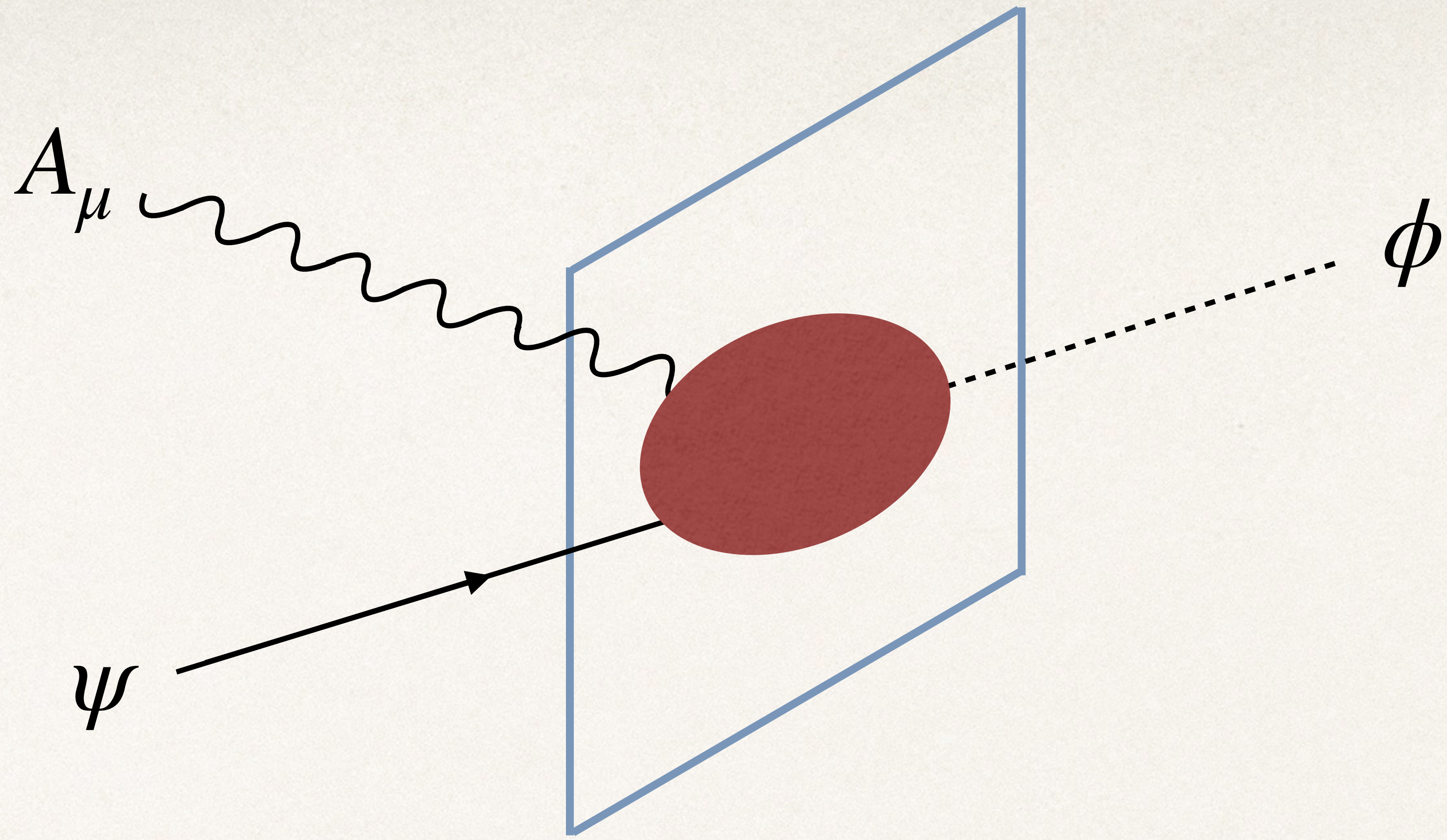

Defect CFTs that are free in the bulk: status report

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A Prologue

The game of the large N and small ϵ expansions.
One way to play is to start with a classically marginal interaction in d dimensions…

$$\int d^6x\,\phi^3 \qquad \int d^4x\,\phi^4 \qquad \int d^3x\,\phi^6 \qquad \int d^2x(\bar{\psi}\psi)^2$$

$$\int d^4x\,\phi\bar{\psi}\psi \qquad \int d^3x\,\phi^2\bar{\psi}\psi \qquad \text{or even} \int d^dx\,J^\mu A_\mu$$

… but maybe this can get boring or maybe it's hard to find new problems or maybe it's hard to find easy problems.

