Gravitational Form Factors, Conformal Symmetry and the Conformal Window

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CENTRE FOR THEORETICAL PHYSICS
Higgs

- **Conformal window:** *non-SUSY (QCD, technicolor …)*
  N=1 SUSY: Seiberg dualities & phases

- **Grav. form factors:** *very brief intro*
  Focus on D-term: druck (mechanical stability?)
  dilaton (IR-conformal symmetry?)

- **Summary & Outlook**

*my work: is **massless QCD** described by **IR-CFT@SSB/IRFP?***
a) specific anomalous dimensions
b) σ-meson = dilaton (goldstone of SSB-CFT)
discussed en passent (refs in the backup)

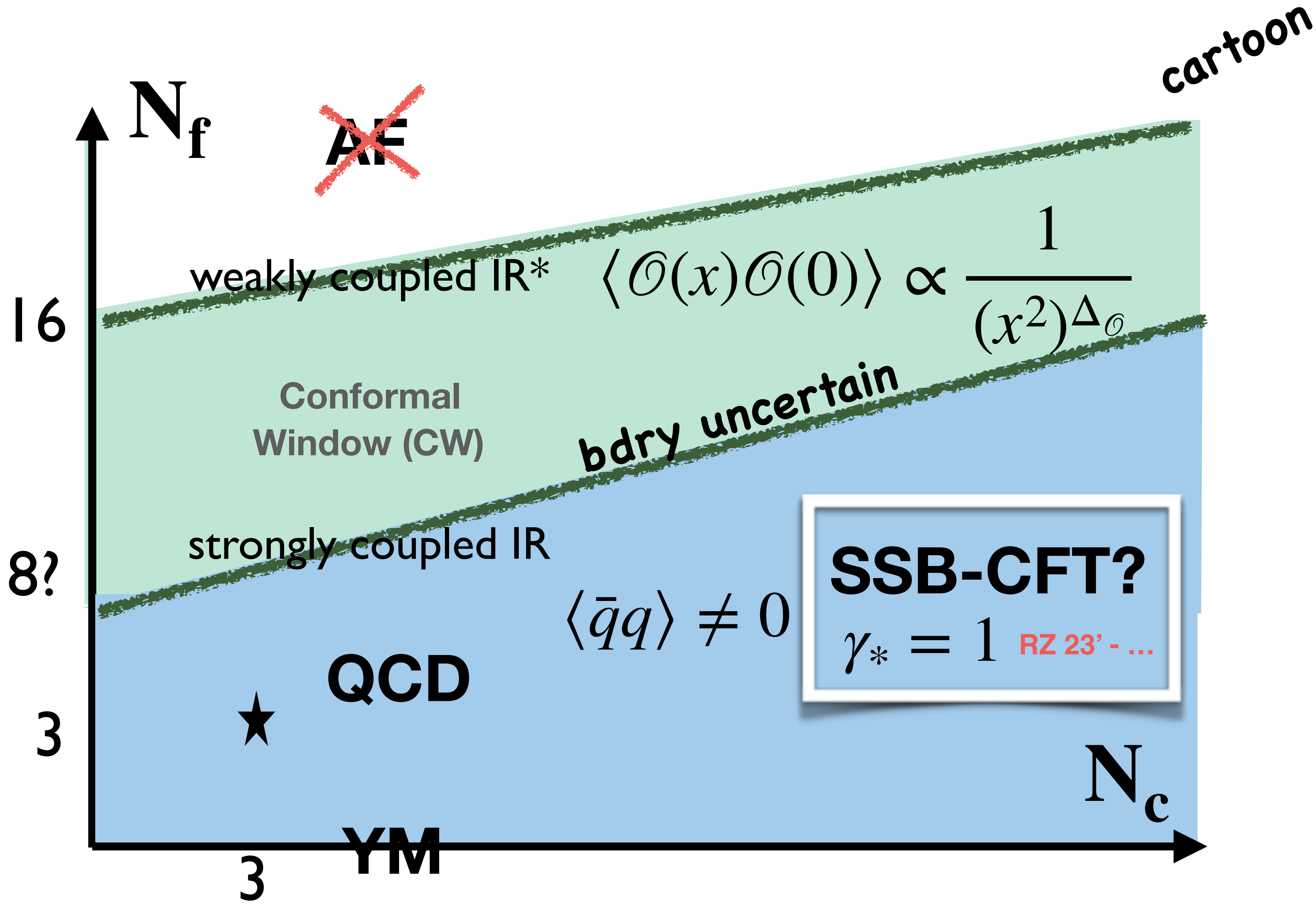
ERG - Sussex - 1-5 Sep 2026

Few topically related talks

- **Non-perturbative news from the conformal window** — A. Pastor Gutierrez *Tue 1 Sep, 17:13, poster*
- **Dense QCD and the moat regime** — Fabian Rennecke. *Tue 2 Sep, 09:00, plenary*
- **Existence of IR fixed points in the QCD chiral transition** — Gergely Fejos *Wed 2 Sep, 09:30, plenary*
- **Strongly coupled 4-fermion fixed points in 4D** — Charlie Cresswell-Hogg *Tue 1 Sep, 13:40*
- **Probing Conformal Windows with Perturbation Theory** — Nahzaan Riyaz *Tue 1 Sep, 15:43, poster*
- **Conformal Sym as an Organizing Principle for the ∂-Expansion** — Gonzalo De Polsi *Tue 1 Sep, 14:00*
- **SO**$(1, d+1)$ **symmetry of the exact RG equation** — Semanti Dutta *Thu 3 Sep, 11:20*
- **Chiral Lagrangian as a proxy for ASQG** — Francesco Del Porro. *Tue 2 Sep, 14:00*

