Cosmological Correlators: Unitarity and renormalisation

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Main message

- The cosmological wavefunction and its coefficients ψ_n determine the natural observables in cosmology, namely correlators.

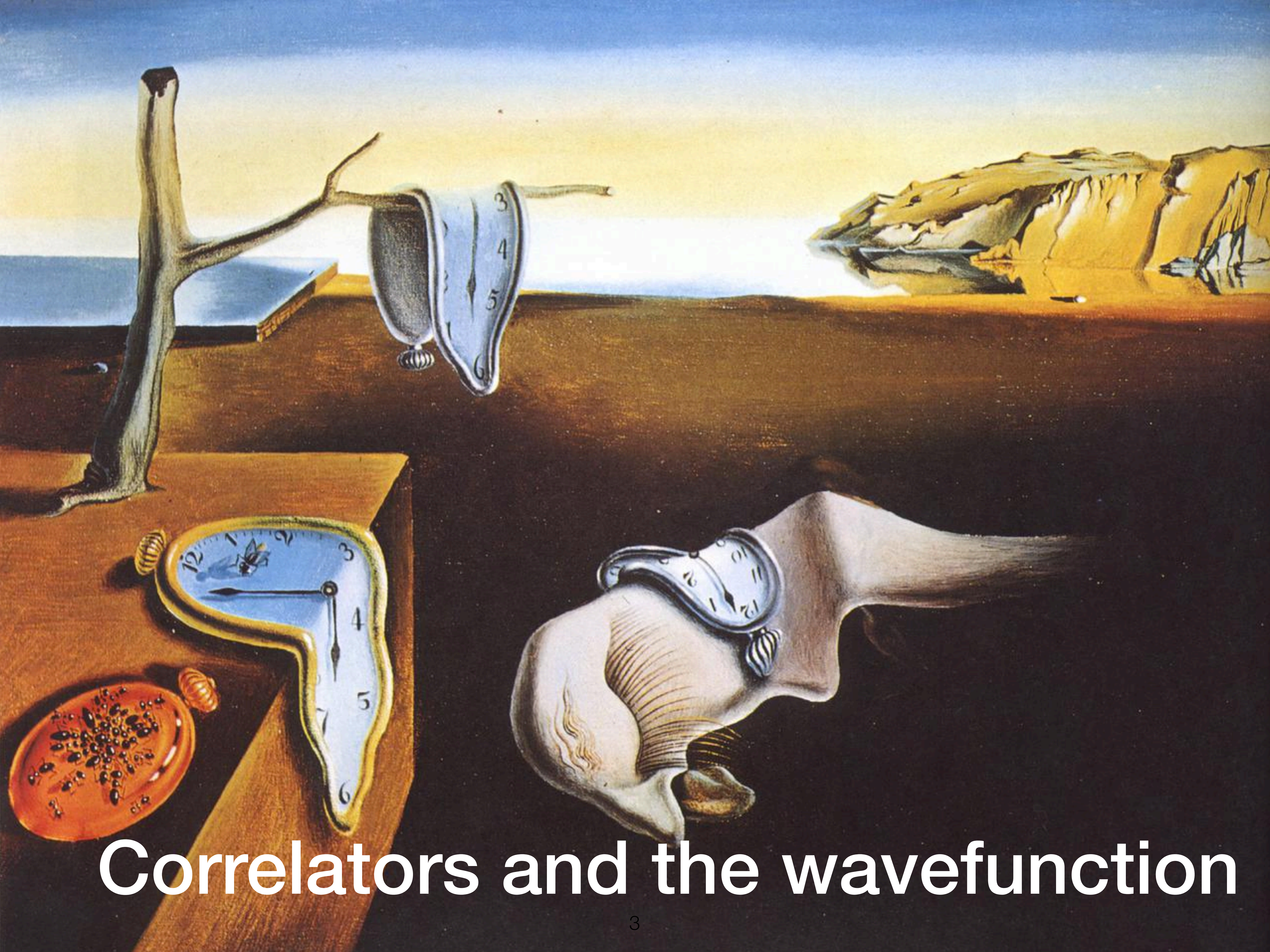
- For massless scalars in de Sitter spacetime, a good approximation to many models of inflation, *the wavefunction is real at tree level* but becomes complex due to *a quantum anomaly in renormalization*. From counter-term renormalization we find an *all-loop relation*

$$\boxed{\mathrm{Im}\, \widehat{\psi}_n = \tan\left(\frac{\pi}{2}\mu\partial_\mu\right) \mathrm{Re}\, \widehat{\psi}_n}$$

