Stochastic observables and Wilson–Itô diffusions

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STOCHFIELDS
Stochastic analysis of quantum fields

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Euclidean QFT turns geometry into local random data

A Euclidean field is typically a random distribution

$$\varphi \in \mathcal{S}'(\mathbb{R}^d), \qquad \langle \mathcal{O} \rangle = \frac{1}{Z} \int \mathcal{O}(\varphi) e^{-S(\varphi)}\, \mathcal{D}\varphi \quad \text{(formally)}.$$

What makes this a *field theory* is its organization by regions. Writing $\mathcal{F}(U)$ for the information in U, take

$$U \longmapsto \mathcal{F}(U), \qquad \mathcal{A}(U) := L^\infty(\mathcal{F}(U)), \qquad U \subset V \Longrightarrow \mathcal{A}(U) \subset \mathcal{A}(V),$$

so $\mathcal{A}(U)$ contains the observables measurable inside U. For disjoint regions, they can be combined without resolving coincident points.

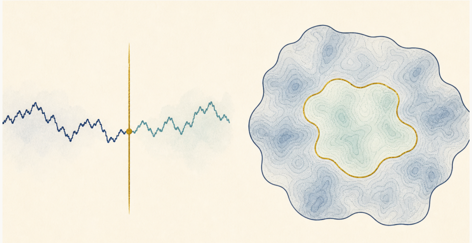
Local Euclidean QFTs extend the Markov property to space

Markov process: a time slice screens past from future.

$$\mathcal{F}_{<t} := \sigma\{X_s : s < t\}, \quad \mathcal{F}_{>t} := \sigma\{X_s : s > t\},$$
$$\mathcal{F}_{<t} \perp\!\!\!\perp \mathcal{F}_{>t} \;\Big|\; \sigma(X_t).$$

Markov field: a boundary screens inside from outside.

$$\mathcal{F}_{\partial D} := \bigcap_{\varepsilon>0} \mathcal{F}\big((\partial D)_\varepsilon\big),$$
$$\mathcal{F}(D) \perp\!\!\!\perp \mathcal{F}(D^{\mathrm{ext}}) \;\Big|\; \mathcal{F}_{\partial D}.$$



a time cut t a spatial cut ∂D

Here $\mathcal{F}_{\partial D}$ is the boundary germ (or trace) of the field.

UV consequence. In non-degenerate local models, fluctuations at every scale accumulate into a rough field.

Fresh Markov fluctuations at every scale force UV divergences

Nested domain-Markov decompositions of the GFF give independent scale increments:

$$\varphi_\varepsilon(x) \simeq \sum_{k=0}^{N(\varepsilon)} \eta_k(x), \qquad \eta_0, \eta_1, \ldots \text{ independent.}$$

Therefore the variances add:

$$C_\varepsilon(0) = \mathbb{E}\big[\varphi_\varepsilon(x)^2\big] = \sum_{k \le N(\varepsilon)} \operatorname{Var}\big(\eta_k(x)\big) \longrightarrow \infty, \qquad C_\varepsilon(0) \sim c \log(\varepsilon^{-1}) \quad (d = 2).$$

Away from coincidence: finite.

$$C_\varepsilon(x - y) \longrightarrow C(x - y), \qquad x \neq y.$$

Separated products have a finite limit.

At coincidence: compensate.

$$\varphi_\varepsilon(x)^2 = C_\varepsilon(0) + :\varphi_\varepsilon^2:(x).$$

The divergent term is supported at $x = y$.

Why counterterms are local

Markov increments create the divergence; coincidence confines it, so its counterterm is local.

Every observation carries an intrinsic resolution

Although the theory specifies a terminal random distribution φ_∞, every actual probe has a finite scale: detector width, lattice spacing or localization length.

Varying this unavoidable scale produces the globally filtered process $\left(\varphi_a, \mathcal{F}_a\right)_{a>0}$, where

$$\mathcal{F}_a := \sigma\{\varphi_b(x) : b \leq a,\ x \in \mathbb{R}^d\}, \qquad \mathcal{F}_a \subseteq \mathcal{F}_b \quad (a \leq b).$$

Study the evolution in resolution

Study the drift, innovations and local dependence of the global filtered process—not only its terminal law.

Observables are martingales in resolution

Definition

A stochastic observable is an adapted process $\mathcal{O} = (\mathcal{O}_a)_{a>0}$ such that

$$\mathbb{E}[\mathcal{O}_b \mid \mathcal{F}_a] = \mathcal{O}_a, \qquad a \le b.$$

Hence $\langle \mathcal{O} \rangle := \mathbb{E}\mathcal{O}_a$ is independent of the resolution.

Locality is encoded by a *germ*: for some local functional $\mathcal{O}_a^*$,

$$\big\|\mathcal{O}_a - \mathcal{O}_a^*(\varphi_a)\big\|_{L^1} \longrightarrow 0, \qquad \operatorname{supp} \mathcal{O}_a^* \subset U_{1/a}.$$

Nonlinear locality + martingality forces renormalization

At coincident points, requiring a nonlinear local observable to be a martingale forces scale-dependent compensating counterterms.

Products exist when singularities cannot meet

Pointwise multiplication does not preserve martingales:

$$\mathrm{d}(\mathcal{O}^1_a\mathcal{O}^2_a) = \mathcal{O}^1_a\,\mathrm{d}\mathcal{O}^2_a + \mathcal{O}^2_a\,\mathrm{d}\mathcal{O}^1_a + \mathrm{d}\langle \mathcal{O}^1, \mathcal{O}^2\rangle_a.$$

If the supports are separated, the cross-variation is integrable and one may define

$$(\mathcal{O}^1 \bullet \mathcal{O}^2)_a := \lim_{b\to\infty} \mathbb{E}\Big[\mathcal{O}^1_b\mathcal{O}^2_b \mid \mathcal{F}_a\Big].$$

Separated regions

A canonical product and stable correlation functions.

Coincident regions

Extra local data: composite-operator renormalization.