Conformal window

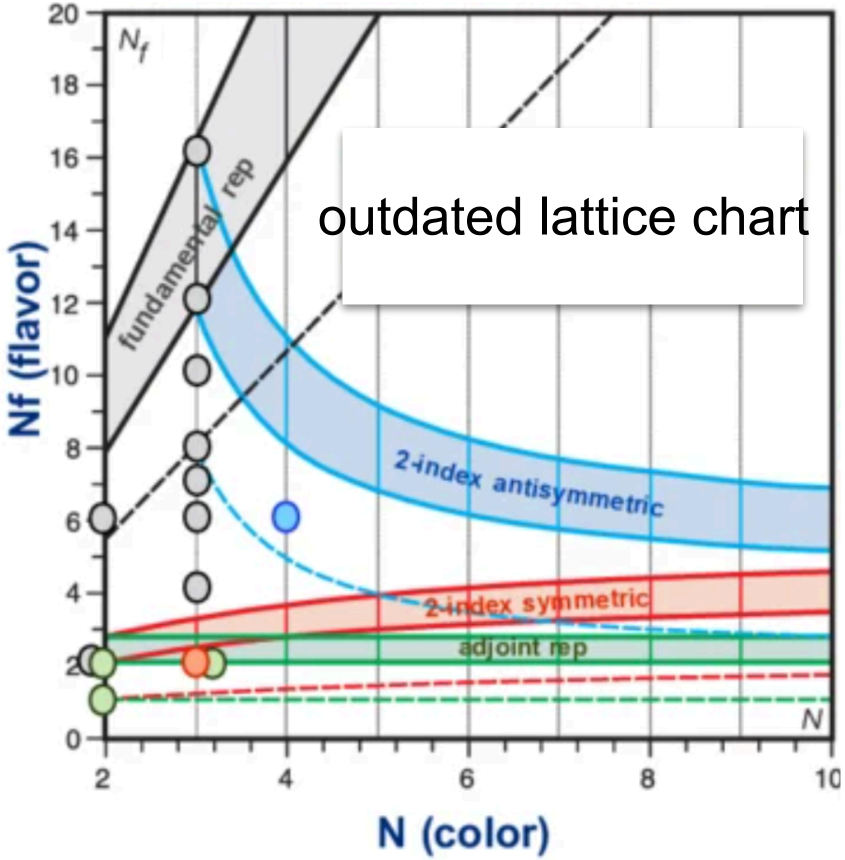
asympt free (AF) gauge theory - what is the **IR-phase**?



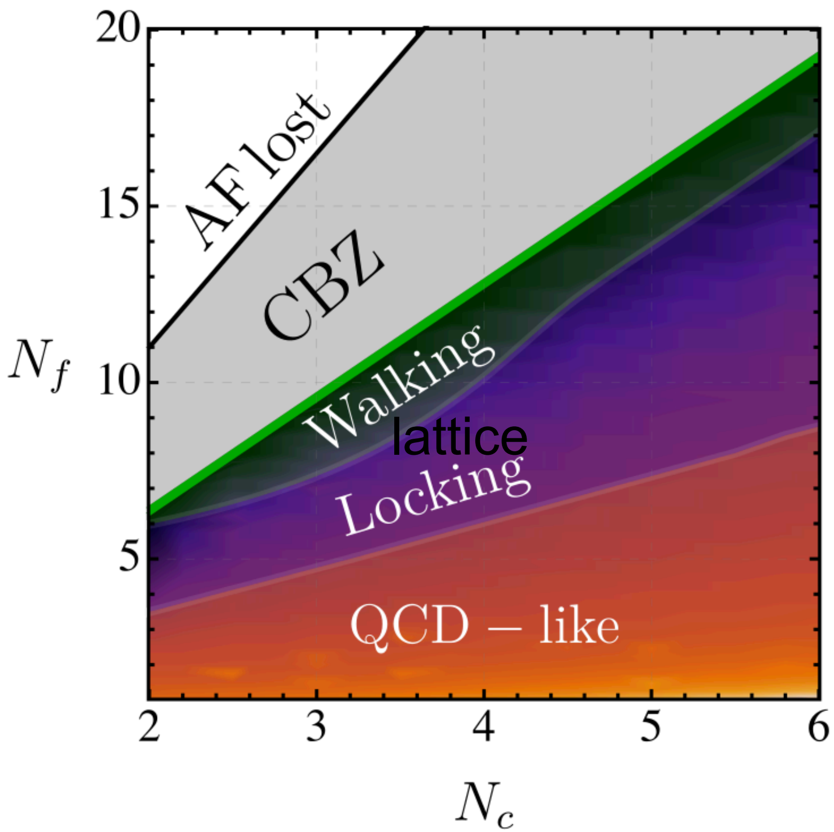
* Banks-Zaks IR fixed point $g^* \ll 1$ (Caswell'74, Belavin Migdal'74, BZ'82)

- First considered to assure QCD was chirally broken.
- Boost: - walking technicolor: boundary? & $\gamma_* = \gamma_{m_q}^{\mu=0} \approx 2?$ (unitarity bound)