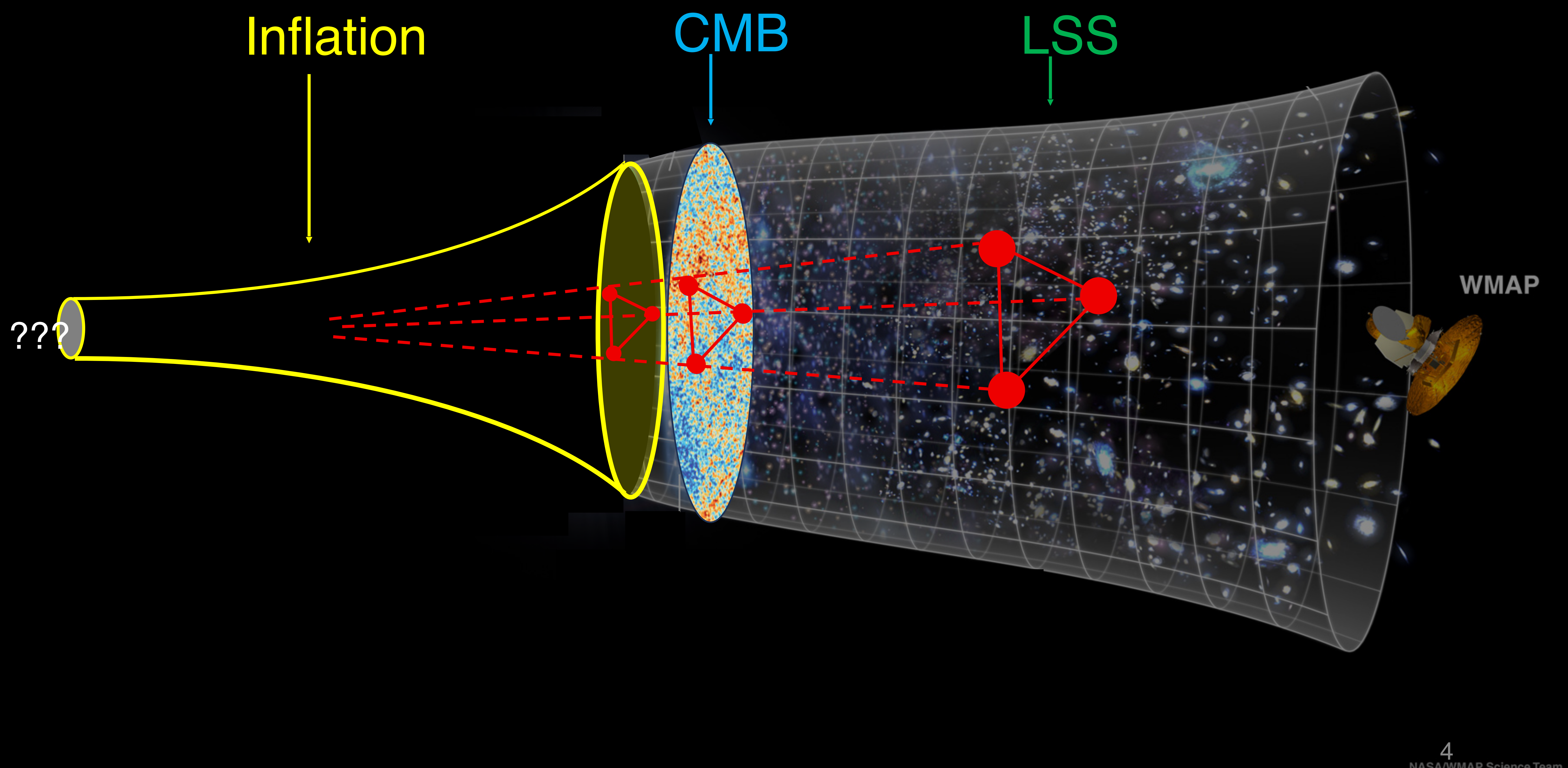
- where μ is the renormalisation scale.

- This quantum contribution is the leading signal in parity-odd correlators and is being searched for in Cosmic Microwave Background and Large Scale Structure data.

- **Open question: Can one derive this from exact/functional RG??**



Correlators and the wavefunction



- Observables = cosmological correlators

(Fig. credit Xi Tong)

Primordial Cosmo = QFT in curved spacetime / Quantum Gravity

- On large scales (>> Mpc) cosmological surveys measure QFT correlators of metric fluctuation

$$\langle \prod_a^n \delta(\mathbf{k}_a) \rangle \sim \left[\prod_a^n \Delta^{(\delta)}(\mathbf{k}_a) \right] \langle \prod_a^n \zeta(\mathbf{k}_a) \rangle$$

CMB temperature
density of galaxies
dark matter, …

QFT / QG
in de Sitter

- Gravitational floor of non-Gaussianity in single-clock inflation: $f_{NL}^{eq} \gtrsim 10^{-2}$
- The goal of primordial cosmology is to understand QFT and QG in (approximately, asymptotically) de Sitter

