Fixed points of the renormalisation group flow of fermion mixing matrices: CP violation in the Standard Model and beyond

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Yukawa couplings (N generations)

Higgs $SU(2)$ doublet Φ, $\Phi_c = -i\sigma_2\Phi^*$;
fermion doublets, Ψ^a_L; singlets, Ψ^a_R and Ψ'^a_R, $a = 1, \dots, N$.

$$L_Y = -\sum_{\bar a,\, b=1}^{N} \left(Y_{\bar a b}\overline{\Psi}^{\bar a}_L \Phi_c \Psi^b_R + Y'_{\bar a b}\overline{\Psi}^{\bar a}_L \Phi \Psi'^b_R \right) + h.c.$$

$$\Psi_R = \begin{cases} (u, c, t)_R \\ (\nu_1, \nu_2, \nu_3)_R; \end{cases} \qquad \Psi'_R = \begin{cases} (d, s, b)_R \\ (e, \mu, \tau)_R, \end{cases}$$

$Y_{\bar a b}$ and $Y'_{\bar a b}$ are arbitrary $N \times N$ complex matrices.

$Z = \frac{1}{4\pi} Y Y^\dagger$ and $Z' = \frac{1}{4\pi} Y' Y'^\dagger$ are Hermitian, diagonalize with U and $U' \in SU(N)$:

$$\Lambda = U^\dagger Z U, \quad \Lambda' = U'^\dagger Z' U', \qquad \Lambda_a = \left(\frac{y_a^2}{4\pi}\right)^2, \ \Lambda'_a = \left(\frac{y'^2_a}{4\pi}\right)^2.$$

Mixing matrix, $V \in SU(N)$

$$V = U^\dagger U'$$

The mixing matrix $(N = 3)$

Physics is independent of global fermion phases:

$$\Psi^a_R \;\rightarrow\; e^{-i\rho_a}\Psi^a_R, \quad \Psi'^a_R \;\rightarrow\; e^{-i\rho'_a}\Psi^a_R \quad \Rightarrow \quad V \in SU(3)/U(1)\times U(1),$$
$$\Psi^a_L \;\rightarrow\; e^{-i\eta_a}\Psi^a_L \quad \Rightarrow$$

Double coset

$$V \in U(1)^2 \backslash SU(3)/U(1)^2$$

$$V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_1 & s_1 \\ 0 & -s_1 e & c_1 \end{pmatrix} \begin{pmatrix} c_2 & 0 & s_2 e^{-i\delta} \\ 0 & 1 & 0 \\ -s_2 e^{i\delta} & 0 & c_2 \end{pmatrix} \begin{pmatrix} c_3 & s_3 e & 0 \\ -s_3 e & c_3 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$c_1 = \cos\theta_{23}$, $s_1 = \sin\theta_{23}$, etc.

β-functions

1-loop β-functions: (K.S. Babu, Z. Phys. C, 35, 35 (1987))

$$\frac{dY}{dt} = \frac{1}{16\pi^2}\left[\frac{3}{2}\left(3YY^\dagger + Y'Y'^\dagger\right) - \boldsymbol{\alpha}\mathbf{1}\right] Y,$$

$$\frac{dY'}{dt} = \frac{1}{16\pi^2}\left[\frac{3}{2}\left(3Y'Y'^\dagger + YY^\dagger\right) - \boldsymbol{\alpha}'\mathbf{1}\right] Y',$$

$$\left(\alpha = 8\alpha_3 + \tfrac{9}{4}\alpha_2 + \tfrac{17}{12}\alpha_1,\ \alpha' = 8\alpha_3 + \tfrac{9}{4}\alpha_2 + \tfrac{5}{12}\alpha_1\right).$$

$$\frac{dZ}{dt} = \frac{1}{2\pi}\left[\frac{9}{2}Z^2 - \frac{3}{4}\left(ZZ' + Z'Z\right) - \boldsymbol{\alpha}Z\right],$$

$$\frac{dZ'}{dt} = \frac{1}{2\pi}\left[\frac{9}{2}Z'^2 - \frac{3}{4}\left(ZZ' + Z'Z\right) - \boldsymbol{\alpha}Z'\right].$$

Running of the mixing matrix ($Z = U\Lambda U^\dagger$, $Z' = U'\Lambda'^2 U'^\dagger$)

$$\frac{dZ}{dt} = U\left(\frac{d\Lambda}{dt}U^\dagger - i[A,\Lambda]\right)U^\dagger, \quad \frac{dZ'}{dt} = U'\left(\frac{d\Lambda'}{dt}U'^\dagger - i[A',\Lambda]\right)U'^\dagger,$$

$$\text{with} \qquad A = iU^\dagger\frac{dU}{dt}, \qquad A' = iU'^\dagger\frac{dU'}{dt}.$$