  N=1 SUSY CW - NSVZ β-function - anomaly matching



interest in other groups SP(4)
composite Higgs / DM



FRG: Goertz, Pastor-Guiterrez,Pawlowski,’24
update: Pastor-Guiterrez ‘26

N=1 supersymmetric QCD

- ***more tools and phases (any of them present in QCD (N=0))***

super-cft-algebra:

$$\Delta_{Q\tilde{Q}} = 2 - \gamma_* = 3(1 - N_c/N_f) \geq 1$$

moduli (turn on **VEVs***)

$$M_i^{\bar{j}} = Q_i \tilde{Q}^{\bar{j}} \quad B \sim Q^{N_c}$$

mesons *baryons*

N_f		
	~~AF~~	
$3N_c$		
	CW	*VEVs = 0[a], finite*
$\frac{3}{2}N_c$		
	IR-free	*VEVs = 0 , finite[b]*
$N_c + 1$	**s-confinement**	*VEVs = 0[c], finite*
N_c	**q-moduli space**	*VEVs = finite*
	runaway	*VEVs = ∞*

Key to understanding:

Seiberg dual(s) [d)]

QCD (N=0) analogues?

a) standard CW

b) QCD (with dilaton?)

c) Hasenfratz et al (SMG)?

d) attempts, new hope with b)

* ever wondered "how to dial a VEV?" (Creswell-Hogg, Litim, RZ'25)

$\mathcal{N} = 1$ SUSY gauge theories (Seiberg duality)

electric theory

$SU(N_c)$ & N_f χ-matter fields Q

magnetic theory

$SU(N_f - N_c)$ & N_f χ-matter fields q

$M_i^{\bar{j}}$ colour neutral **meson field** (adjoint)

Dual IR? (a) some operators known to match (b) **global symmetries match IR**

(a) e.g. $\tilde{Q}^{\bar{j}} Q_i \leftrightarrow M_i^{\bar{j}}$

(b) e.g. $\langle T^\rho{}_\rho(x) T^\alpha{}_\alpha(0)\rangle_{\text{el}} \overset{\text{IR}}{\longleftrightarrow} \langle T^\rho{}_\rho(x) T^\alpha{}_\alpha(0)\rangle_{\text{mag}}$

non-trivial checks:

*global symmetries, **moduli spaces**, decoupling, susycft index*

*NSVZ β-fct, **anomaly matching** (next slide)*

… applying Seiberg duality

(a) $$2-\gamma_* = \Delta_{\tilde{Q}Q} = \Delta_M = 1 \Leftrightarrow \quad \gamma_* = 1$$

$\Rightarrow$ It does really **make sense** to extend IRFP-scenario in N=1 SUSY **below boundary** of the **conformal window.**

RZ '23 (last)

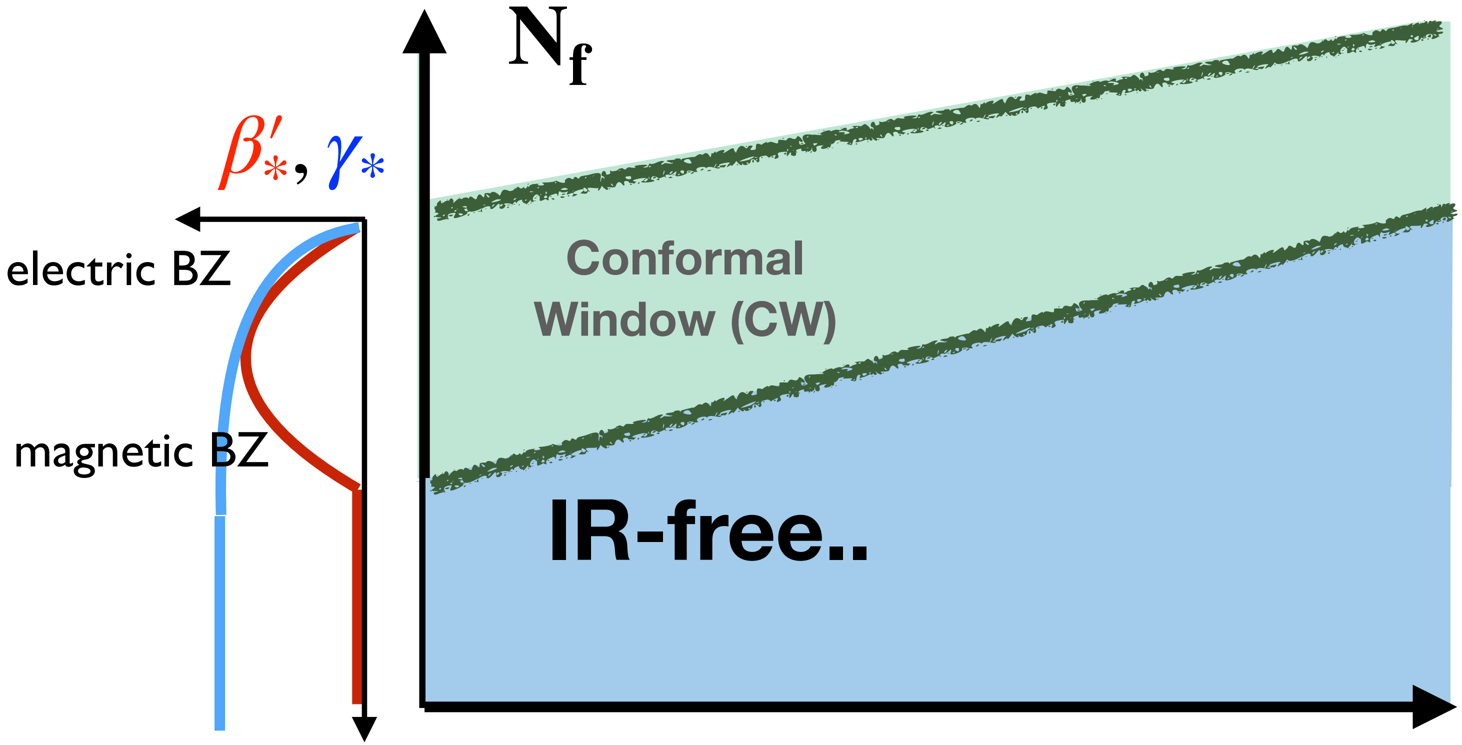
(b) matching correlator: $\langle T^\rho{}_\rho(x) T^\rho{}_\rho(0)\rangle \propto \dfrac{1}{(x^2)^{4+\beta'_*}}$