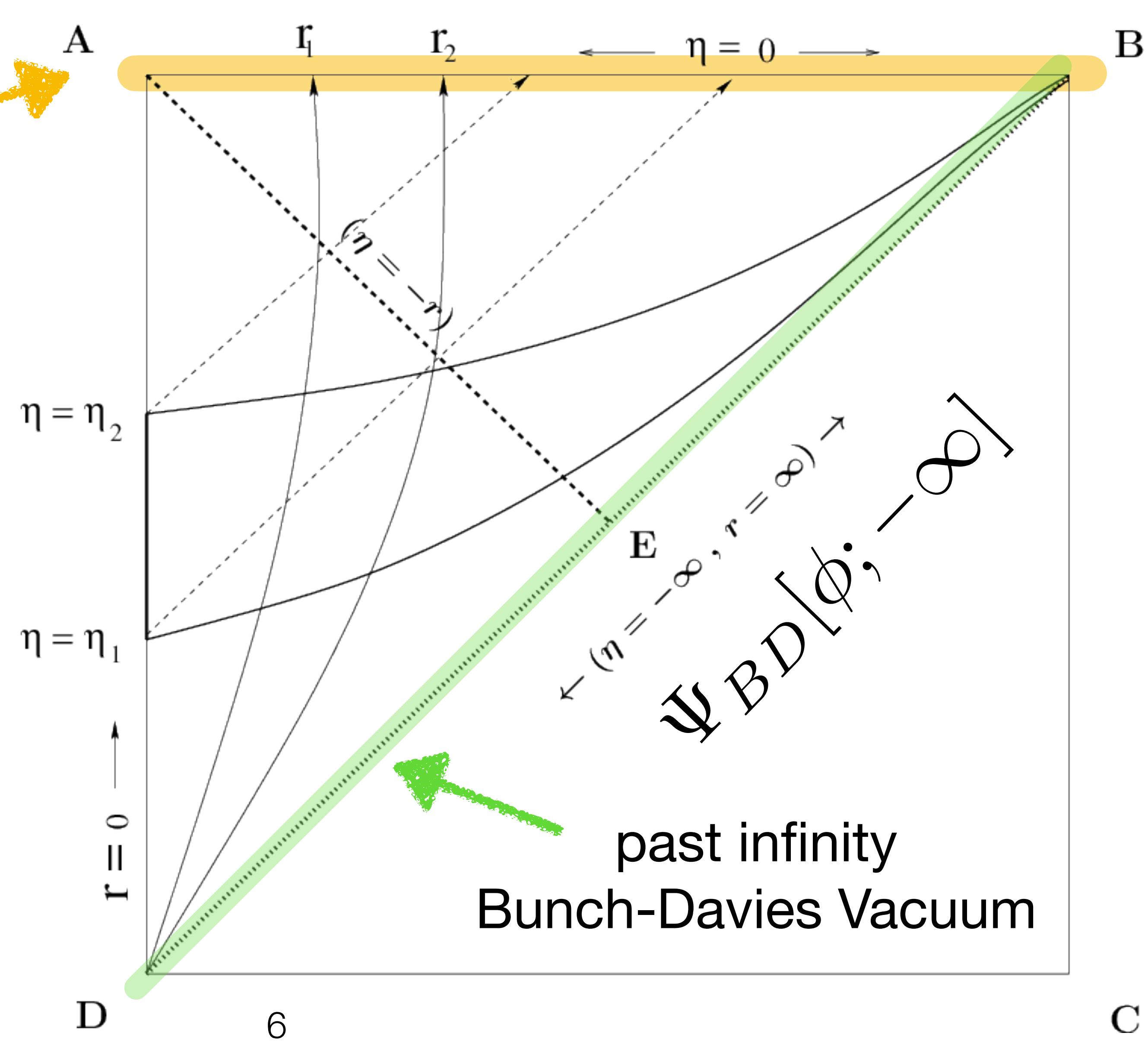
Penrose diagram

- We work in (the Poincaré patch of) de Sitter spacetime with constant Hubble parameter H being the characteristic energy scale of the problem

$$ds^2 = -\,dt^2 + e^{2Ht}d\mathbf{x}^2 = \frac{-d\tau^2 + d\mathbf{x}^2}{(\tau H)^2}$$

$$\boxed{\Psi[\phi;\eta \to 0]}$$

The future (conformal) boundary can be thought as the reheating surface after inflation and determines the statistics of *Large Scale Structures and Cosmic Microwave Background observations*



The wavefunction

- The observables of cosmology are correlators of equal-time local operators $\mathcal{O}$ at $\eta \to 0$
$$\lim_{\eta\to 0} \left\langle \Omega \left| \prod_{a=1}^{n} \mathcal{O}(\mathbf{k}_a, \eta) \right| \Omega \right\rangle \equiv \left\langle \prod_{a=1}^{n} \mathcal{O}(\mathbf{k}_a) \right\rangle \equiv \langle \mathcal{O}^n \rangle \, .$$