Consistency across resolutions singles out martingales

At resolution $a \le b$, a finer cutoff expression N_b is accessible only through

$$\mathcal{O}_a^{(b)} := \mathbb{E}[N_b \mid \mathcal{F}_a].$$

Conditional expectation forgets ultraviolet scheme dependence:

$$\big\|\mathbb{E}[N_b \mid \mathcal{F}_a] - \mathbb{E}[\widetilde{N}_b \mid \mathcal{F}_a]\big\|_{L^1} \le \|N_b - \widetilde{N}_b\|_{L^1}.$$

Thus cutoff prescriptions with the same UV asymptotics give the same limit. If $\mathcal{O}_a = \lim_{b\to\infty} \mathcal{O}_a^{(b)}$ exists, the tower property gives

$$\boxed{\mathbb{E}[\mathcal{O}_b \mid \mathcal{F}_a] = \mathcal{O}_a, \qquad a \le b.}$$

Observables for the GFF

Let $X_a = \int_0^a \dot{C}_s^{1/2}\,\mathrm{d}W_s$ be a scale decomposition of the massive GFF, so $\mathbb{E}[X_a(x)X_a(y)] = C_a(x-y)$. One explicit heat-kernel cutoff is

$$C_a = (1-\Delta)^{-1}e^{-(1-\Delta)/a^2}, \qquad C_a(z) = \int_{\mathbb{R}^d} \frac{e^{ip\cdot z}e^{-(1+|p|^2)/a^2}}{1+|p|^2}\frac{\mathrm{d}^d p}{(2\pi)^d}.$$

Wick powers

$$:X_a^2:(x) = X_a(x)^2 - C_a(0),$$
$$\mathrm{d}:X_a^n:(x) = n:X_a^{n-1}:(x)\,\mathrm{d}X_a(x).$$

Wick exponentials

$$\mathcal{E}_a^r(x) = \exp\left(rX_a(x) - \frac{r^2}{2}C_a(0)\right),$$
$$\mathrm{d}\mathcal{E}_a^r(x) = r\mathcal{E}_a^r(x)\,\mathrm{d}X_a(x).$$

Renormalization is required by the martingale property

Here $C_a(0) = \langle X(x), X(x)\rangle_a$; the counterterms cancel exactly the Itô drift.

Nested observations turn Ising samples into scale trajectories

Apply nested probes $h_{a,x}$ ($a \uparrow$ finer) to each Ising configuration:

$$\varphi_a^{(n)}(x) := \langle \sigma^{(n)}, h_{a,x} \rangle, \qquad \Delta_a \varphi^{(n)} := \varphi_{a+\Delta a}^{(n)} - \varphi_a^{(n)}.$$

Measure the newly revealed information

Condition on the local $\mathcal{F}_a$-information $z = \mathcal{N}_R(\varphi_a)$, then estimate

$$\widehat{B}_a(z) = \frac{1}{\Delta a} \mathbb{E}[\Delta_a \varphi \mid z],$$

$$\widehat{\Xi}_a(z) = \frac{1}{\Delta a} \operatorname{Cov}(\Delta_a \varphi \mid z).$$

Test a local diffusion

- B_a: predictable increment;
- Ξ_a: innovation covariance;
- test locality, translations and Ising parity: B_a odd, Ξ_a even.

$$\mathrm{d}\varphi_a \approx \widehat{B}_a(\varphi_a)\, \mathrm{d}a + \widehat{\Xi}_a(\varphi_a)^{1/2} \mathrm{d}W_a.$$

Wilson–Itô diffusions make the RG dynamical

A Wilson–Itô diffusion is a scale SDE for the resolved field:

$$\mathrm{d}\varphi_a = A_a(\mathbf{f}_a)\,\frac{\mathrm{d}a}{a} + A_a^{1/2}\boldsymbol{\sigma}_a\,\frac{\mathrm{d}W_a}{\sqrt{a}}.$$

The force $\mathbf{f}_a$ and diffusivity $\boldsymbol{\sigma}_a^2$ are local observable fields, while A_a averages over length a^{-1}. The quadratic variation is

$$\mathrm{d}\langle\varphi\rangle_a = A_a^{1/2}\boldsymbol{\sigma}_a^2 A_a^{1/2}\,\frac{\mathrm{d}a}{a}.$$

Wilson plus Itô

Wilson: integrate information scale by scale. Itô: enforce the exact compensations needed for observables.

Force-free shells formally seek a flat measure

Set $\mathbf{f}_a = 0$ and let the diffusivity be deterministic, $\boldsymbol{\sigma}_a = \boldsymbol{\sigma}_0$. Then

$$\mathrm{d}\varphi_a = A_a^{1/2}\boldsymbol{\sigma}_0 \frac{\mathrm{d}W_a}{\sqrt{a}}, \qquad \varphi_a = X_a^A := \boldsymbol{\sigma}_0 \int_0^a A_b^{1/2} \frac{\mathrm{d}W_b}{\sqrt{b}}.$$

At finite resolution the field is smooth and Gaussian, with

$$\mathrm{Cov}(\varphi_a) = \boldsymbol{\sigma}_0^2 \, \mathcal{A}_{a,0}, \qquad \mathcal{A}_{a,0} := \int_0^a A_b \frac{\mathrm{d}b}{b}.$$

Each shell adds variance

With no restoring force, every newly resolved direction receives an independent fluctuation.

Nonintegrable shells diverge

Formally,

$$\mathrm{Cov}(\varphi_\infty) = \infty \cdot \boldsymbol{\sigma}_0^2 \delta(x-y).$$

Field-space interpretation

For such shells, noise alone formally approaches $\mu(\mathrm{d}\varphi) \propto \mathcal{D}\varphi$: a nonexistent translation-invariant measure on infinite-dimensional field space.

