nu}{\beta}\tau}\right]}$$

$\hat\varphi,\ \varphi$: trivial winding

$$\frac{2\pi\nu}{\beta}r^2\int_\tau \partial_\tau\hat\vartheta = 0$$

Pull out radial meanfield

$$\int_\tau \left[\frac{m}{2}\frac{(2\pi\nu)^2}{\beta^2} - i\frac{\nu}{\beta}\theta\right]r^2 = \left[\frac{m}{2}\frac{(2\pi\nu)^2}{\beta} - i\nu\theta\right]r^2$$

$$\Downarrow$$

$$S\left[\varphi\, e^{\frac{2\pi i\nu}{\beta}\tau}\right] = S(\varphi) + \left[\frac{m}{2}\frac{(2\pi\nu)^2}{\beta} - i\nu\theta\right]r^2$$

Summation to V_{top}

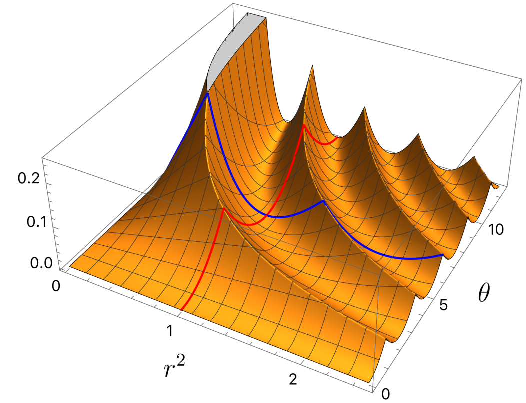
$$\mathcal{Z} \simeq \int [\mathcal{D}\vartheta]\, e^{-S[\bar r,\vartheta]}$$

$$= \int [\mathcal{D}\delta\vartheta] e^{-S[\bar r,\delta\vartheta]} \sum_\nu e^{-\beta\left(\frac{m}{2}\frac{(2\pi\nu)^2}{\beta^2} - i\frac{\nu}{\beta}\theta\right)\bar r^2}$$

$$\Downarrow$$

$$\mathcal{Z} \simeq \int [\mathcal{D}\delta\vartheta] e^{-S[\bar r,\delta\vartheta]}\, e^{-\beta\Delta V_{\text{top},\beta}(r,\theta)}$$

$$\Delta V_{\text{top},\beta}(\theta, r) = -\frac{1}{\beta}\log\left[\sum_\nu e^{-\beta\left(\frac{m}{2}\frac{(2\pi\nu)^2}{\beta^2} - i\frac{\nu}{\beta}\theta\right)\bar r^2}\right]$$



Appendices

- Sector-wise and integrated flows

Sector-resolved 1PIEA

$$e^{-\Gamma_k[\varphi]} = \lim_{\beta\to\infty}\sum_n e^{-\Gamma_{n,k}[\varphi]}$$

Flow equation of integrated EA (besides for cusps)

$$\partial_t\Gamma_k[\varphi] = \lim_{\beta\to\infty}\sum_n p_n\partial_t\Gamma_{n,k}[\varphi]$$

$$p_n = \frac{e^{-\Gamma_{n,k}[\varphi]}}{\sum_m e^{-\Gamma_{m,k}[\varphi]}}$$

$\beta\to\infty$ Selects the minimal potential value

$$\partial_t\Gamma_k[\varphi] = \partial_t\Gamma_{n^*,k}[\varphi]$$

$$n^*(\varphi,k) = \arg\min_n V_{n,k}(\varphi)$$

Flow "at cusps"

Variation in RG time and the zero-temperature limit cannot be exchanged

Sector 1

$$\delta_t\Gamma_k[\varphi_k^{\text{cusp}}] \nearrow \lim_{\beta\to\infty}\sum_n\Gamma_{n,k+\delta k}[\varphi_k^{\text{cusp}}] - \lim_{\beta\to\infty}\sum_n\Gamma_{n,k}[\varphi_k^{\text{cusp}}]$$

$$\delta_t\Gamma_k[\varphi_k^{\text{cusp}}] \searrow \lim_{\beta\to\infty}\sum_n\Gamma_{n,k}[\varphi_k^{\text{cusp}}] - \lim_{\beta\to\infty}\sum_n\Gamma_{n,k-\delta k}[\varphi_k^{\text{cusp}}]$$

Sector 2

Flow equation change from two sides, not well defined

Moving cusps (infinite volume β)

Cusp condition: $$V_k^-(\varphi_k^{\text{cusp}}) = V_k^+(\varphi_k^{\text{cusp}})$$

Vary with scale:

$$\partial_t\varphi_k^{\text{cusp}}\left[V_k^{+,(1)}(\varphi_k^{\text{cusp}}) - V_k^{-,(1)}(\varphi_k^{\text{cusp}})\right] = -\left[\partial_t V_k^+(\varphi_k^{\text{cusp}}) - \partial_t V_k^-(\varphi_k^{\text{cusp}})\right]$$

$$\partial_t\varphi_k^{\text{cusp}} = -\left(\frac{\partial_t V_k^+ - \partial_t V_k^-}{V_k^{+,(1)} - V_k^{-,(1)}}\right)_{\varphi_k^{\text{cusp}}}$$