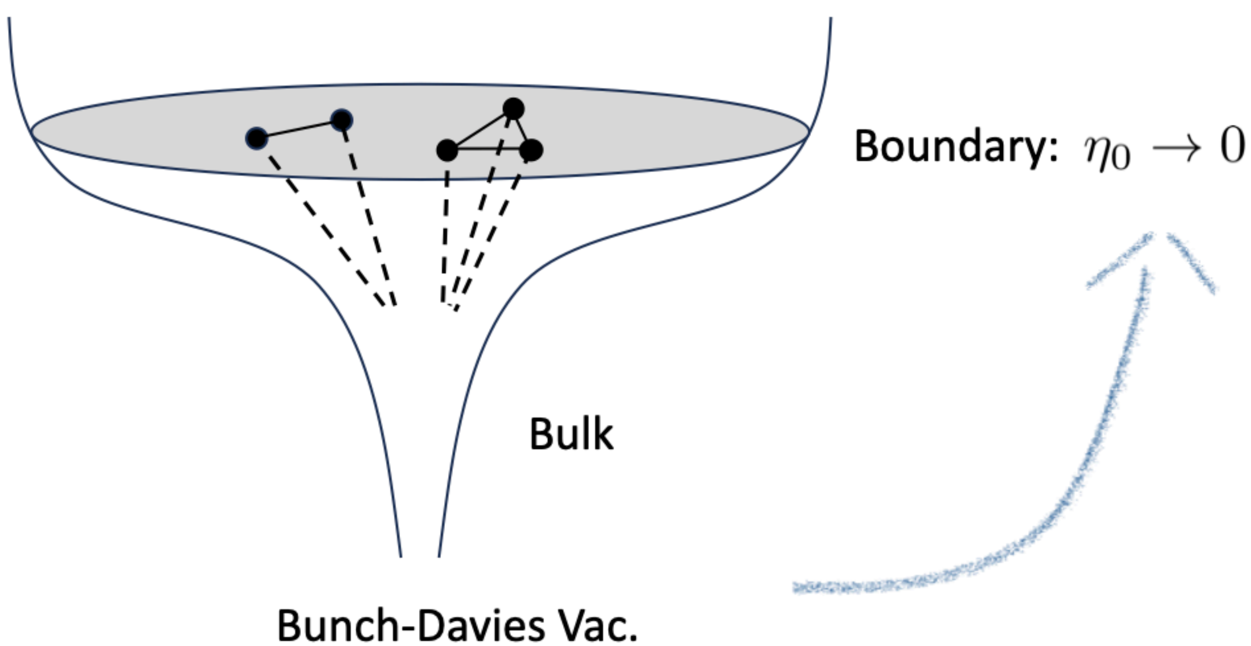
- To compute correlators we use the *wavefunction* (Schroedinger picture QFT), i.e. the projection of the quantum state $|\Psi\rangle$ of the system onto eigenstates $|\phi\rangle$ of the field operators, $\hat{\phi}(x,\eta)\,|\,\phi\rangle = \phi(x,\eta)\,|\,\phi\rangle$, namely $\Psi[\phi,\eta] \equiv \langle \phi \,|\, \Psi, \eta \rangle$

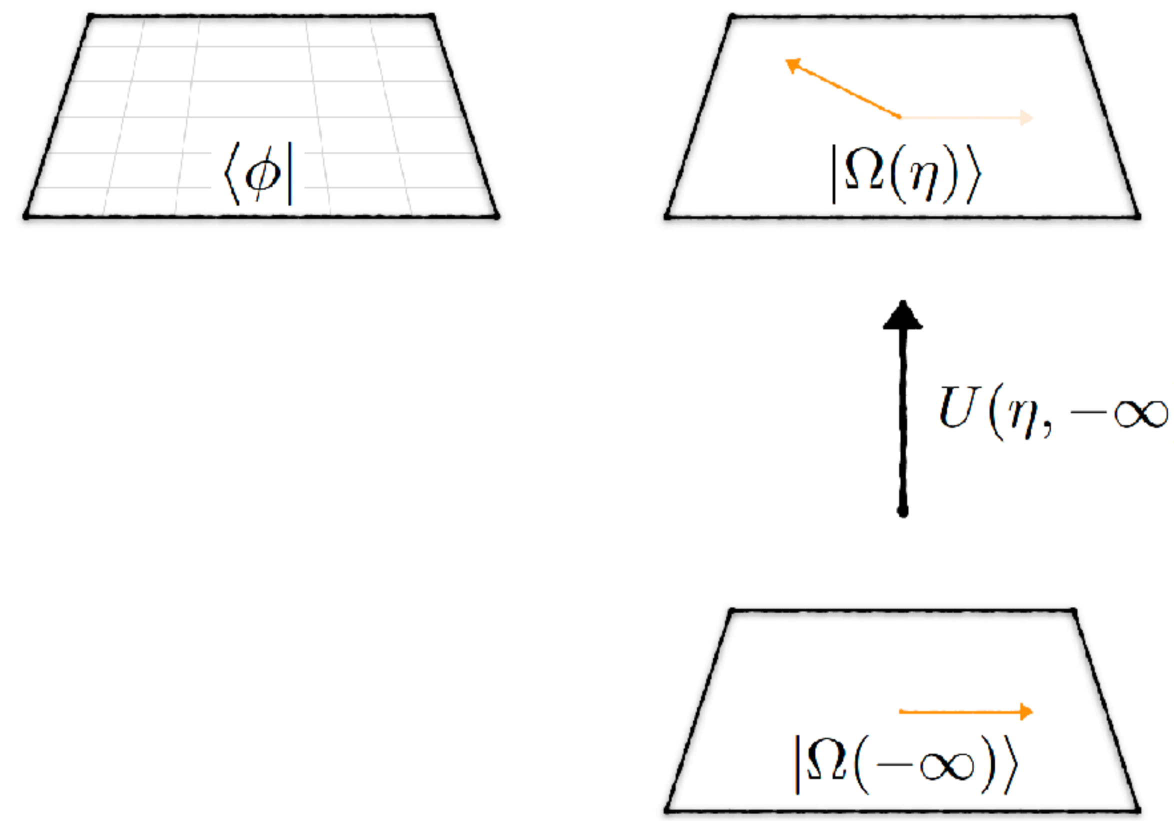
- It is a functional of the all fields in the theory (including the metric) at some time. It can be written in terms of wavefunction coefficients ψ_n
$$\Psi[\phi,\eta] = \exp\left[\sum_{n=2}^{\infty} \int_{\mathbf{k}} \psi_n(\mathbf{k}_1, \ldots, \mathbf{k}_n; \eta) \prod_{a}^{n} \phi(\mathbf{k}_a)\right]$$

- The wavefunction coincides with the large volume limit of the wavefunction of the universe in canonical quantum gravity, which solves the Wheeler de Witt equation.