Integrating the linear force produces the effective covariance

At terminal scale T, in the commuting symmetric setting, split the force into

$$f_a = -L\Lambda_a^T + \nu_a, \qquad \Lambda_a^T := \mathbb{E}[\varphi_T \mid \mathcal{F}_a], \qquad \nu_a := \mathbb{E}[h(\varphi_T) \mid \mathcal{F}_a].$$

Exact linear integration

$$\mathcal{A}_{T,a} = \int_a^T A_b \frac{\mathrm{d}b}{b}, \qquad R_{T,a} = \mathrm{Id} + \mathcal{A}_{T,a} L.$$

With $\widetilde{\varphi}_a = R_{T,a}^{-1}\varphi_a$,

$$\mathrm{d}\widetilde{\varphi}_a = \dot{Q}_a \nu_a \,\mathrm{d}a + (\dot{Q}_a)^{1/2}\mathrm{d}W_a,$$

$$\boxed{\dot{Q}_a = R_{T,a}^{-1} \frac{A_a}{a} R_{T,a}^{-1}}.$$

Heat-kernel scale decomposition

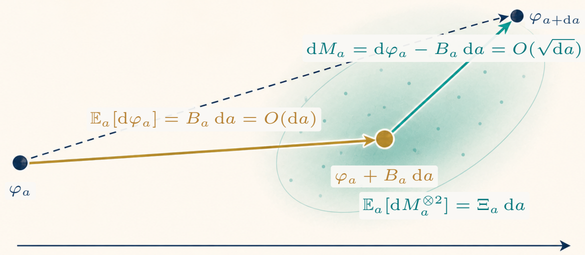
Take $L = 1 - \Delta$ and prescribe

$$Q_a = L^{-1} e^{-L/a^2},$$
$$\dot{Q}_a = 2a^{-3} e^{-L/a^2}.$$

Then $X_a^Q = \int_0^a (\dot{Q}_b)^{1/2}\mathrm{d}W_b$, and $Q_\infty = L^{-1}$.

The Gaussian linear propagation is encoded exactly by $\dot{Q}_a$; only the nonlinear perturbation ν_a remains in the transformed drift.

A diffusion separates prediction from innovation



increasing resolution $a \longrightarrow a + \mathrm{d}a$

The next increment has the Doob–Meyer split

$$\mathrm{d}\varphi_a = B_a\,\mathrm{d}a + \mathrm{d}M_a, \qquad \mathbb{E}_a[\mathrm{d}M_a] = 0.$$

Drift: conditional mean

$$B_a = \frac{1}{a}A_a(\mathbf{f}_a), \qquad \mathbb{E}_a[\mathrm{d}\varphi_a] = B_a\,\mathrm{d}a.$$

Diffusion: covariance

$$\mathrm{d}M_a = A_a^{1/2}\boldsymbol{\sigma}_a\frac{\mathrm{d}W_a}{\sqrt{a}},$$

$$\Xi_a = \frac{1}{a}A_a^{1/2}\boldsymbol{\sigma}_a^2 A_a^{1/2}.$$

Hence $\mathrm{d}\langle M\rangle_a = \Xi_a\,\mathrm{d}a$.

The martingale equation is the observable RG equation

Write the WIDE locally as $\mathrm{d}\varphi_a = B_a\,\mathrm{d}a + \mathrm{d}M_a$, with $\mathrm{d}\langle M\rangle_a = \Xi_a\,\mathrm{d}a$. Its generator is

$$\mathcal{L}_a\mathcal{O} = D\mathcal{O}[B_a] + \frac{1}{2}\,\mathrm{Tr}_{\Xi_a}\,D^2\mathcal{O}.$$

For a scale-dependent functional $\mathcal{O}_a$, Itô gives

$$\mathrm{d}\mathcal{O}_a(\varphi_a) = (\partial_a + \mathcal{L}_a)\mathcal{O}_a(\varphi_a)\,\mathrm{d}a + D\mathcal{O}_a(\varphi_a)\,\mathrm{d}M_a.$$

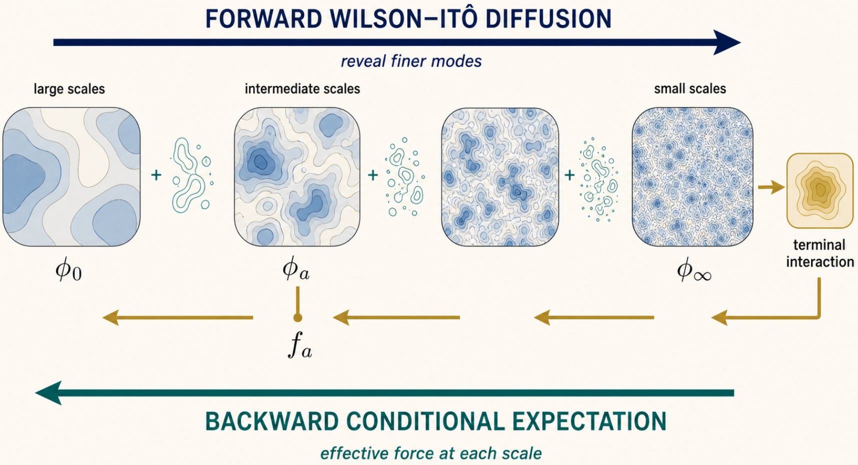
Observable criterion

Under the usual regularity assumptions, $\mathcal{O}_a(\varphi_a)$ is a local martingale when

$$\boxed{(\partial_a + \mathcal{L}_a)\mathcal{O}_a = 0}.$$

Integrability or uniform integrability upgrades it to a true martingale.

Forward reveals modes; backward expectation builds the force



The propagated covariance yields Wilson–Polchinski

After integrating the linear endpoint force, relabel the reduced field $R_{T,a}^{-1}\varphi_a$ as φ_a. Its covariance profile is

$$\dot{Q}_a = R_{T,a}^{-1}\frac{A_a}{a}(R_{T,a}^{-1})^*, \qquad Q_a = \int_{a_0}^{a}\dot{Q}_b\,\mathrm{d}b.$$

Thus $\mathrm{d}\langle\varphi\rangle_a = \dot{Q}_a\,\mathrm{d}a$: the covariance shell is the raw noise propagated through the linear resolvent. For a residual gradient force $h = -DV_T$, closure gives

$$\nu_a = -DV_a(\varphi_a), \qquad \mathrm{d}\varphi_a = -\dot{Q}_a DV_a(\varphi_a)\,\mathrm{d}a + \dot{Q}_a^{1/2}\mathrm{d}W_a.$$

With the increasing-resolution convention, the effective potential obeys

$$\boxed{\partial_a V_a = -\frac{1}{2}\,\mathrm{Tr}_{\dot{Q}_a}\,D^2V_a + \langle DV_a, \dot{Q}_a DV_a\rangle},$$

the Wilson–Polchinski equation for the propagated covariance Q_a.