Mixing matrix running, $V = U^\dagger U'$

$$\frac{dV}{dt} = i(AV - VA')$$

$$\frac{d\Lambda}{dt} = \frac{1}{2\pi}\left[\frac{9}{2}\Lambda^2 - \frac{3}{4}\left(\Lambda V\Lambda' V^\dagger + V\Lambda' V^\dagger\Lambda\right) - \alpha\Lambda\right] - i[A,\Lambda],$$

$$\frac{d\Lambda'}{dt} = \frac{1}{2\pi}\left[\frac{9}{2}\Lambda'^2 - \frac{3}{4}\left(\Lambda' V^\dagger\Lambda V + V^\dagger\Lambda V\Lambda'\right) - \alpha'\Lambda'\right] - i[A',\Lambda'].$$

Fixed points of V at 1-loop

$\theta_1, \theta_3 = 0$ or $\frac{\pi}{2}$ and

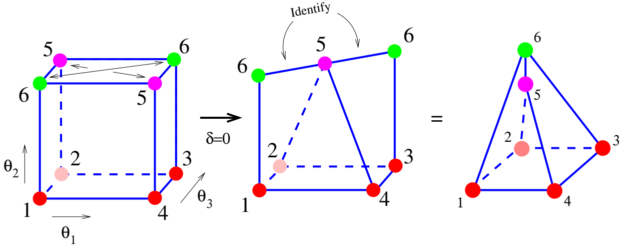
$$\underset{\delta=0,\pi}{\theta_2 = 0}: \; V^* = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}, \begin{pmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & -1 & 0\end{pmatrix}, \begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0\end{pmatrix},$$

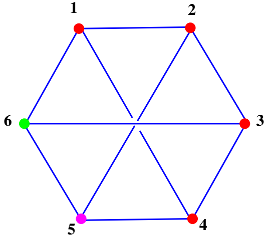
$$\underset{\delta=0}{\theta_2 = \tfrac{\pi}{2}}: \; V^* = \begin{pmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ -1 & 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0 & 1\\ 0 & -1 & 0\\ 1 & 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0 & 1\\ -1 & 0 & 0\\ 0 & -1 & 0\end{pmatrix}, \begin{pmatrix}0 & 0 & 1\\ -1 & 0 & 0\\ 0 & -1 & 0\end{pmatrix},$$

$$\underset{\delta=\pi}{\theta_2 = \tfrac{\pi}{2}}: \; V^* = \begin{pmatrix}0 & 0 & -1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0 & -1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0 & -1\\ 1 & 0 & 0\\ 0 & -1 & 0\end{pmatrix}, \begin{pmatrix}0 & 0 & -1\\ -1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}.$$

6 physically distinct fixed points, no CP violation at any fixed point.

Fixed Points





Permutahedron

Fixed points

Fixed points of RG flow, V^*

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\ \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix},\ \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix},$$
$$\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix},\ \begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix},\ \begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}.$$

- The group of permutations on three objects, S_3.

  R. Alkofer *et al*, Annals of Physics, **421** (2020) 168282, arXiv:2003.08401 [hep-ph]
- Remains true to all orders in perturbation theory.

  BPD, arXiv:2608.28149 [hep-ph]
- These are also fixed points of the left action of $U(1)^2$ on $SU(3)/U(1)^2$: S_3 is the Weyl group of $SU(3)$.

$$U(1)^2\backslash SU(3)/U(1)^2 \approx S^4$$

V.M. Buchstaber and S. Terzić, *Sbornik: Mathematics*, **210** (2019) 508

Non-perturbative fixed points

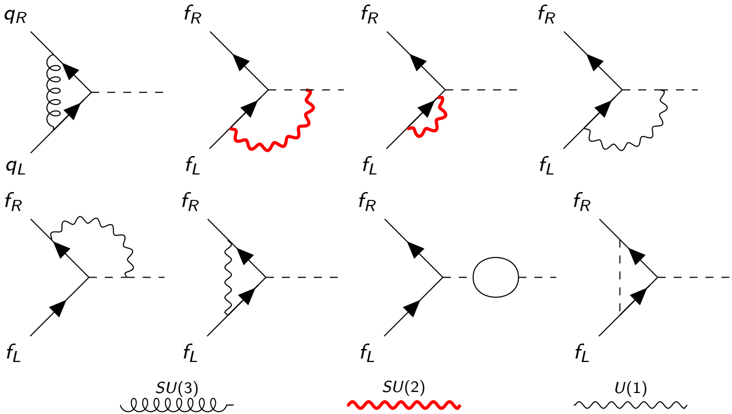
- Assume the RG flow on $U(1)^2 \backslash SU(3)/U(1)^2$ lifts to a flow on the flag manifold, $SU(3)/U(1) \times U(1)$, $(\chi = 6)$.
- Assume the left action of $U(1) \times U(1)$ commutes with this RG flow.
- Let $s \in S_3$ and $f \in SU(3)/U(1) \times U(1)$ be a point in the flag manifold which is a fixed point of s: $s(f) = f$.
- Let $\epsilon : f \to \epsilon(f)$ be an infinitesimal RG transformation on f.
- s commutes with $\epsilon \;\Rightarrow\; s(\epsilon(f)) = \epsilon(s(f)) = \epsilon(f)$
  $\Rightarrow \quad \epsilon(f)$ is a fixed point of s.
- There are 6 isolated fixed points of $U(1) \times U(1)$
  $\Rightarrow \quad$ there are no fixed points of s infinitesimally close to f
  $\Rightarrow \quad \epsilon(f) = f$ and f is a fixed point of the RF flow.