The wavefunction

$$\frac{\text{wavefunction}}{\text{correlators}} \sim \frac{\text{amplitudes}}{\text{cross sections}}$$





semiclassical picture

Quantum picture

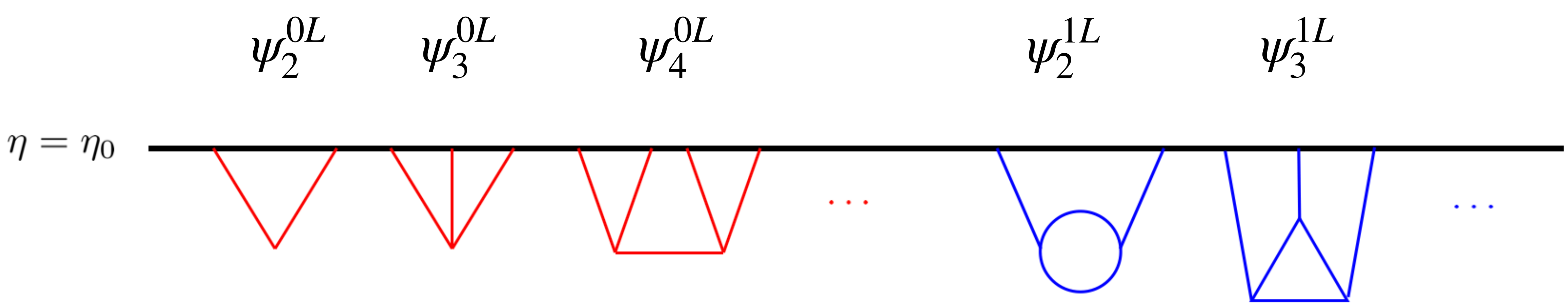
Feynman Diagrams

- Given a model, the wavefunction can be computed perturbatively from a path integral

$$\Psi[\varphi] = \int_{\phi(-\infty)=\mathrm{BD}}^{\phi(\eta_0)=\varphi} \mathscr{D}\phi\, e^{iS[\phi]} = e^{iS[\phi_{\mathrm{cl}}]} \times \int_{\delta\phi(-\infty)=\mathrm{BD}}^{\delta\phi(\eta_0)=0} \mathscr{D}\delta\phi\, e^{i\left(S[\phi_{\mathrm{cl}}+\delta\phi]-S[\phi_{\mathrm{cl}}]\right)}$$

classical saddle
trees

quantum contributions
loops



From Ψ to correlators

- Correlators can be computed from Ψ as in quantum mechanics via the Born rule

$$\langle \mathcal{O} \rangle = \frac{\int d\phi \Psi^*[\phi]\, \mathcal{O}\, \Psi[\phi]}{\int d\phi\, |\Psi[\phi]|^2}$$

- Introduce the real and imaginary part: $\psi_n(\mathbf{k}) = \frac{1}{2}\rho_n(\mathbf{k}) + i\gamma_n(\mathbf{k})$
- Then in perturbation theory, one finds: $\langle \phi(\mathbf{k})\phi(-\mathbf{k})\rangle' = -\frac{1}{\rho_2(\mathbf{k})} + \ldots$ and

$$\langle \prod^n \phi(\mathbf{k}_a)\rangle \propto \frac{\psi_n(\mathbf{k}) + \psi_n^*(-\mathbf{k})}{\prod^n \rho_2(\mathbf{k}_a)} + \ldots = \begin{cases} \dfrac{\rho_n}{\prod^n \rho_2(\mathbf{k}_a)} + \ldots & \text{Parity Even} \\ \dfrac{i\gamma_n}{\prod^n \rho_2(\mathbf{k}_a)} + \ldots & \text{Parity Odd} \end{cases}$$

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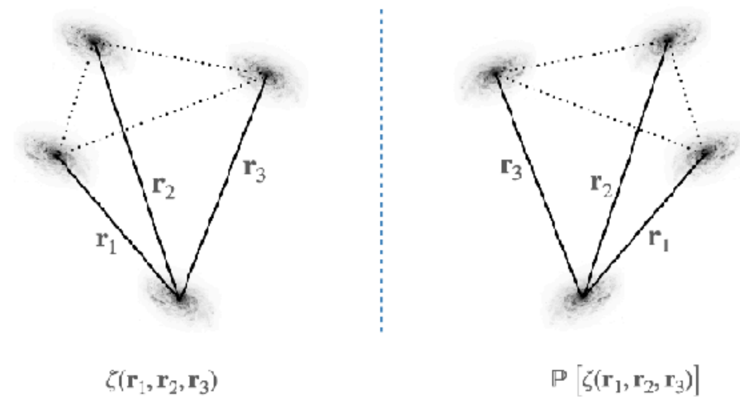
Wavefunction reality and parity-odd correlators

Reality and parity



- All wavefunction coefficients ψ_n^{0L} are real at tree-level! [Liu et al 19; Cabass, Jazayeri, EP & Stefanyszyn '22; Stefanyszyn, Tong & Zhu, 2023]

- This assumes external massless scalars or gravitons in exact dS with IR-finite interactions exchanging light fields in the complementary series ($m \leq 3H/2$), as in many models of inflation

- Reality can be seen directly from the perturbation expressions or from unitarity in the form of the Cosmological Optical Theorem (COT) [Goodhew, Jazayeri & Pajer '20].