A terminal interaction determines a forward–backward SDE

Suppose

$$\mu(\mathrm{d}\phi) = Z^{-1} e^{-V_\infty(\phi)} \gamma_{Q_\infty}(\mathrm{d}\phi), \qquad Q_a = \int_0^a \dot{Q}_s \,\mathrm{d}s.$$

The unresolved Gaussian increment

$$\eta_a := X_\infty^Q - X_a^Q \sim \mathcal{N}(0, Q_\infty - Q_a)$$

is independent of $\mathcal{F}_a$. The effective potential is the conditional free energy

$$V_a(\varphi) = -\log \mathbb{E}\left[e^{-V_\infty(\varphi+\eta_a)}\right].$$

Its gradient along the resolved field satisfies

$$DV_a(\varphi_a) = \mathbb{E}_a[DV_\infty(\varphi_\infty)],$$

and the Polchinski forward–backward equation is

$$\mathrm{d}\varphi_a = -\dot{Q}_a \,\mathbb{E}_a[DV_\infty(\varphi_\infty)]\,\mathrm{d}a + \dot{Q}_a^{1/2}\mathrm{d}W_a.$$

The forward–backward equation is variational

For an adapted control $u = (u_a)$, define

$$\zeta_\infty^u = \int_0^\infty \dot{Q}_a^{1/2} u_a \, \mathrm{d}a + \int_0^\infty \dot{Q}_a^{1/2} \mathrm{d}W_a.$$

The stochastic action is

$$\boxed{\mathcal{J}(u) = \mathbb{E}\left[V_\infty(\zeta_\infty^u) + \frac{1}{2} \int_0^\infty \|u_a\|^2 \, \mathrm{d}a \right].}$$

For every adapted variation v, conditioning at scale a gives

$$D\mathcal{J}(u)[v] = \mathbb{E} \int_0^\infty \left\langle u_a + \dot{Q}_a^{1/2} \mathbb{E}_a[DV_\infty(\zeta_\infty^u)], v_a \right\rangle \mathrm{d}a.$$

Euler–Lagrange equation

$$\boxed{u_a^* = -\dot{Q}_a^{1/2} \mathbb{E}_a[DV_\infty(\zeta_\infty^{u^*})]}$$

The minimizer has exactly the drift of the Polchinski FB-SDE.

The variational formulation survives beyond densities

Formally,

$$-\log \mathbb{E} e^{-V_\infty(X_\infty^Q)} = \inf_{u \text{ adapted}} \mathbb{E}\left[V_\infty(\zeta_\infty^u) + \frac{1}{2}\int_0^\infty \|u_a\|^2 \,\mathrm{d}a\right].$$

The right-hand side uses only noise, adapted controls and finite-resolution fields.

- It is nonperturbative: no expansion in the coupling is required.
- It provides coercive estimates through the quadratic control cost.
- Variants can produce limiting laws beyond absolutely continuous perturbations of the GFF.

Conceptual point

The scale dynamics and its variational principle can remain meaningful when a cutoff-free path-integral density does not.

From additive field space to nonlinear targets

The scalar theory lives in a fixed linear space: fluctuations at different resolutions can be added using a prescribed covariance profile Q_a. On a nonlinear target M, the fluctuation at x belongs to $T_{\Phi_a(x)}M$, so tangent spaces must first be compared.

$$\text{drift:} \quad \underbrace{A_a f_a \frac{\mathrm{d}a}{a}}_{\text{fixed vector-space drift}} \longrightarrow \underbrace{\mathcal{A}_a(\Phi_a) f_a[\Phi_a] \frac{\mathrm{d}a}{a}}_{\substack{\text{force mapped to a tangent vector} \\ \in\Gamma(\Phi_a^* TM)}},$$

$$\text{martingale:} \quad \underbrace{A_a^{1/2} \boldsymbol{\sigma}_a \frac{\mathrm{d}W_a}{\sqrt{a}}}_{\text{fixed vector-space noise}} \longrightarrow \underbrace{\Sigma_a(\Phi_a) \circ \frac{\mathrm{d}W_a}{\sqrt{a}}}_{\substack{\text{full tangent noise coefficient} \\ \in\Gamma(\Phi_a^* TM)}}.$$

The coefficients become geometric data

$\mathcal{A}_a$ maps the covector force to a tangent drift, while Σ_a transports the noise; their field dependence enters the force flow.

Sigma-model data are geometric

Let $\Phi\colon \Sigma \to (M, g_a)$, with two-dimensional worldsheet Σ. The running action begins with the two-derivative sector

$$S_a[\Phi] = \frac{1}{4\pi\alpha'} \int_\Sigma g_{a,ij}(\Phi)\, \partial_\mu \Phi^i \partial^\mu \Phi^j \, \mathrm{d}^2 x + \mathrm{h.o.t.}$$

The natural local force is a *covector field*:

$$f_{a,i}[\Phi] = g_{a,ij}(\Phi)\, \partial^\mu \partial_\mu \Phi^j + \Gamma_{i,jk}(g_a; \Phi)\, \partial_\mu \Phi^j \partial^\mu \Phi^k + \cdots .$$

Variationally,

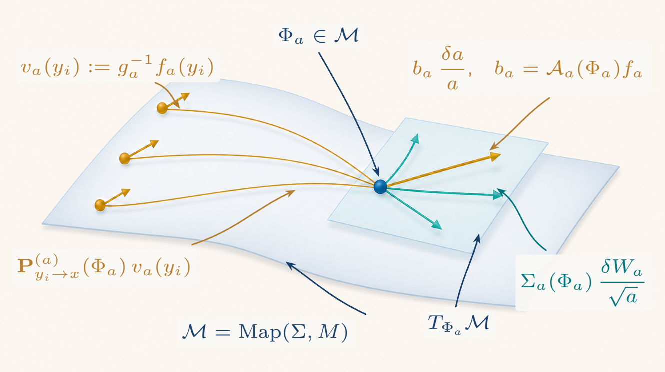
$$f_{a,i}[\Phi] = -2\pi\alpha' \, \frac{\delta S_a}{\delta \Phi^i},$$

so the force is intrinsically a one-form on field space.

Geometric constraint

Field coordinates are not target-covariant observables. Averaging must compare tangent spaces by parallel transport.

