Weak to strong wave turbulence: RG, epsilon expansion, and large N

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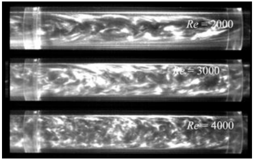
V. R., N. Vladimirova, G. Falkovich, “Universal regimes of strong turbulence in the multi-component Gross-Pitaevskii model”, arXiv: 2511.14068, *Commun. Phys., to appear*

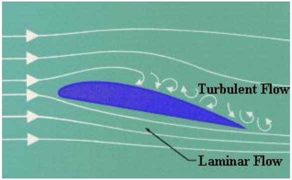
V. R., M. Smolkin, “Renormalization Group in far-from-equilibrium states”, arXiv:2504.06243, *Phys. Rev. D*

Turbulence

Turbulence is about redistribution of energy in nonlinear systems with many interacting degrees. The most familiar case is fluids.

Quantity of interest is moments of velocity $\langle(\delta v)^n\rangle$, which are related to transport coefficients



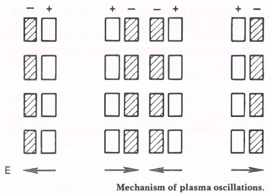


Wave turbulence

Turbulence is not confined to fluids.

Can have collective excitations — waves. Want to understand redistribution of energy among them. Goal is to find the second moment—n_k: number of waves of wavenumer k

Ocean waves



Langmuir waves.

Weak and strong wave turbulence

Weak

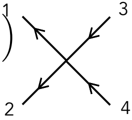
Strong

What is n_k?

Weak turbulence: kinetic theory

The quantum Boltzmann equation:

$$\frac{\partial n_1}{\partial t} = \sum_{p_2, p_3, p_4} |\mathcal{A}|^2 \Big((n_1+1)(n_2+1) n_3 n_4 - (n_3+1)(n_4+1) n_1 n_2 \Big)$$
$$\delta(\omega_{p_1} + \omega_{p_2} - \omega_{p_3} - \omega_{p_4}) \delta(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4)$$



The $n_k \gg 1$ limit is the wave kinetic equation

Kolmogorov-Zakharov cascade: $n_k \sim \left(\frac{Q}{\lambda^2}\right)^{1/3} k^{-\gamma}$. A stationary solution of the kinetic equation that carries constant flux Q.
$Q(k) \equiv \int^k d\vec{p}\, \frac{\partial n_p}{\partial t}$.

Strong turbulence: critical balance

No theory. *Heuristically* claim nonlinear term must saturate and not exceed linear term. [Goldreich & Sridhar, '95]

E.g., Nonlinear Schrödinger equation:

$$H = \int d\mathbf{r} \left[|\nabla\psi|^2 + \lambda|\psi|^4 \right]$$

Set,

$$1 \sim \frac{|\psi|^4}{|\nabla\psi|^2} \sim \frac{(n_k k^d)^2}{k^2(n_k k^d)} , \quad \Rightarrow n_k \sim k^{2-d}$$

where $n_k \equiv \langle |\psi_k|^2 \rangle$.

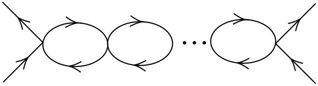
A kinetic theory of weak and strong turbulence

To do strong turbulence: Take the large N limit: $\psi \to \vec{\psi}$ having $N \gg 1$ components

$$\mathcal{H} = \int d\mathbf{r} \left[|\nabla \vec{\psi}|^2 + \frac{\lambda}{N} (\vec{\psi}^* \cdot \vec{\psi})^2 \right] ,$$

[R. Kraichnan, '67], [R. Walz, K. Boguslavski, J. Berges, '17], [Chantesana, A. Pineiro Orioli, and T. Gasenzer, '18];

The quasiclassical approximation is still true.
We sum just the bubble diagrams.

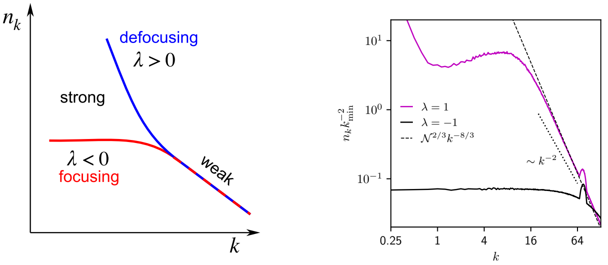


This gives an effective (renormalized) interaction:

$$\mathcal{A} = \frac{\lambda}{1 - \lambda \Pi^R(\omega, \mathbf{k})} , \quad \Pi^R(\omega, \mathbf{k}) = \int \frac{d^d q}{(2\pi)^d} \frac{n_{\mathbf{q}} - n_{\mathbf{k}-\mathbf{q}}}{\omega - q^2 + (\mathbf{k}-\mathbf{q})^2 + i\epsilon}$$

where $\mathbf{k} = \vec{p}_4 - \vec{p}_2$ and $\omega = \omega_{p_4} - \omega_{p_2}$

The renormalized interaction can get either weaker or stronger with scale, which causes n_k to get larger or smaller



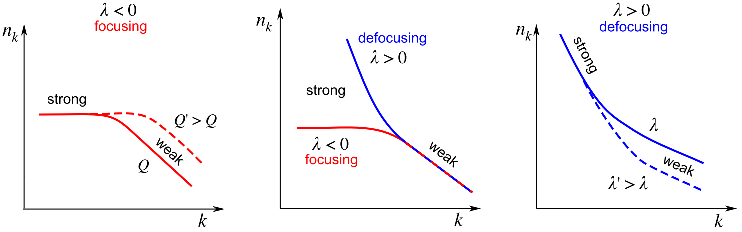
Sketch on the left; Direct Numerical Simulation ($N = 1$) on the right.