$$\beta'_*|_{\rm el} = \beta'_*|_{\rm mag} \quad \Leftrightarrow \quad \langle T^\rho{}_\rho(x) T^\alpha{}_\alpha(0)\rangle_{\rm mag} \stackrel{\rm IR}{\longleftrightarrow} \langle T^\rho{}_\rho(x) T^\alpha{}_\alpha(0)\rangle_{\rm el}$$

$\Rightarrow \beta'_* \to 0$ by continuous **matching** to **N=1 SUSY** conformal window @boundary, again can be extended below!

Anselmi, Grisaru, Johanson 97' Shifman RZ '23

… on possible implications if true in QCD (N=0)



- deduced $\beta'_* = 0$ for QCD@IRFP a) hyerscaling with a (additional) scale $\Lambda \propto F_\pi$
  b) EFT and QCD matching of $\langle T^\rho_\rho(x) T^\rho_\rho(0) \rangle$

- do **no**t need **trace anomaly matching** in EFT (will come from loops)

- within this hypothesis a **Seiberg dual** in **QCD** is **more likely**:
  β'_* @CW-bdry suggests a weakly coupled dual (in itself)

’t Hooft anomaly matching

$SU_L(N_f)^3$ *in electric* ***and magnetic*** *theory*

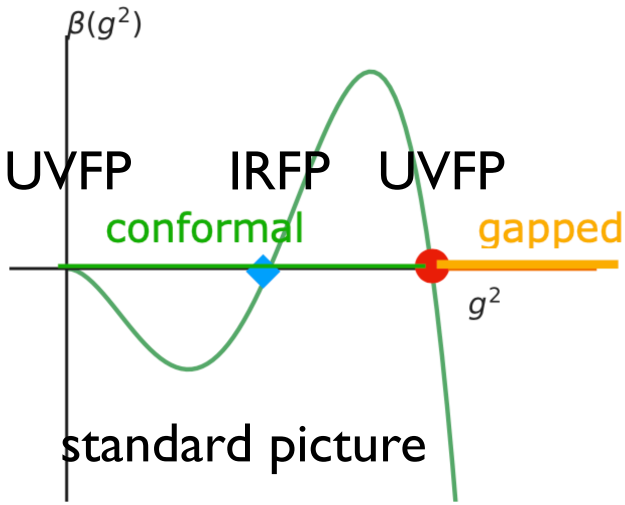
N_f		~~VEVs~~	VEVs	
$3N_c$	**CW**	squarks	squarks + GBs	$W = \tilde{q}Mq$
$\frac{3}{2}N_c$	**IR-free**	squarks	squarks + GBs	$W = \tilde{q}Mq$
$N_c + 1$	**s-confinement**	Mesons, Baryons	Mesons, Baryons + GBs	$SU(N_{mag} = 1)$
N_c	**q-moduli space**		GBs	$\det M - B\tilde{B} = \Lambda^{2N_c}$
$< N_c$	**runaway**		GBs	$W \propto \frac{1}{\det M} \propto \frac{1}{F_\pi}$ weakly coupled

- ***black boxes*** *analogues of QCD (N=0)*
- ***IR-free*** *with VEVs … hidden local symmetry* $SU(N_f - N_c)$ Komargodski’10 Abel,Barnard’12
- ***purple box*** *phase discussed by* Hasenfratz et al ’24 - .. ?

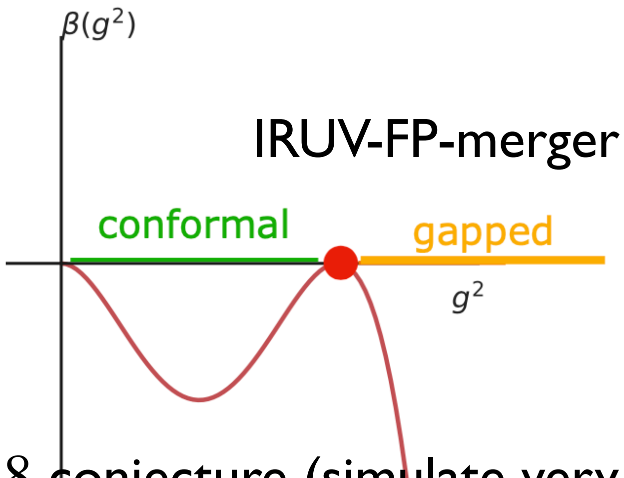
confined-gapped (no GBs) [unknown UV theory]

Hasenfratz'22 Hasenfratz Witzel'24, Butt, Caterall, Hasenfratz'25 Hasenfratz Chu'26

- $SU_c(3)\,, N_f = 8$ *lattice no* $SU_L(8)^3$ *t'Hooft anomalies but discrete version (cancelled for* $N_f = 8$*)*
- ***paradigm** for **symmetric mass generation** (massive fermions but unbroken flavour symmetry)*



standard picture



$N_f = 8$ conjecture (simulate very strong coupling)

- ***lattice artefact?** (staggered fermion)*
  *Maybe **not** as phase transition seems continuous and **continuum limit expected to exist***

- *confined but no GBs **similar to s-confinement (VEV=0)** but different with respect to 't Hooft anomalies*