Stratonovich dynamics is tangent to the field manifold



At a configuration $\Phi_a \in \mathcal{M}$, write

$$b_a = \mathcal{A}_a(\Phi_a) f_a[\Phi_a].$$

The Stratonovich increment is

$$\delta\Phi_a = \underbrace{b_a \frac{\delta a}{a}}_{\text{drift}} + \underbrace{\Sigma_a(\Phi_a)\frac{\delta W_a}{\sqrt{a}}}_{\text{noise}}.$$

Intrinsicity requires

$$b_a \in T_{\Phi_a}\mathcal{M}, \qquad \mathrm{Im}\,\Sigma_a(\Phi_a) \subset T_{\Phi_a}\mathcal{M}.$$

Why Stratonovich?

Both terms are tangent vectors, and field-coordinate changes obey the ordinary chain rule.

The metric beta function is a projection of the force martingale

Force martingality is the generator identity for the geometric Itô SDE:

$$B_a^{\mathrm{Ito}} = \mathcal{A}_a(\Phi_a) f_a + \frac{1}{2}\mathcal{S}_a, \qquad \Xi_a = \Sigma_a(\Phi_a)\Sigma_a(\Phi_a)^*,$$

$$\boxed{0 = a\partial_a f_a + Df_a[B_a^{\mathrm{Ito}}] + \frac{1}{2}\operatorname{Tr}_{\Xi_a} D^2 f_a}.$$

Here $\mathcal{S}_a$ is the Stratonovich correction. For the local two-derivative projector Π_{g_a}, write

$$f_a = u[g_a] + w_a, \qquad \Pi_{g_a} f_a = u[g_a], \qquad \Pi_{g_a} w_a = 0.$$

$$0 = \underbrace{Du[g_a][a\partial_a g_a]}_{\text{metric beta function}} + \underbrace{\Pi_{g_a}\left(a\partial_a w_a + Df_a[B_a^{\mathrm{Ito}}] + \frac{1}{2}\operatorname{Tr}_{\Xi_a} D^2 f_a\right)}_{\text{mixing with the complementary force}}.$$

What is universal at one loop?

The shell fixes $\frac{1}{2}\Pi_{g_a}\operatorname{Tr}_{\Xi_a} D^2 u[g_a]$; drift and remainder terms stay model dependent.

The formal local contraction produces the Ricci term

In target normal coordinates and for a centered isotropic shell,

$$\Xi_a^{\mathrm{loc}} = \alpha' \kappa_a g_a^{-1} + \text{nonlocal/higher derivatives}, \qquad g_{a,ij}(y) = \delta_{ij} - \frac{1}{3} R_{a,ikj\ell} y^k y^\ell + O(|y|^3).$$

The universal two-derivative part of the Itô correction is

$$\frac{1}{2}\Pi_{g_a}\left(\mathrm{Tr}_{\Xi_a} D^2 u_i[g_a]\right) = -\alpha' \kappa_a \left(R_{a,ij} \Delta \Phi^j + \Gamma_{i,jk}(\mathrm{Ric}(g_a))\, \partial_\mu \Phi^j \partial^\mu \Phi^k\right) + \cdots .$$

Here $\Gamma(h)$ denotes the Christoffel-type polarization of a symmetric tensor h. Hence

$$\boxed{a\partial_a g_a = \alpha' \kappa_a \,\mathrm{Ric}(g_a) + \cdots , \qquad \kappa_a = \frac{1}{2\pi} \qquad (d = 2)}.$$

Universal local contribution

The traced Itô variation closes on the two-derivative force and fixes the Ricci coefficient; remainder and higher-derivative terms are contained in the ellipsis.

What is rigorous, and what remains a program

Rigorous anchors

- Gaussian and Wick observables;
- variational constructions of scalar models;
- Sine–Gordon and exponential interactions;
- ϕ^4_2, ϕ^4_3 and fermionic examples.

Programmatic extensions

- covariance connections and exact local WIDEs beyond gradient models;
- gauge-covariant and manifold-valued fields;
- nonlinear sigma-model force flow;
- inverse reconstruction from observables.

Formal identities identify the structure to preserve; constructive estimates decide which theories realize it.

Take-home message

1. Local Euclidean QFTs are spatial Markov fields; in non-degenerate continuum models, fine-scale fluctuations make fields distribution-valued.
2. Stochastic observables are martingales in resolution; for nonlinear local fields, renormalization supplies the required Itô compensation.
3. Wilson–Itô diffusions turn RG into a local stochastic dynamics and recover the Polchinski flow with its forward–backward variational structure.
4. In the programmatic geometric calculation, field-dependent averaging and the force martingale formally yield

$$a\partial_a g_a = \frac{\alpha'}{2\pi}\,\mathrm{Ric}(g_a) + \cdots .$$

Additional material

Technical derivations, comparisons and examples

Locality and OPE · Gaussian and SPDE details · rescaling · gauge covariance · sigma-model examples

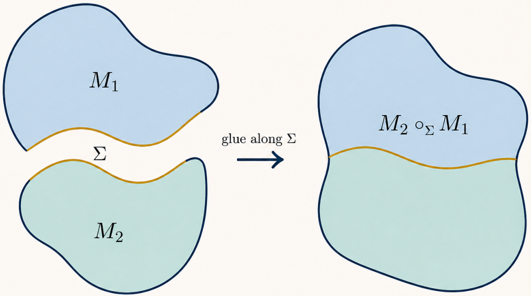
Segal locality is composition under sewing

Assign states to boundaries and amplitudes to bordisms:

$$\Sigma \longmapsto \mathcal{H}_\Sigma,$$
$$M : \Sigma_{\text{in}} \to \Sigma_{\text{out}} \longmapsto Z(M).$$

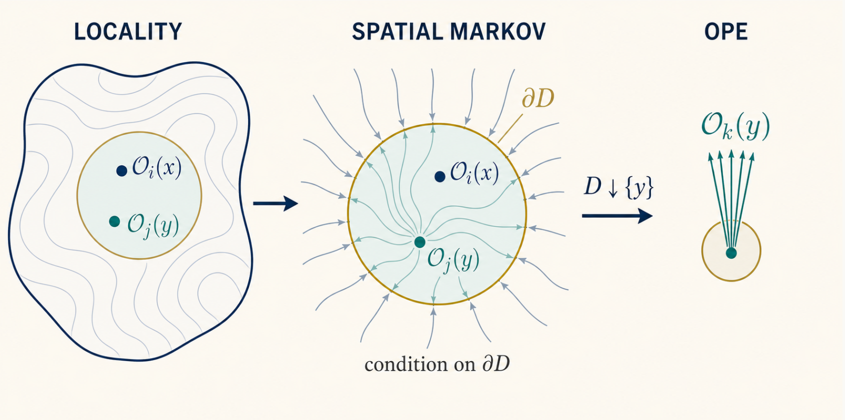
Sewing along a common boundary becomes composition:

$$M = M_2 \circ_\Sigma M_1,$$
$$Z(M) = Z(M_2)Z(M_1).$$



Global from local. The glued amplitude is determined by its pieces.