Defects and Boundaries

One way to change the game is to add a defect or boundary. This is a flat, codimension one or greater space sitting inside or adjacent to the ambient spacetime. Fixed points can possess conformal symmetry and there are condensed matter applications — Kondo problems, graphene…

In 4d alone, one can now consider the classically marginal interactions

$$\int d^p x \, \phi^p \ , \quad p = 1,2,3 \qquad\qquad \int d^3 x \, \bar{\psi} \psi$$

But wait, there's more. We can introduce degrees of freedom on the defect…

Coupling to Stuff on the Defect

- Maxwell in the bulk: $\int \mathrm{d}^p x\, J^\mu A_\mu$ where J^μ is a conserved current on the defect.
- Codimension two defect with fermions: $g\int \mathrm{d}^{d-2}x\,(\bar{\psi}\chi + \bar{\chi}\psi)$ where ψ is a bulk spinor and χ is a defect spinor.
- 4d/3d systems with scalars: $g\int \mathrm{d}^3 x\,(\partial_\perp \Phi)\phi^2$ where Φ is a bulk scalar and ϕ is a boundary scalar.

Potentially an incredibly rich system of defect QFTs that are under perturbative control…

Specializing to free bulk fields and conformal fixed points

What can we learn if the bulk is free (and massless)?

$$\Box \langle \phi(x) \cdots \rangle = 0$$

(Behan, di Pietro, Lauria, Liendo, van Rees, Zhao '20, '20, '21)

$$\gamma^\mu \partial_\mu \langle \psi(x) \cdots \rangle = 0$$

(work in progress)

$$\partial_\mu \langle F^{\mu\nu}(x) \cdots \rangle = 0$$

(Herzog, Huang '17;
di Pietro, Gaiotto, Lauria, Wu '19;
Bartlett-Tisdall, Herzog, Schaub '23; Herzog, Shrestha '22)

Free Field Lessons: Constraints on the CFT data

The free bulk can give us an exactly controlled sector of an otherwise interacting RG fixed point.

- $\langle \phi(x) \hat{O}(y) \rangle$: constraints on bulk-defect correlation functions
  - anomalous dimensions of $\hat{O}(y)$
- $\langle \phi(x) \hat{O}(y) \hat{O}'(y') \rangle$: constraints on bulk-defect-defect correlation functions
  - anomalous dimensions of $\hat{O}(y)$ or three-point function coefficient constraints

Where do these constraints come from?

Boundary Theory for Scalars

Let's take $d = 4$, $p = 3$ and the free bulk scalar Φ.

This is a boundary theory. We have a single Klein-Gordon equation to solve with two boundary conditions.

The boundary values are two defect operators, $\Phi|_{\text{bry}} = \phi$ and $\partial_\perp \Phi_{\text{bry}} = \partial_\perp \phi$ with $\Delta = 1$ and 2 respectively.

Claim: $\Box \Phi = 0$ implies the only nonzero bulk-defect two point functions with Φ are of the form $\langle \Phi \phi \rangle$ and $\langle \Phi \partial_\perp \phi \rangle$.

The Shadow Transform

From a classical point of view, if we know $\Phi|_{\text{bry}} = \phi$ at the boundary and have a regularity condition in the interior, we can figure out $\partial_\perp \Phi|_{\text{bry}} = \partial_\perp \phi$ (Dirichlet-Neumann map) and vice versa.

From a CFT point of view, this is a shadow transform. Given $\langle \phi \ \cdots \rangle$, we can figure out $\langle \partial_\perp \phi \ \cdots \rangle$ and vice versa.

Schematically $$\langle \phi(x) \ \cdots \rangle \sim \int \mathrm{d}^3 y \langle \phi(x) \phi(y) \rangle \langle \partial_\perp \phi(y) \ \cdots \rangle \ .$$

$$= \int \mathrm{d}^3 y \frac{1}{|x-y|^2} \langle \partial_\perp \phi(y) \ \cdots \rangle$$

Where does this come from?