Gravitational form factor

$$\frac{1}{2m_N}\bar{u}(p)\Theta_{\mu\nu}(q)u(p') = \langle N(p')\,|\,T_{\mu\nu}\,|\,N(p)\rangle$$

$$q \doteq p' - p$$

$$\boxed{\Theta_{\mu\nu}(q) = a_{\mu\nu}A(q^2) + d_{\mu\nu}D(q^2) + j_{\mu\nu}J(q^2)}$$

$$a_{\mu\nu} = 2\mathcal{P}_\mu\mathcal{P}_\nu\ , \quad j_{\mu\nu} = 2i\,\mathcal{P}_\mu\sigma_{q\nu} + \mu \leftrightarrow \nu\ , \quad d_{\mu\nu} = \frac{1}{2}(q_\mu q_\nu - q^2\eta_{\mu\nu})$$

$A^N(0) = 1$	$J^N(0) = \frac{1}{2}$,
Energy	Spin

model-independent

$D^N(0) \; =??$

"The last unknown global property of the nucleon"

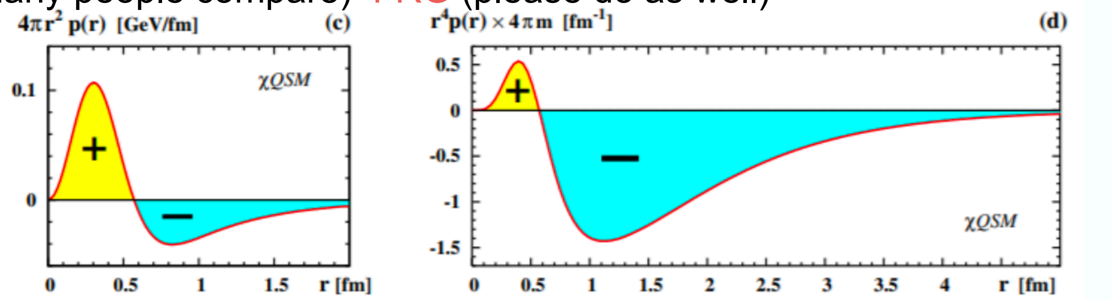
Polyakov, Schweitzer'18 (review)

some history

Kobzarev Okun'62 (defined), Donoghue, Gasser, Leutwyler'90 Okun'62 (light Higgs ($=\sigma \to \pi\pi$))
Ji'95 (nucleon mass decomposition - split quark & gluon part rel. pdfs) Ji'97 (same spin - GFF era begins)
Polyakov'02 (pressure interpretation - **mechanical stability D(0) < 0** Ji et al (counterexamples hydrogen atom)
MIT lattice group 19' '21' 23' (many form factors - many people compare) FRG (please do as well)

$$\int d^3r p(r) = 0 \quad \textit{von Laue}$$

$$D(0) = M\int d^3r\, r^2 p(r) \; < 0$$



Is the σ-meson a dilaton (massless QCD)? Well..

- **logical possibility:** *idea older than QCD* **PhD thesis: J.Ellis & R,Crewther'70 Formal approach Isham, Salam, Strathdee …**
- **lattice results:** 0^{++}*-state lighter than "excepted"* **LSD, LatKMI, Kuti et al …**
- **N=1 SUSY-QCD:** *seems there* **Komargodski'10, Abel, Barnard'12 not emphasised RZ'23 pointed out (more work needed) & useful elsewhere**
- **a-thm applied QCD:** *IR-divergent without dilaton* **Prochazka & RZ'17**
- **dilaton & SU(3)-χPT:** *improve predictions?* **Crewther, Tunstall 12-15 NLO needed**
- **GFF support it :** *remaining talk*
- **Skyrme-dilaton :** *remaining talk works well* **Fischbacher RZ in progress**
- **Higgs a dilaton:** *seems possible* ***if*** $F_\sigma/F_\pi \approx 1$ **Cata, Crewther, Tunstall'18 RZ'23 unclear why - mild evidence this work**

Soft-pion theorem and Gravitational form factor

$$\Theta^{\pi}_{\mu\nu} = \langle \pi(p') | T_{\mu\nu} | \pi(p) \rangle = 2\mathcal{P}_\mu \mathcal{P}_\nu \, A^\pi(q^2) + \frac{1}{2}(q_\mu q_\nu - q^2 \eta_{\mu\nu}) \, D^\pi(q^2)$$

$$\boxed{\text{Since } P_\mu = \int d^3x T^0_\mu \quad \Rightarrow \quad A^\pi(0) = 1}$$

1.soft-pion theorem: $\Theta^\pi(q^2) = 2m_\pi^2 + q^2 + \mathrm{NLO}$

Dolgov Voloshin'80
Donoghue, Leutwyler'91

2.contracting def. : $\Theta^\pi(q^2) \; = \; 2m_\pi^2 \, A^\pi(q^2) - \frac{q^2}{2}\,(A^\pi(q^2) + 3D^\pi(q^2))$

3.LO-$d\chi PT$: $\mathscr{L}^{EFT} \supset \frac{1}{4} m_\pi^2 F_\pi^2 \, \chi^{\Delta_{\bar{q}q}} \mathrm{Tr}[U + U^\dagger]$ $\quad \hat{\chi} = \exp(-\sigma/F_\sigma)$ compensator

$$A^\pi(q^2) = 1 \, , \quad D^\pi(q^2) = \frac{2}{3} \frac{q^2 + (\Delta_{\bar{q}q} - 2) m_\pi^2}{q^2 - m_\sigma^2} - 1$$