Locality and spatial Markov make the OPE natural



$$\mathcal{O}_i(x)\mathcal{O}_j(y) \sim \sum_k C_{ij}^{\ k}(x-y)\mathcal{O}_k(y), \qquad x \to y.$$

Conditioning on ∂D screens the exterior from the interior; shrinking D then expands the induced interior state in local fields at y.

Renormalized $U(1)$ exponentials acquire a running radius

At a fixed point x, let $U_a(x) := e^{i\beta X_a(x)} \in U(1) \subset \mathbb{C}$, with $\mathrm{d}\langle X(x)\rangle_a = \mathrm{d}C_a(0)$.
The bare phase lies on the unit circle, but is not an ambient martingale.

$$\mathrm{d}U_a = i\beta U_a\,\mathrm{d}X_a - \frac{\beta^2}{2} U_a\,\mathrm{d}C_a(0).$$

The second term is the inward Itô drift generated by tangential noise.

Wick compensation produces an ambient martingale.

$$\mathcal{U}_a := e^{\frac{\beta^2}{2}C_a(0)} U_a = \mathcal{E}_a^{i\beta}, \qquad \mathrm{d}\mathcal{U}_a = i\beta \mathcal{U}_a\,\mathrm{d}X_a.$$

Its deterministic modulus satisfies

$$|\mathcal{U}_a| = \rho_a = e^{\frac{\beta^2}{2}C_a(0)}, \qquad \mathrm{d}\log\rho_a = \frac{\beta^2}{2}\,\mathrm{d}C_a(0).$$

Bridge to the circle sigma model. The phase remains $U(1)$-valued, but its martingale lives on $\rho_a U(1) \subset \mathbb{C}$: intrinsic radius and observable radius are different data.

Intrinsic and extrinsic viewpoints track different data

Intrinsic geometry on M

$$\Phi_a : \Sigma \to (M, g_a), \qquad f_a \in \Gamma(\Phi_a^* T^* M).$$

Drift and noise are sections of $\Phi_a^* TM$; parallel transport compares their values at different points.

running datum: $\quad g_a.$

Extrinsic geometry in $\mathbb{R}^N$

$$F_a : M \to \mathbb{R}^N, \qquad X_a = F_a \circ \Phi_a.$$

A chosen nonlinear ambient observable may itself require renormalization; when it is an embedding, its bending is measured by II.

renormalized datum: $\quad F_a.$

Same sigma model, different questions

Intrinsic flow renormalizes the target metric; extrinsic flow renormalizes a chosen nonlinear ambient observable.

SPDEs acquire a resolution dynamics

Write the physical-time equation as

$$\mathsf{P}\phi = \mathcal{N}(\phi) + \xi, \qquad \mathsf{P} := \partial_t - \Delta.$$

For resolution filters $K_0 = 0$, $K_\infty = \mathrm{Id}$, set

$$\varphi_a = \mathbb{E}_a[\phi], \qquad f_a = \mathbb{E}_a[\mathcal{N}(\phi)], \qquad \xi_a = \mathbb{E}_a[\xi].$$

Its evolution in resolution is

$$\begin{aligned} \mathsf{P}\,\mathrm{d}\varphi_a &= \dot{K}_a f_a\,\mathrm{d}a + \mathrm{d}\xi_a, \\ \mathrm{d}\varphi_a &= \dot{Q}_a \mathsf{P}^* f_a\,\mathrm{d}a + \dot{Q}_a^{1/2}\mathrm{d}W_a, \qquad Q_a = \mathsf{P}^{-1} K_a (\mathsf{P}^{-1})^*. \end{aligned}$$

Here t is physical time, while a reveals progressively finer information.

New SPDE feature

The terminal nonlinearity contains the random distribution ξ; closing its scale evolution generally requires a functional BSDE.

Stochastic localization uses the same probabilistic coordinates

Start from a Gaussian reference law γ_{Q_∞} and tilt it:

$$\mu(\mathrm{d}\phi) = Z^{-1} e^{-V_\infty(\phi)} \gamma_{Q_\infty}(\mathrm{d}\phi).$$

Revealing Gaussian directions progressively produces a score martingale

$$u_a = \mathbb{E}\left[D \log \frac{\mathrm{d}\mu}{\mathrm{d}\gamma_{Q_\infty}}(\varphi_\infty) \,\middle|\, \mathcal{F}_a \right]$$

and a drifted observation process

$$\mathrm{d}\varphi_a = \dot{Q}_a u_a \,\mathrm{d}a + \dot{Q}_a^{1/2} \mathrm{d}W_a.$$

Common core

Conditional scores, Föllmer drifts and effective forces are different coordinates for transporting a measure through a filtration.

Comparison with score-based diffusion models

Score-based diffusion

Forward noising:

$$\mathrm{d}X_t = b_t(X_t)\,\mathrm{d}t + g_t\,\mathrm{d}W_t, \qquad t : 0 \longrightarrow T.$$

Reverse-time generation:

$$\begin{aligned}\mathrm{d}X_t = {} & \left[b_t(X_t) - g_t^2 s_t(X_t)\right]\mathrm{d}t \\ & + g_t\,\mathrm{d}\bar{W}_t, \qquad t : T \longrightarrow 0,\end{aligned}$$

with a learned score $s_t = \nabla \log p_t$.