BPD, Phys. Rev. **13** (2026) 095043, arXiv:2601.02452 [hep-ph].

Discussion

- The space of all possible mixing matrices in the Standard Model is $U(1)^2\backslash SU(3)/U(1)^2 \approx S^4$;
- The RG generates a diffeomorphism of S^4 with 6 fixed points, which are a unitary representation of the Weyl group of $SU(3)$;
- $\delta = 0, \pi$ at all 6 fixed points $\Rightarrow$ CP $\checkmark$;
- In the normal hierarchy $V_1^* = 1$ is attractive in the IR;
- None of this is physically relevant to the Standard Model, (with the measured values of the Yukawa couplings, the running is too slow!);
- Possible relevance to CP-violation in dark matter, if it is a gauge theory with N generations of $SU(2)_W$ chiral fermions?
- N generations, $U(1)^{N-1}\backslash SU(N)/U(1)^{N-1}$: $\frac{1}{2}(N-1)(N-2)$ $\not{C}\not{P}$ phases and $N!$ fixed points (Weyl group, S_N).

1-loop Feynman diagrams



1-loop β-functions

Define

$$z_1 = \frac{y_3^2 - y_2^2}{y_3^2 + y_2^2}, \qquad z_2 = \frac{y_3^2 - y_1^2}{y_3^2 + y_1^2}, \qquad z_3 = \frac{y_2^2 - y_1^2}{y_2^2 + y_1^2},$$

$$z'_1 = \frac{y_3'^2 - y_2'^2}{y_3'^2 + y_2'^2}, \qquad z'_2 = \frac{y_3'^2 - y_1'^2}{y_3'^2 + y_1'^2}, \qquad z'_3 = \frac{y_2'^2 - y_1'^2}{y_2'^2 + y_1'^2}$$

with $-1 \leq z_a, z'_a \leq 1$, not all independent e.g. $z_3 = \frac{z_2 - z_1}{1 - z_1 z_2}$.

$$\frac{d\theta_a}{dt} = \frac{3}{32\pi^2} \sum_{b,c=1}^{3} \left[\frac{y_3^2 z_c P_a^{bc}}{z'_b(1+z_c)} + \frac{y_3'^2 z'_c Q_a^{bc}}{z_b(1+z'_c)} \right]$$

$$\frac{d\delta}{dt} = \frac{3\sin\delta}{32\pi^2} \sum_{b,c=1}^{3} \left[\frac{y_3^2 z_c P_\delta^{bc}}{z'_b(1+z_c)} + \frac{y_3'^2 z'_c Q_\delta^{bc}}{z_b(1+z'_c)} \right]$$

with P_a^{bc}, Q_a^{bc}, P_δ^{bc}, Q_δ^{bc} all (low order) polynomials of trigonemtric functions of $(\theta_1, \theta_2, \theta_3, \delta)$.

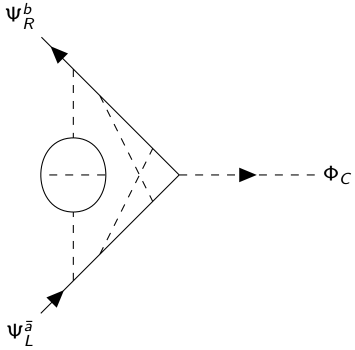
72 functions in all!