This implies: $\boxed{\Delta_{\bar{q}q} = 3 - \gamma_* = 2 \quad \Rightarrow \quad \gamma_* = 1}$

RZ 2306.12914

Conformal WI: $T^{\rho}_{\rho} = 0 + \text{contact-terms}$

THE TENSION...

textbook formula

$$\Theta^{\rho}_{\rho}(q)\,|_{q\to 0} = 2m_N^2$$

(IR) conformal WI

$$\Theta^{\rho}_{\rho}(q)\,|_{q\to 0} = 0$$

… resolved by dilaton

- *Taking the trace* $$\Theta^{\rho}_{\rho}(q^2) = 2m_N^2 A(q^2) - \frac{q^2}{2}(A(q^2) - 2J(q^2) + 3D(q^2))$$

- Now, **IF:** $$\boxed{D(q^2) = \frac{4}{3}\frac{m_N^2}{q^2} + O(1)}$$ $$\boxed{\Rightarrow \Theta^{\rho}_{\rho}(0) = 0 \quad \& \quad m_N \neq 0}$$

“dilaton hides the nucleon mass” , Goldstone restores WI (same pion)

“Massive Hadrons in a Conformal Phase” Del Debbio, RZ’21

… mechanism: dilaton Goldberger-Treiman formula

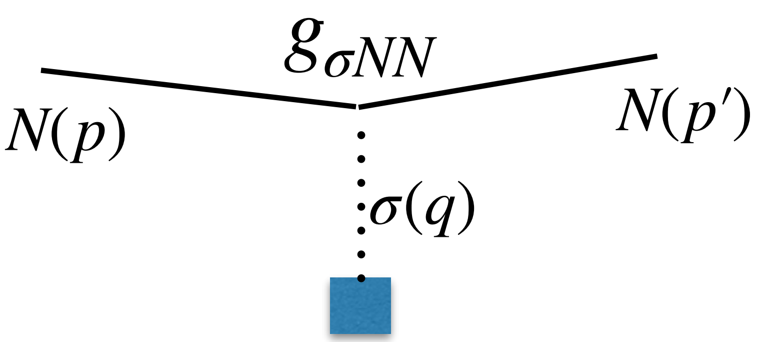
Gell-Mann 69', Carruthers'71 …. Del Debbio RZ'21, RZ'23

pions:		***dilaton***:
$g_{\pi NN} = g_A \frac{m_N}{F_\pi}$	*ensures WI*	$g_{\sigma NN} = \frac{m_N}{F_\sigma}$
$\langle 0 | A^a_\mu | \pi^a \rangle = i p_\mu F_\pi$	*decay const.*	$\langle 0 | J^D_\mu | \sigma \rangle = i p_\mu F_\sigma \qquad J^D_\mu = x^\nu T_{\mu\nu}$
$U = e^{i\pi/F_\pi}$	*coset field*	$\hat{\chi} = e^{-\sigma/F_\sigma}$

EFT-level:

$$\mathscr{L}^*_{EFT} \supset \boxed{\frac{F_\sigma^2}{2}(\partial\hat{\chi}^2 + \frac{1}{6}R\hat{\chi}^2)} + \boxed{N i\gamma^\mu\partial_\mu N} - \boxed{\hat{\chi} m_N \bar{N} N}$$

all three locally Weyl-invariant



$$T^R_{\mu\nu} \propto (\partial_\mu\partial_\nu - \eta_{\mu'nu}\partial^2)\sigma$$

$$\Rightarrow \quad \boxed{D(q^2) = \frac{4}{3}\frac{m_N^2}{q^2} + O(1)}$$

* simplified: no Weyl-derivatives & $m_q = 0$

The σ-meson as a dilaton confronted with lattice data

Stegeman, RZ' 25'25

- *fit for residue with **background** and **effective σ-mass** $m_{E,\sigma}$*

$$D(q^2) = \frac{r_{E,\sigma}}{q^2 - m_{E,\sigma}^2} + b(q^2)$$

$1\,\mathrm{GeV} \leftrightarrow f_0(980)..$

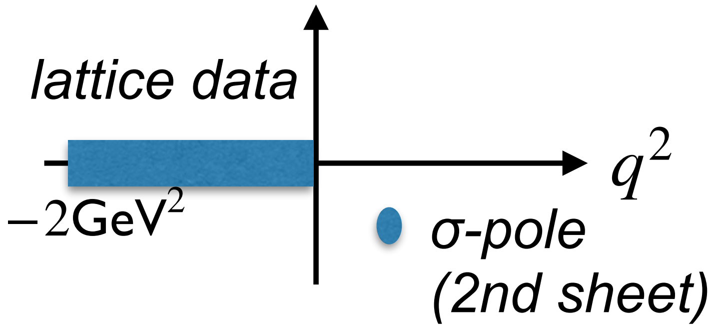
background: $b(q^2) = b + b'q^2 + b''q^2 + \dfrac{r_{eff}}{q^2 - m_{eff}^2}$

accounts higher spectrum & avoid overfitting

effective σ-mass $m_{E,\sigma}$ = *550(50)MeV :* empiric knowledge in Euclidean region

Donoghue'06 OBEs

- extensively tested in linear σ-model as case study
- and **multipole expansion in momentum space**
- $r_{E,\sigma} \approx r_{EFT}$ holds in linear σ-model (assumed in QCD as well)