Trade the bulk Φ for its nonlocal boundary version ϕ.
Formally, we can write the relevant part of the 3d action

$$\int \mathrm{d}^3x \left(\frac{1}{2}\phi\mathscr{D}\phi + g\phi(\partial_\perp\phi + \ldots) \right)$$

operators that mix with $\partial_\perp\phi$

where formally $\mathscr{D} = \sqrt{-\Box}$ gives a nonlocal kinetic term and the equation of motion $\mathscr{D}\phi + g(\partial_\perp\phi + \ldots) = 0$ is the origin of this shadow transform.

Generically we get relations between for example 3-point functions.

Dirichlet/Neumann

$$\langle \phi(x)\ \cdots\rangle \sim \int \mathrm{d}^3y \frac{1}{|x-y|^2}\langle \partial_\perp \phi(y)\ \cdots\rangle\ .$$

D boundary conditions $\phi = 0 \implies$ 2 possibilities

- $\langle \partial_\perp \phi(y)\ \cdots\rangle = 0$
- shadow transform kills it $\implies$

for 3-pt fns, only $\langle \partial_\perp \phi(x)\, \hat{O}(y) :\!\partial_\perp \phi \,\Box^n\, \hat{O}(z)\!:\rangle$ are nonzero

There is an inverse transform $\langle \partial_\perp \phi(x)\ \cdots\rangle \sim \int \mathrm{d}^3y \frac{\langle \phi(y)\ \cdots\rangle}{|x-y|^4}\ .$

Can run the same argument in reverse for N boundary conditions

This was just a boundary theory. What about defect theories?

Convert the General Defect Problem to a Boundary Problem

If we are interested in CFT fixed points, then $\mathbb{R}^d$ with a $\mathbb{R}^p$ defect, $p < d$, can be Weyl rescaled to $AdS_{p+1} \times S^{d-p-1}$ where AdS is anti-de Sitter space, S^p is a p-dimensional sphere, and the defect lives at the boundary of AdS.

We can make a kind of Kaluza-Klein reduction on the sphere. The free field, e.g. the scalar, breaks up into a tower of massive spinning fields with different values of angular momentum on the sphere.

We are left with a boundary field theory problem in AdS_{p+1}.

Looking at the Bottom of the Tower

The higher the transverse angular momentum of the KK mode, the higher the mass of the field in AdS. Each field in AdS satisfies a massive free field equation with two possible asymptotic behaviors. The boundary conditions have the reinterpretation as defect operators on the boundary.

Unitarity forces us to choose Dirichlet boundary conditions for almost every member of the tower. (Gets rid of operators with negative scaling dimension!)

Interesting stuff happens at the bottom of the tower, where the dimensions are still RG relevant and there can be two available boundary conditions/ operators above the unitarity bound.

Broader Landscape

Free Maxwell Field, 4d/3d setup

The boundary values of $F_{\mu\nu}$ are conserved currents.

$$J_a^{(1)} = F_{\perp a} \qquad\qquad J_a^{(2)} = \frac{1}{2}\epsilon_a{}^{bc} F_{bc}$$

The shadow transform relates correlation functions of these currents. Current two-point functions control conductivities and so there are constraints on conductivities and transport.

di Pietro, Gaiotto, Lauria, Wu ’19;
Bartlett-Tisdall, Herzog, Schaub ’23

Free Maxwell Field, 4d/2d setup

We have only the Dirichlet boundary condition available.

If we want the correlation functions involving $F_{\mu\nu}$ not to vanish, then we need to restrict the operator dimensions for three point functions to be in the kernel of the shadow transform.

Let $\hat{\mathscr{W}}$ have a nonzero two-point function with $F_{\mu\nu}$. Let $\hat{O}$ be another defect operator. Then the only nonzero three-point functions look schematically like

$$\langle \hat{\mathscr{W}}(x)\hat{\mathscr{O}}(y) :\hat{\mathscr{W}}\partial^n\hat{\mathscr{O}}(z):\rangle$$