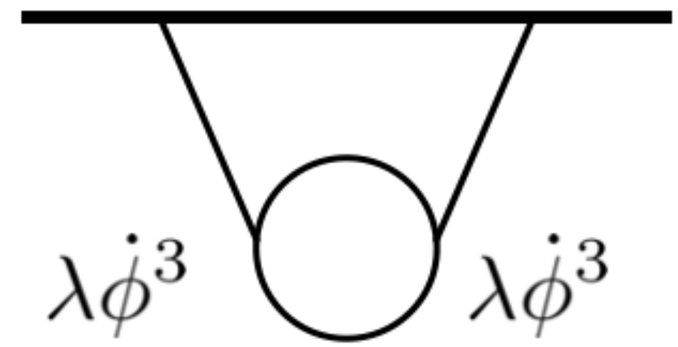
- Parity odd correlators: $B_4^{PO} \propto \dfrac{\psi_4(\mathbf{k}) + \psi_4^*(-\mathbf{k})}{\prod^4 \rho_2(\mathbf{k}_a)} \propto \mathrm{Im}\, \psi_4$

- Hence $B_4^{PO} = 0$ at tree level (despite ∞-many quartic PO interactions).

- Parity violation in cosmological data sets: Hints in the E-B monopole cross-correlation of the CMB have been found [Komatsu et al 2010s]; and in the galaxy trispectrum have been found but not in the CMB [Hou, Slepian Cahn 22; Philcox '22; 100's of papers]

- Counterterm renormalisation at 1-loop generates an imaginary part! Let's understand this better

The toy model



- For simplicity assume a single massless scalar with IR-finite interaction (results actually generalise to any number of intermediate light fields of any spin)

$$S = \int d^4x \, \sqrt{-g} \left[-\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{\lambda}{3\,!} (n^\mu \partial_\mu \phi)^3 \right] \; .$$

- Since $\dot{\phi}^3 \to 0$ as $\eta \to 0$, all time-integrals are IR finite (and we can Wick rotate :).
- There are UV divergences since $\dot{\phi}^3$ is dim 6. Once all such UV-divs are renormalised in the wavefunction, no more UV-divs appear in correlators.
- Regularisation methods: dimensional regularisation (dim reg), mass and dimensional regularisation (m&d reg), η-regulators
- To gain intuition we will compute ψ_2^{1L}, generalising to all n and all loops at the end. All regularisations give the remarkable result (μ is the renormalisation scale)

$$\boxed{\widehat{\psi}\,_2^{1\mathrm{L}}(k, \eta = 0) = \frac{1}{15} \frac{\lambda^2 H^2}{(4\pi)^2} k^3 \left(\frac{i\pi}{2} + \log \frac{\mu}{H} \right)}$$

- **The renormalised wavefunction acquired a universal finite imaginary part!**

Summary

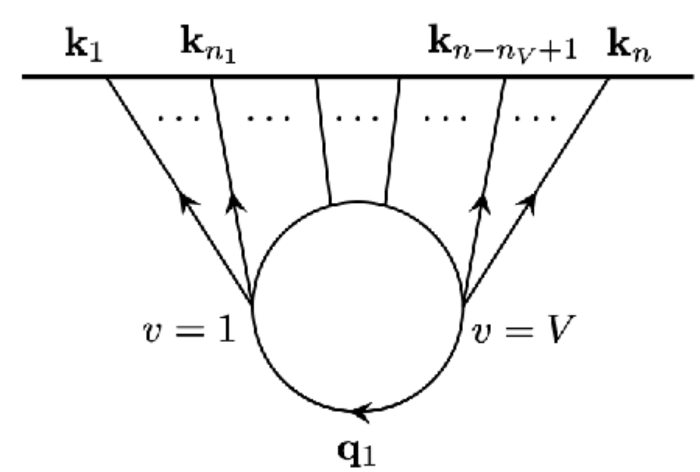
Regularisation schemes	Scale inv.	Im. part	Time-indep.
Dim reg	✗ → ✓	$0 \to +\pi/2$	✓
$m\&d$ reg	✓	$+\pi/2$	✗ → ✓
η reg (unitary & analytic)	✓	$+\pi/2$	✓

- Different renormalisation methods agree but with very different intermediate steps. This suggests there is a more direct derivation based on unitarity
- Not only Im$\widehat{\psi_2} \neq 0$, but also Im$\widehat{\psi_2}$ is fixed by the $\log \mu$ on the renormalisation scale μ. This is the *universal part* of renormalisation, commonly giving RG running.
- The same derivation generalises this to all 1-loop diagrams