Wilson–Itô diffusion

Increasing resolution:

$$\begin{aligned}\mathrm{d}\varphi_a = {} & A_a(\mathbf{f}_a)\,\frac{\mathrm{d}a}{a} \\ & + A_a^{1/2}\boldsymbol{\sigma}_a\,\frac{\mathrm{d}W_a}{\sqrt{a}}, \qquad a : 0 \longrightarrow \infty.\end{aligned}$$

Here $\mathbf{f}_a$ and $\boldsymbol{\sigma}_a^2$ are local stochastic observables constrained across scales.

Same stochastic skeleton, different target

Reverse diffusion samples a terminal law; WIDE resolves a field theory together with its local observables and renormalization identities.

Source: Y. Song et al., ICLR 2021, arXiv:2011.13456.

The equation of motion has a finite-resolution field

Consider the gradient-type WIDE

$$\mathrm{d}\varphi_a = \dot{Q}_a f_a \,\mathrm{d}a + \dot{Q}_a^{1/2}\mathrm{d}W_a, \qquad f_a := F_a(\varphi_a),$$

where F_a is the effective-force functional and f_a its observable process. Its associated field observable is

$$\boldsymbol{\Phi}_a := \mathbb{E}_a[\varphi_\infty] = \varphi_a + (Q_\infty - Q_a) f_a.$$

If $LQ_\infty = \mathrm{Id}$ and $J_a := LQ_a$, define the scale equation-of-motion field

$$\boldsymbol{\Theta}_a := L\boldsymbol{\Phi}_a - f_a = L\varphi_a - J_a f_a.$$

Point

The equation of motion is itself a scale-indexed observable, not only a formal identity at φ_∞.

Dyson–Schwinger holds before removing the cutoff

Let $\mathcal{O}_a$ solve the backward observable equation

$$\partial_a \mathcal{O}_a + D\mathcal{O}_a[\dot{Q}_a F_a] + \frac{1}{2}\operatorname{Tr}_{\dot{Q}_a} D^2\mathcal{O}_a = 0.$$

Assume the force derivative is symmetric,

$$\langle h, D_k F_a\rangle = \langle k, D_h F_a\rangle,$$

as for $F_a = -DV_a$. Then for every test field h,

$$\boxed{\mathbb{E}[D_{J_a h}\mathcal{O}_a(\varphi_a)] = \mathbb{E}[\mathcal{O}_a(\varphi_a)\langle h, L\varphi_a - J_a F_a(\varphi_a)\rangle]\,.}$$

Both sides are regular finite-resolution quantities.

The terminal identity is recovered as a limit

At the final scale $J_\infty = \mathrm{Id}$. In the gradient case $F_\infty = -DV_\infty$, so

$$0 = \mathbb{E}[D_h \mathcal{O}_\infty(\varphi_\infty) - \mathcal{O}_\infty(\varphi_\infty)\langle h, L\varphi_\infty + DV_\infty(\varphi_\infty)\rangle]\,,$$

the usual Dyson–Schwinger integration-by-parts formula. At the level of observables the finite-scale identity reads

$$\langle D_{\mathsf{J}h}\mathcal{O} - \mathcal{O} \bullet \langle h, L\mathbf{\Phi} - \boldsymbol{f}\rangle\rangle = 0.$$

Here $\bullet$ is the renormalized observable product, so the statement remains meaningful even when terminal products are singular.

Coherent germs reconstruct observables from local data

Let $\mathcal{O}_a^*(\varphi_a)$ be a local germ and define

$$R_a^{\mathcal{O}} := \mathcal{O}_a - \mathcal{O}_a^*(\varphi_a).$$

If the generator error is integrable, then

$$R_a^{\mathcal{O}} = \mathbb{E}_a\left[\int_a^\infty (\partial_c + \mathcal{L}_c)\mathcal{O}_c^*(\varphi_c)\,\mathrm{d}c\right],$$

hence

$$\mathcal{O}_a = \mathcal{O}_a^*(\varphi_a) + \mathbb{E}_a \int_a^\infty (\partial_c + \mathcal{L}_c)\mathcal{O}_c^*(\varphi_c)\,\mathrm{d}c.$$

Local-to-global principle

A coherent germ plus its controlled generator defect determines the full observable.

Renormalization is a rescaling problem

For a scale-covariant averaging family, rescale space and field amplitude by

$$\phi_a^\lambda := \lambda^\alpha \rho_\lambda \phi_{\lambda^{-1}a}, \qquad (\rho_\lambda \phi)(x) := \phi(\lambda x).$$

After normalizing the noise, the microscopic force becomes

$$F^\lambda(\psi) := \lambda^{d-\alpha} \rho_\lambda F\left(\lambda^{-\alpha}\rho_\lambda^{-1}\psi\right), \qquad \mathrm{d}\phi_a^\lambda = A_a \mathbb{E}_a[F^\lambda(\phi_\infty^\lambda)] \frac{\mathrm{d}a}{a} + A_a^{1/2} \frac{\mathrm{d}W_a^\lambda}{\sqrt{a}}.$$

Renormalization criterion

Choose the scale-dependent force—including counterterms—so that F^λ has a nontrivial local limit as $\lambda \to \infty$.

For the free force $F(\phi) = -(-\Delta)\phi$, scale invariance requires

$$\lambda^{d-2\alpha-2}(-\Delta)\phi^\lambda = O(1) \quad \Longrightarrow \quad \boxed{\alpha = \frac{d-2}{2}},$$

the canonical field dimension.

Formal flatness makes covariance transport path independent

Let $Q_t(\theta)$ interpolate between fixed endpoint covariances and solve

$$\partial_t F_t(\theta) = X_{\partial_t Q_t(\theta)}(F_t(\theta)), \qquad F_T(\theta) = F_T.$$

Flatness transports the θ-variation horizontally, so at the fixed lower endpoint

$$\boxed{\partial_\theta F_S = 0} \qquad \Longrightarrow \qquad F_S = F_S(F_T, Q_T - Q_S).$$

Additive gradient case

Covariance directions are fixed and commute: there is no covariance holonomy.

Field-dependent case

Geometric averaging depends on Φ; derivatives of $\mathcal{A}_a(\Phi)$ can produce a non-flat connection.

Do not confuse the curvatures

Covariance-connection curvature and target Ricci curvature are distinct.