Herzog, Shrestha '22

Free Fermion, 4d/2d setup

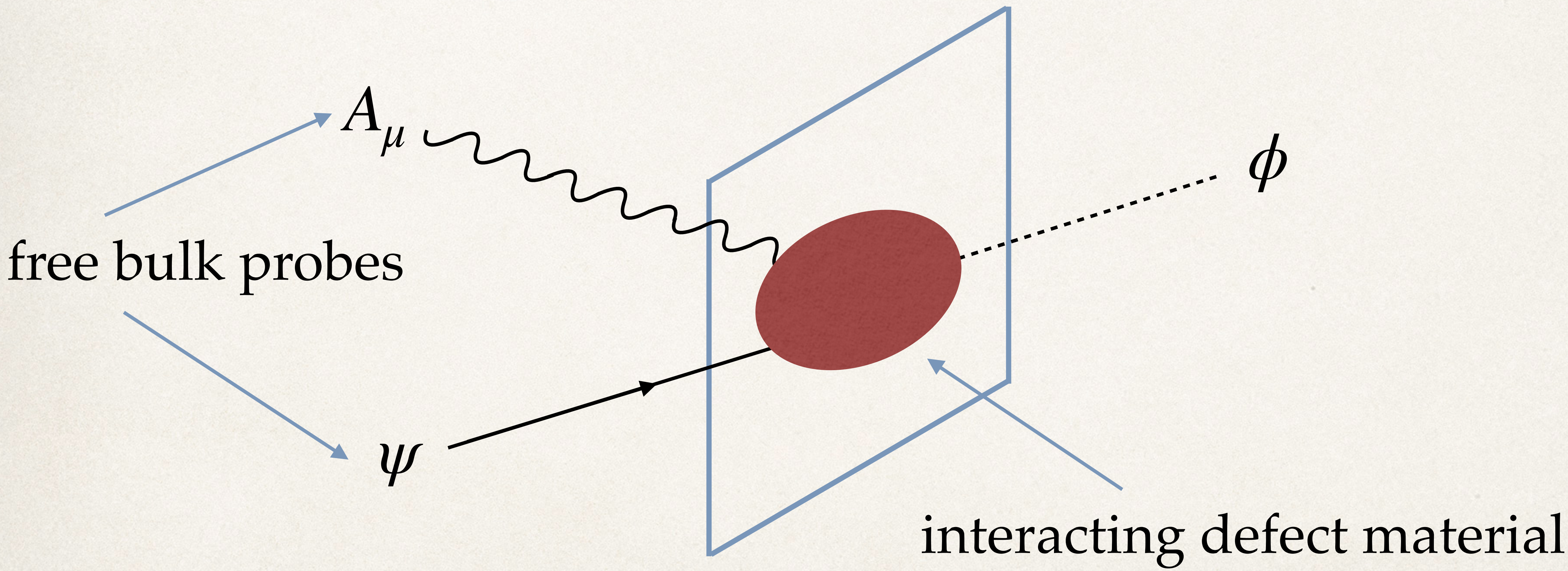
In the standard case, only Dirichlet boundary conditions are available. However, using the Aharonov-Bohm effect, we can thread the surface defect with a magnetic field and shift the scaling dimensions at the bottom of the KK tower to get a shadow pair

$$\Delta_d = \frac{1}{2} + \nu \qquad\qquad \Delta_u = \frac{3}{2} - \nu$$

(Bianchi, Chalabi, Prochazka, Robinson, Sisti '21; Herzog, Lloyd-Jones in progress)

Next Steps: A Theoretical Experiment

Can we use defect QFT to describe the material and the experiment used to interrogate it within the same theoretical framework?



what is the response?

Summary

- Strong constraints on free-in-the-bulk defect CFTs that can be understood using a shadow transform.
- Emerging picture for the free-in-the-bulk scalar, Maxwell field, and now the fermion.
- Theoretical experiment: build the measuring apparatus into the CFT or QFT

Extra Slides

The CFT data

- Two and three point correlation functions are fixed up to numbers.
- A CFT is defined by its spectrum of operator dimensions (critical exponents) and three-point function coefficients (response functions)
- With a defect, there are both bulk and defect operators, each with their own spectrum and three-point function coefficients, as well as defect-bulk coefficients (boundary conditions for bulk fields)

Looking at the Bottom of the Tower

For Φ, $(mL)^2 = \Delta(\Delta - p)$ where m is the mass (determined by the spin) and L is the radius of AdS. Importantly m^2 can be negative in AdS, but already for $m^2 > 0$, the smaller solution for Δ is below the unitarity bound $\dfrac{p-2}{2}$ for scalar operators and the larger Δ is RG irrelevant.