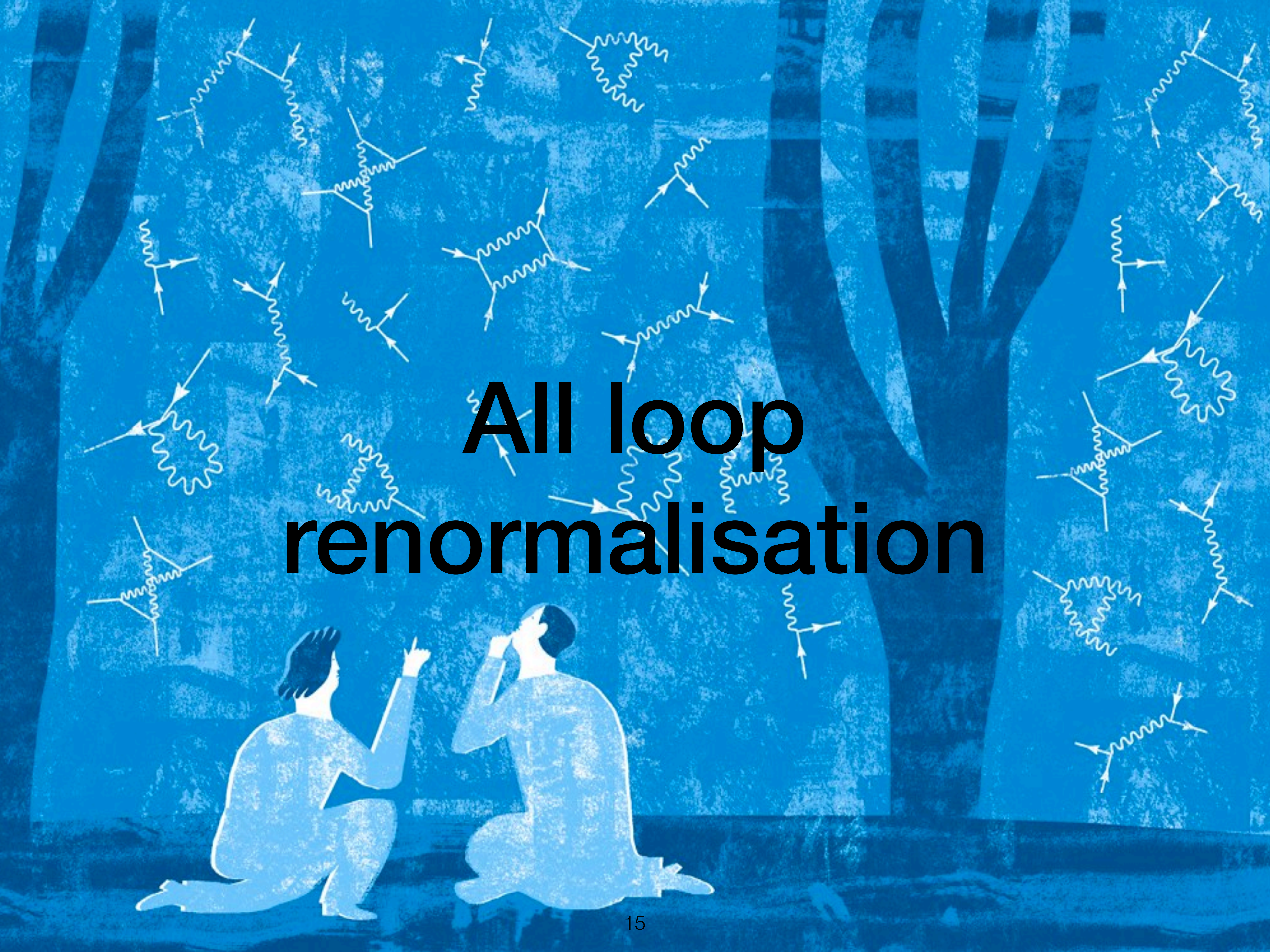
$$\widehat{\psi}_n^{1L}(\{k\},\{s\},\{\mathbf{k}\}) = f_0(\{k\},\{s\},\{\mathbf{k}\})\left(i\frac{\pi}{2} + \log\frac{\mu}{H} + g(\{k\},\{s\},\{\mathbf{k}\})\right)$$

- More suggestively, we find a 1-loop universality relation

$$\boxed{\left(\mu\frac{\partial}{\partial\mu} - \frac{2}{\pi}\,\mathrm{Im}\right)\widehat{\psi}_n^{1L} = 0}$$



- This is very reminiscent of the Renormalization Group



All loop renormalisation

All-loop renormalisation

- This 1-loop universality raises many questions. Here I focus on: *what happens at higher loops*?

- We found that *the imaginary part is fixed to all loops in terms of the* $\log \mu$ *dependence* (here $a_l \in \mathbb{R}$ depend on kinematics)

$$\widehat{\psi}_n^{(L)}(\{\mathbf{k}_i\}) = \sum_{\ell=0}^{L} a_\ell(\{\mathbf{k}_i\}) \left(\log \frac{\mu}{H} + \frac{i\pi}{2}\right)^\ell$$

- Assumptions: massless external scalars (or gravitons) in dS and internal light fields of any spin in the complementary series ($m \leq 3H/2$)

- This can also be written in the suggestive form

$$\boxed{\mathrm{Im}\,\widehat{\psi}_n = \tan\left(\frac{\pi}{2}\mu\partial_\mu\right) \mathrm{Re}\,\widehat{\psi}_n}$$

- At tree level there is no μ-dependence and we recover tree-level reality Im $\psi_n^{0L} = 0$

- Expanding the $\tan$ to linear order we recover the 1-loop universality relation

Derivation in dim reg

- The derivations computes loops and counterterms using Feynman rules

$$\psi_n^{(L)} = \int_{\mathbf{q}_1 \cdots \mathbf{q}_L} \int_{-\infty}^{0} \left[\prod_{v=1}^{V} d\eta_v V_v \right] \left[\prod_{e=1}^{n} K_e \right] \left[\prod_{\mathrm{i}=1}^{I} G_{\mathrm{i}} \right]$$