Strong nonlinearity: For $\lambda > 0$ drop the 1; For $\lambda < 0$ set $\mathcal{L} \approx 1$.

$$\mathcal{A} \sim \begin{cases} k^{-2} \\ k^{2} \end{cases} \quad n_k \sim \begin{cases} \lambda^{-1}, & \lambda < 0 \\ \frac{Q^{1/3}}{\mathcal{N}^{2/3}} k^{-8/3}, & \lambda > 0 \end{cases}$$

Weak to strong: a sketch

Set up an inverse cascade (of wave action), in $2d$. Nonlinearity grows with increasing scale (small k)



Weak (large k):

$$n_k = (Q/\lambda^2)^{1/3} k^{-2}$$

Strong (small k)

$$n_k \sim \begin{cases} \lambda^{-1}, & \lambda < 0 \text{ (focusing)} \\ \frac{Q^{1/3}}{\mathcal{N}^{2/3}} k^{-8/3}\ , & \lambda > 0 \text{ (defocusing)} \end{cases}$$

Universality emerges in strong turbulence: no flux dependence for focusing, no coupling dependence for defocusing.

Strong wave turbulence

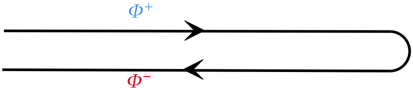
Problem: How to get the critical balance scaling for an inverse cascade in nonlinear Schrödinger in the IR analytically?

This an RG problem, but it is with a relevant interaction (not close to marginal).

QFT in excited states

The path integral naturally computes in-out correlation functions, not expectation values.

Solved with the Keldysh contour.



$$\mathcal{L} = \mathcal{L}(\psi^+) - \mathcal{L}(\psi^-)$$

After a field rotation, $\psi = \frac{1}{2}(\psi^+ + \psi^-)$, $\eta = \psi^+ - \psi^-$

$$\mathcal{L} = \eta^* \text{ e.o.m} \ , \qquad \text{e.o.m} = (i\partial_t - \nabla^2 - \lambda|\psi|^2)\psi$$

$$G^K_{k,\omega} = n_k \delta(\omega - k^2) \ , \qquad G^R_{k,\omega} = \frac{i}{\omega - k^2 + i\epsilon}$$

The state $n_k \sim k^{-\gamma}$ is something you pick a posteriori, so that there is a stationary solution.

Marginal case

$$H = \int d^d x \left(|\nabla^2 \psi|^2 + \lambda |\nabla^2 \psi|^4 \right) \ .$$

Weak turbulent state is

$$n_k = k^{-d-4}$$

Interaction is marginal in this state. The beta function is,

$$\mu \frac{d\lambda}{d\mu} = 2 S_d (d+4) \lambda^2$$

Positive coupling $\Rightarrow$ coupling decreases in the IR — the interaction is marginally irrelevant. Means the one-loop corrections make n_k steeper than the background state $n_k \sim k^{-\gamma}$.

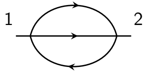
Negative coupling $\Rightarrow$ coupling increases in the IR — the interaction is marginally relevant. Means the one-loop corrections make n_k less steep than the background state $n_k \sim k^{-\gamma}$.

Random coupling (SYK) model

Take N complex scalar fields $\Psi_i(x)$,

$$\mathcal{L} = \Psi_i^*(i\partial_t + \nabla^2)\Psi_i - J_{ijkl}\Psi_i^*\Psi_j^*\Psi_k\Psi_l$$

Sums melons



The spectral density will no longer be $\delta(\omega - k^2)$. Therefore, the kinetic equation for $n_k(t)$ involves the propagator at earlier times.

Assume scale invariance in the IR

$$G^K(\mu^{-z}\tau, \mu k) = \mu^{-\gamma} G^K(\tau, k) \ ,$$

Using $\Sigma = G^3$, and looking for a stationary solution of kinetic equation, gives $\gamma = d$ and $z = 0$ [V.R. & Schubring]

Summary

- We set up an inverse turbulent cascade, pump in UV, dissipate in IR
- In the UV, the coupling is weak — know the state
- In the IR, numerically know it (for 2d): critical balance. Found it from large N
- Can also find state for marginal/nearly-marginal interactions. And for SYK-like models

Nonlinear Schrödinger

Problem: How to get the critical balance scaling for an inverse cascade in nonlinear Schrödinger in the IR analytically? (in 2d)

Two issues:

1. Have to do RG for a relevant interaction. Assume FRG solves this.
2. Our framework has been to do the calculation by placing ourselves in a state that is stationary, which we find a posteriori. This only works if the state is close to Gaussian.

To solve 2), just do what an experiment does — add stochastic pumping in the UV, dissipation in the IR. This adds an $|\eta_k|^2$ term to the action. While doing RG, the pumping will get transmitted to lower scales.