Gauge theory replaces connections by gauge orbits

For a principal G-bundle P, the field is a connection $\varphi \in \mathrm{Conn}(P)$. A gauge transformation acts by

$$g\cdot\varphi = \mathrm{Ad}_g\,\varphi - (\mathrm{d}g)g^{-1}, \qquad \text{physical state: } [\varphi] \in \mathrm{Conn}(P)/\mathcal{G}.$$

Infinitesimally, the vertical directions are

$$T_\varphi(\mathcal{G}\cdot\varphi) = \mathrm{Im}\,d_\varphi, \qquad d_\varphi h = \mathrm{d}h + [\varphi, h].$$

Physical observable

$$\mathcal{O}(g\cdot\varphi) = \mathcal{O}(\varphi).$$

Only the orbit should be measurable.

Equivariant force

$$f_a(g\cdot\varphi) = g\cdot f_a(\varphi).$$

The dynamics must descend to the quotient.

Scope

This is a gauge-compatibility test for WIDEs, not yet a construction of a Yang–Mills measure.

Gauge fixing changes representatives, not the orbit law

Add an adapted vertical drift to choose a scale-dependent representative:

$$\mathrm{d}\varphi_a^{(h)} = \Big(A_a(\varphi_a^{(h)})f_a + d_{\varphi_a^{(h)}} h_a\Big)\,\frac{\mathrm{d}a}{a} + A_a^{1/2}(\varphi_a^{(h)})\frac{\mathrm{d}W_a}{\sqrt{a}}.$$

Solve $\dot{g}_a g_a^{-1} = a^{-1}h_a$ and set $\varphi_a^{[h]} := g_a\cdot\varphi_a^{(h)}$. Gauge covariance gives

$$\mathrm{d}\varphi_a^{[h]} = A_a(\varphi_a^{[h]})f_a\frac{\mathrm{d}a}{a} + A_a^{1/2}(\varphi_a^{[h]})\frac{\mathrm{d}W_a^h}{\sqrt{a}},$$

where W^h is again Brownian. Consequently,

$$\boxed{\mathrm{Law}\big([\varphi^{(h)}]\big) = \mathrm{Law}\big([\varphi^{(0)}]\big).}$$

Gauge fixing is a stochastic coordinate choice

The vertical drift selects representatives; gauge-invariant observables and the induced orbit law are unchanged.

A fixed embedding turns manifold preservation into a constraint

Let $\iota : M \hookrightarrow \mathbb{R}^N$, $X_a = \iota(\Phi_a)$, and push the intrinsic coefficients forward:

$$\delta X_a = \widetilde{B}_a[X_a]\,\frac{\delta a}{a} + \widetilde{\Sigma}_a[X_a]\,\frac{\delta W_a}{\sqrt{a}}, \qquad \widetilde{B}_a,\ \mathrm{Im}\,\widetilde{\Sigma}_a \subset T_{X_a}M.$$

If locally $M = \{F = 0\}$, the Stratonovich chain rule gives

$$\delta F(X_a) = DF(X_a)\,\delta X_a = 0, \qquad X_{a_0} \in M \implies X_a \in M.$$

Stratonovich

Tangent drift and tangent noise preserve the constraint without an additional normal term.

Naive Itô coordinates

The ambient formula adds

$$\frac{1}{2}\,\Xi_a^{ij}\,\mathrm{II}_{ij} \in N_{X_a}M,$$

which must be compensated by a normal drift.

Renormalizability requires control of the nonlocal remainder

Decompose the exact force into its local two-derivative part and its complement:

$$f_a = u[g_a] + w_a, \qquad \mathbf{C}_{g_a}(w_a) = 0, \qquad \Pi_{g_a} := \mathcal{L}_{g_a}\mathbf{C}_{g_a}.$$

The force-martingale equation splits into an implicit local beta relation

$$\mathfrak{M}_{g_a, w_a}(\beta_a^g) = -\mathbf{C}_{g_a}[\mathcal{N}_a(u[g_a] + w_a)],$$

and a complementary evolution

$$\nabla_a^{\perp} w_a = -(\mathbf{1} - \Pi_{g_a})\mathcal{N}_a(u[g_a] + w_a).$$

With $\tau = \log a$, the UV-normalized solution has the mild form

$$w_\tau = -\int_\tau^\infty \mathcal{U}(\tau, \eta)\big[\mathcal{R}_\eta + \mathcal{Q}_\eta(w_\eta, w_\eta)\big]\, \mathrm{d}\eta.$$

WIDE renormalizability criterion

Finite local closure + UV-integrable mild forcing + no free relevant or marginal data in w. Then the running couplings determine the remainder.

Example: the round sphere closes on one coupling

Take the target $S^n \subset \mathbb{R}^{n+1}$ with

$$g_a = R_a^2 g^{\mathrm{rd}}, \qquad \mathrm{Ric}(g_a) = \frac{n-1}{R_a^2} g_a.$$

The WIDE metric flow therefore becomes

$$a\partial_a R_a^2 = \frac{\alpha'(n-1)}{2\pi} + O(R_a^{-2}).$$

With the inverse radius $\lambda_a := R_a^{-2}$,

$$\boxed{a\partial_a \lambda_a = -\frac{\alpha'(n-1)}{2\pi}\lambda_a^2 + O(\alpha'^2 \lambda_a^3)}.$$

Checks

$n > 1$: one-loop asymptotic freedom. $n = 1$: the intrinsic circle beta function vanishes.

The circle separates intrinsic and extrinsic running

Fix the intrinsic circle metric and consider the ambient observable

$$g = R^2\,\mathrm{d}\theta^2, \qquad X_a = \rho_a\big(\cos\theta_a, \sin\theta_a\big) \in \mathbb{R}^2.$$

Intrinsic coupling

Since every one-dimensional target is Ricci-flat,

$$\mathrm{Ric}(g) = 0, \qquad a\partial_a g = 0 \quad \text{at one loop.}$$

Extrinsic observable radius

Making X_a an ambient martingale requires

$$a\partial_a \rho_a = \frac{\alpha'}{4\pi R^2}\,\rho_a, \qquad h_a^{\mathrm{obs}} = \rho_a^2\,\mathrm{d}\theta^2.$$

The radius R belongs to the sigma-model action; ρ_a is the renormalized modulus of a nonlinear embedding observable. Their beta functions answer different questions.