LO EFT predictions for completeness

- *Spins* $0,\frac{1}{2},1,\frac{3}{2}$ *(pion special as Goldstone)*

$$D^{\pi}(q^2) = \frac{r_\sigma^\pi q^2}{q^2 - m_\sigma^2} - 1\,, \qquad D^N(q^2) = \frac{r_\sigma^N}{q^2 - m_\sigma^2}$$

$$D^{\rho}(q^2) = \frac{r_\sigma^\rho}{q^2 - m_\sigma^2} - 1\,, \qquad D^\Delta(q^2) = \frac{r_\sigma^\Delta}{q^2 - m_\sigma^2}$$

- *residues include: LO* m_q*-corrections*

$$r_\sigma^\pi = \frac{2}{3}\,, \qquad r_\sigma^N = \frac{4}{3}\bar{m}_N^2\,, \qquad r_\sigma^\rho = \frac{4}{3}(\bar{m}_\rho^2 - \frac{1}{2}\delta m_\rho^2)\,, \qquad r_\sigma^\Delta = \frac{4}{3}\bar{m}_N^2$$

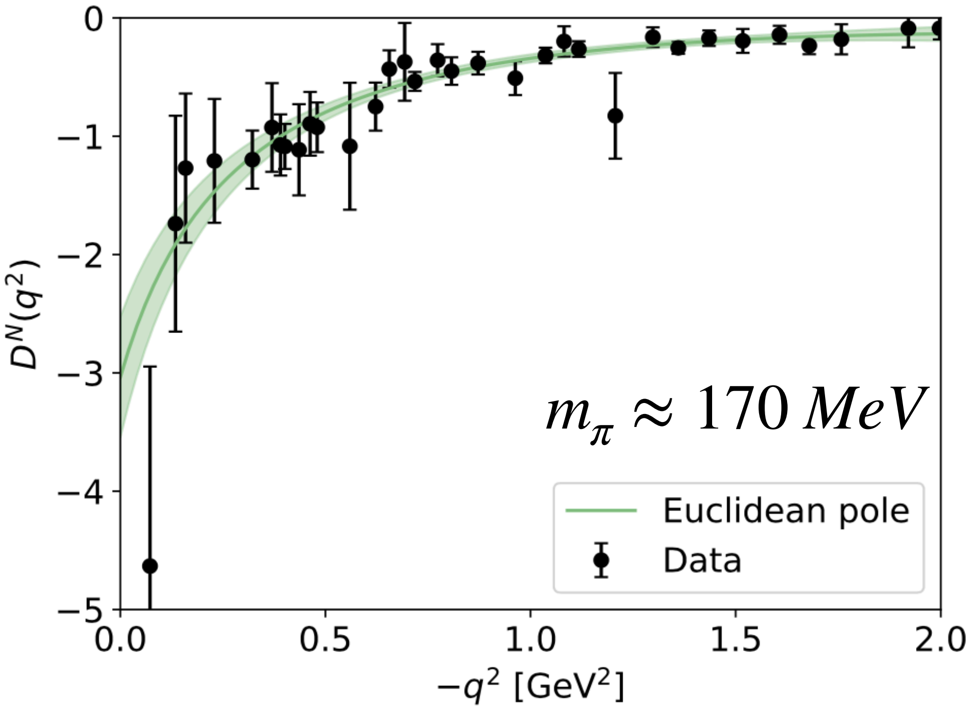
SU(3) chiral limit

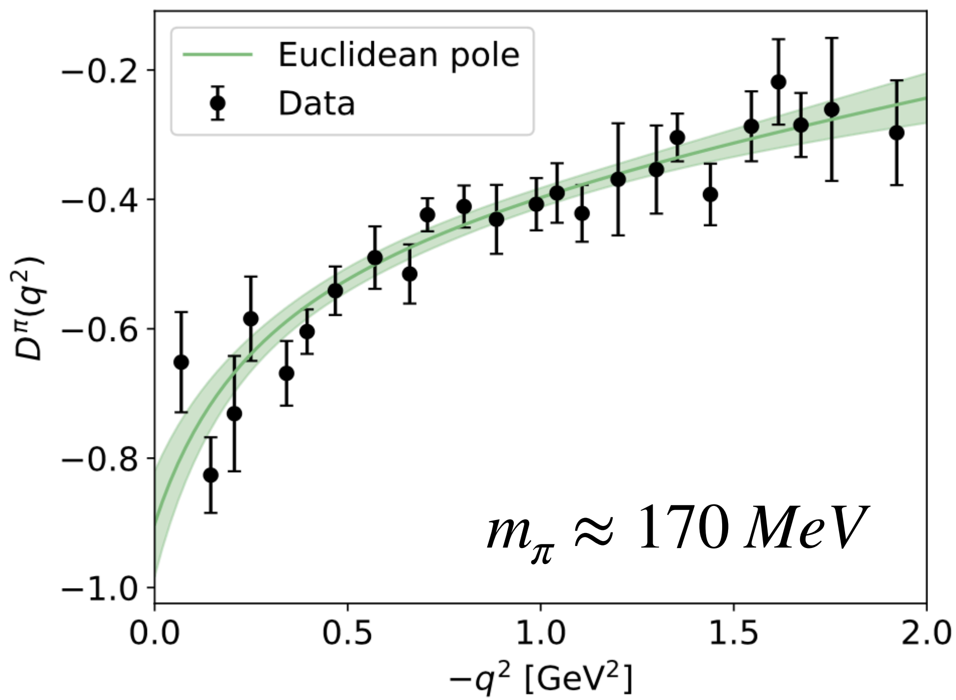
Compare MIT-lattice data @$m_\pi = 170$MeV

$$r_\sigma^N|_{\text{dilaton}} = 0.91(12)\ \text{GeV}^2 \qquad d\chi PT \qquad r_\sigma^\pi|_{\text{dilaton}} = \frac{2}{3} \pm 0.1$$

$$r_{\text{E},\sigma}^N = 1.13(26)(20)\ \text{GeV}^2 \qquad \text{fit} \qquad r_{\text{E},\sigma}^\pi = 0.53(16)(3)$$