- Then we Wick rotate time $\eta = iz$ and use known analytic properties of the propagators K and G. Tracking the imaginary part and the μ-dependence gives

$$\widehat{\psi}_n^{(L)}(\{\mathbf{k}_i\}) = \sum_{\ell=0}^{L} a_\ell(\{\mathbf{k}_i\}) \left(\log \frac{\mu}{H} + \frac{i\pi}{2} \right)^{\ell}$$

- Our result is confirmed by a double Wick rotation to Euclidean Anti-de Sitter (EAdS)
  $\eta \to iz$ and $H^{-1} = L_{\text{dS}} \to -iL_{\text{AdS}}$,
  (only this rotation is compatible with the Witten-Kontsevich-Segal criterion, $\sum |g_{\mu\mu}| < \pi$) which gives a purely real partition function in EAdS

Correlators

- Our wavefunction relations imply ∞-many correlators relations. Here are two 1-loop examples assuming parity.

- Recall our definition: $\widehat{\psi}_n \equiv \widehat{\rho}_n/2 + i\widehat{\gamma}_n$

- Define the correlators of the field φ and its momentum conjugate Π

$$\widehat{B}_2(k) \equiv \langle \varphi_{\mathbf{k}} \varphi_{-\mathbf{k}} \rangle' \,, \qquad \widehat{C}_2(k) \equiv \langle \{ \varphi_{\mathbf{k}}, \Pi_{-\mathbf{k}} \} \rangle' \,,$$
$$\widehat{B}_3(k_1, k_2, k_3) \equiv \langle \varphi_{\mathbf{k}_1} \varphi_{\mathbf{k}_2} \varphi_{\mathbf{k}_3} \rangle' \,, \qquad \widehat{C}_3(k_1, k_2; k_3) \equiv \langle \varphi_{\mathbf{k}_1} \varphi_{\mathbf{k}_2} \Pi_{\mathbf{k}_3} \rangle' \,.$$

- The Born rule gives at 1-loop

$$\widehat{B}_2^{(0)} = -\frac{1}{\widehat{\rho}_2^{(0)}}, \qquad \widehat{C}_2^{(0)} = -\frac{\widehat{\gamma}_2^{(0)}}{\widehat{\rho}_2^{(0)}} \,, \qquad \widehat{B}_2^{(1)} = \frac{\widehat{\rho}_2^{(1)}}{\left(\widehat{\rho}_2^{(0)}\right)^2} + \text{classical loops} \,, \qquad \widehat{C}_2^{(1)} = -\frac{\widehat{\gamma}_2^{(1)}}{\widehat{\rho}_2^{(0)}}$$

where "classical loops" are finite and so independent of μ

- Inverting these expressions and using the universality relation we find

$$\mu \partial_\mu \widehat{B}_2^{(1)}(k) = \frac{4}{\pi} \widehat{C}_2^{(1)}(k)\, \widehat{B}_2^{(0)}(k) \,, \qquad \mu \partial_\mu \widehat{C}_2^{(1)}(k) = 0$$

Relation to amplitudes

- One should complain that all we computed so far does not depend on kinematics. How can I see the μ dependence other than creating a new dS universe with a different cosmo constant??

- Our results have close analogs in amplitudes. For example, for renormalised massless 2-2 amplitude at $|s| \sim |t| \sim |u| \gg \mu^2$, it is believed that ($m_L$ real and independent of kinematics)

$$\mathscr{M}_4^{(L)} = \sum_{0 \le k_1, k_2, k_3 \le L} m_{L,\{k_1,k_2,k_3\}} \log^{k_1}\left(\frac{\mu^2}{-s}\right) \log^{k_2}\left(\frac{\mu^2}{-t}\right) \log^{k_3}\left(\frac{\mu^2}{-u}\right)$$

- Since the wavefunction contains flat space amplitudes at some of its poles, we expect the μ -dependence to be accompanied by kinematics, probably partial energies (work in prog.)

- We noticed that unitarity constrains the final result, which emerged after many cancellations. In [Chavda, MacLoughlin, Mizera & Staunton '25] it was shown that, for $\lambda\phi^4$ in $D=4$, the renormalisation of 2-2 amplitudes to all loops and LO+NLO is fixed by the Opt. Theorem. We suspect the same can be true for $\widehat{\psi_n}$ from the Cosmo Opt. Theorem.

- Even earlier, in [Caron-Huot & Wilhelm 16], similar expression to ours was found for form factors of composite operators. They used unitarity to derive anomalous dimensions and β-functions

Outlook

- The cosmological wavefunction is related to correlators that are probed by cosmological surveys

- For massless scalars and gravitons in dS, it must be real at tree-level. Quantum loops generate non-vanishing contributions that are completely fixed in terms of the renormalisation scale

- Using counter-term renormalisation to all orders we found

$$\text{Im}\, \widehat{\psi}_n = \tan\left(\frac{\pi}{2}\mu\partial_\mu\right)\ \text{Re}\, \widehat{\psi}_n$$

- This result makes no reference to the perturbative expansion used to derive it. It looks like some exact RG equation.

- Q: Can this be derived from a new form of exact RG for the wavefunction?