1-loop β-functions

33 are identically zero, 39 remain, *e.g.*

$$
\begin{aligned}
P_3^{11} &= \sin 2\theta_3 \sin^2\theta_1 \sin^2\theta_2 - \cos\delta \cos^2\theta_3 \sin 2\theta_1 \sin\theta_2, \\
P_3^{21} &= -\sin 2\theta_3 \sin^2\theta_1 \sin^2\theta_2 - \cos\delta \sin^2\theta_3 \sin 2\theta_1 \sin\theta_2, \\
P_3^{31} &= \sin 2\theta_3 (\sin^2\theta_1 \sin^2\theta_2 - \cos^2\theta_1) - \cos\delta \cos 2\theta_3 \sin 2\theta_1 \sin\theta_2, \\
P_3^{12} &= -\sin 2\theta_3 \sin^2\theta_2, \\
P_3^{22} &= \sin 2\theta_3 \sin^2\theta_2, \\
P_3^{32} &= \sin 2\theta_3 \cos^2\theta_2, \\
Q_3^{21} &= -\sin 2\theta_3 \sin^2\theta_1 - \cos\delta \sin^2\theta_3 \sin 2\theta_1 \sin\theta_2, \\
Q_3^{31} &= -\sin 2\theta_3 \cos^2\theta_1 + \cos\delta \sin^2\theta_3 \sin 2\theta_1 \sin\theta_2, \\
Q_3^{22} &= \sin 2\theta_3 \sin^2\theta_1 - \cos\delta \cos^2\theta_3 \sin 2\theta_1 \sin\theta_2, \\
Q_3^{32} &= \sin 2\theta_3 \cos^2\theta_1 + \cos\delta \cos^2\theta_3 \sin 2\theta_1 \sin\theta_2.
\end{aligned}
$$

Balzereit, Hansmann, Mannel and Plümper,
Eur. Phys. J. **C9** (1999) 197, hep-ph/9810350.



$\mathbf{\Delta Y_{\bar{a}b}}$

contributes

Z^3Y, $Z^2Z'Y$, $ZZ'ZY$, $Z'Z^2Y$, Z'^2ZY, $Z'ZZ'Y$, ZZ'^2Y and Z'^3Y

to $\dot{Y}$, with factors $Tr(Z^2)$, $Tr(ZZ')$ or $Tr(Z'^2)$, when summed over internal lines.

- $\dot{Z} = U\Big(\dot{\Lambda} - i[A,\Lambda]\Big)U^\dagger$ will be a sum of terms of the form

$$Z^{q_1}Z'^{q_2}Z^{q_3}\dots Z'^{q_{n-1}}Z^{q_n}\{\prod_{l=1}^{L} Tr\big(Z^{p_{l,1}}Z'^{p_{l,2}}Z^{p_{l,3}}\dots Z'^{p_{l,r_l}}\big)\}\mathcal{G}_{\mathbf{p},\mathbf{q}},$$

  $\mathcal{G}_{\mathbf{p},\mathbf{q}}$ a polynomial in gauge and Higgs couplings and lepton Yukawa couplings.

- At V^*_α

$$Z^{q_1}Z'^{q_2}Z^{q_3}\dots Z'^{q_{n-1}}Z^{q_n} = U(\Lambda^{q_1}V^*_\alpha\Lambda'^{q_2}V^{*\dagger}_\alpha\Lambda^{q_3}\dots V^*_\alpha\Lambda'^{q_{n-1}}V^{*\dagger}_\alpha\Lambda^{q_n})U^\dagger.$$

- $V^*_\alpha\Lambda'^k V^{*\prime}_\alpha$ is diagonal
  (V^*_α merely permutes the diagonal entries of Λ'^k),
  $\Rightarrow [A,\Lambda]$ is diagonal $\Rightarrow [A,\Lambda]=0 \Rightarrow A\in U(1)\times U(1)$ at V^*.
- Physically V^*_α are fixed points of the CKM matrix to all orders in perturbation theory.

Anomalous Dimensions

Eigenvalues of $\partial_a \beta^b$ at V^*_α, with $y_t^2 \gg y_c^2 \gg y_u^2$, $y_b^2 \gg y_s^2 \gg y_d^2$.

Fixed Point	$\bar\lambda_1$	$\bar\lambda_2$	$\bar\lambda_3$	$\bar\lambda_J$
1	$y_t^2 + y_b^2$	$y_t^2 + y_b^2$	$y_c^2 + y_s^2$	$2(y_t^2 + y_b^2)$
2	$y_t^2 + y_b^2$	$y_t^2 + y_b^2$	$-(y_c^2 + y_s^2)$	$2(y_t^2 + y_b^2)$
3	$-(y_t^2 + y_b^2)$	y_b^2	$-y_t^2$	$-2y_t^2$
4	$-(y_t^2 + y_b^2)$	y_b^2	y_t^2	$y_c^2(1 + y_b^2/y_t^2) + y_s^2(1 + y_t^2/y_b^2)$
5	y_t^2	$-y_b^2$	$-(y_t^2 + y_b^2)$	$-2y_b^2$
6	$-y_t^2$	$-(y_t^2 + y_b^2)$	$-y_b^2$	$-2(y_t^2 + y_b^2)$

BPD, Phys. Rev. **13** (2026) 095043, 2601.02452 [hep-ph].