Stegeman, RZ 2508.18537





nucleon: Hackett, Pefkou, Shanahan;PRL 23

pion: Oare, Hackett, Pefkou, Shanahan;PRD 23

$\chi^2/dof = 0.55$

$\chi^2/dof = 1.2$ soft-pion thm obeyed by all fits

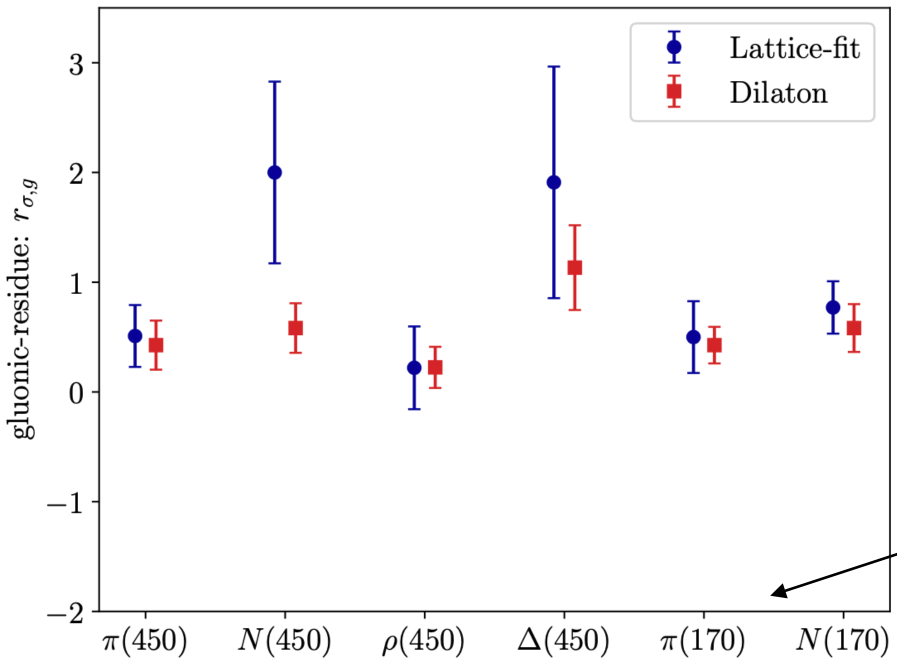
Compare MIT-lattice data @$m_\pi = 450$MeV

Stegeman, RZ 2512.12315

- 2 new (unrelated) aspects: a) σ, ρ, Δ become stable

  Pefkou Hackett,, Shanahan;PRD 21 b) MIT group gluonic part of EMT only

$$r_{\sigma,gluon} \to z_\sigma r_\sigma \qquad z_\sigma^* = \frac{F_{\sigma,gluon}}{F_\sigma}\Big|_{MS,\mu=2GeV} = 0.64(20)$$



- works well within uncert.
- also included $m_\pi = 170MeV$

* *estimated by asymptotic RG-value (analogous to gluon mom-fraction $A^{N,g}(0) = \langle x_g \rangle$)*

Some more data

skip

	$\pi(450)$	$N(450)$	$\rho(450)$	$\Delta(450)$	$\pi(170)$	$N(170)$
j	0	$\frac{1}{2}$	1	$\frac{3}{2}$	0	$\frac{1}{2}$
$r_{\sigma,g}\vert_{\text{dilaton}}$	0.42(24)	0.58(25)	0.22(19)	1.13(39)	0.43(13)	0.58(17)
$r_{\sigma,g}\vert_{\text{fit}}$	0.51(28)(3)	2.00(74)(37)	0.22(37)(7)	1.91(98)(39)	0.50(13)(2)	0.77(13)(12)
pull	0.24	1.64	0.00	0.69	0.38	0.78
b_g or $\hat{b}_g$	−0.25(10)	0.96(66)	−0.27(35)	0.84(86)	−0.067(66)	0.71(17)
b'_g [GeV^{-2}]	−0.12(07)	0.12(22)	−0.27(12)	0.08(27)	−0.037(35)	0.230(67)
$\rho_{r,b}$	−0.85	0.99	0.98	0.99	−0.94	0.98
$\rho_{r,b'}$	0.95	0.95	0.95	0.94	0.86	0.92
$\rho_{b',b}$	−0.74	0.98	0.98	0.98	−0.66	0.98
N_{dat}	25	16	22	16	24	33
χ^2/d.o.f.	1.3	1.8	2.0	2.2	1.06	0.88

Summary

- Rich **N=1 SUSY conformal window** (many phases)…

  *can we **learn more?***   *a) analytic methods*

  b) numerical methods (lattice,FRG,DSE, bootstrap)

- Please **do non-QCD gauge theories** (happening already…)

- Gravitational form factors vibrant field (please **do** them as well…)

- **σ-meson** seems to **couples like** a **dilaton**

  Aspects of **pions physics are consistent with IRFP** with , $\gamma_* = 1 \;\; \beta'_* = 0$

  What happens to σ-pole when $m_s \to 0$? (please **study**)

- D(0) < 0 argued mechanical stability   Polyakov'02 ..

  New alternative: D(0) < 0 if σ dominates over background

End

BACKUP

Some refs

more refs in my papers and surely more as an old and interesting topic

Del Debbio, RZ	JHEP'22 2112.1364	Dilaton new phase?
RZ	PRD, **2306.06752**	broken χ-sym.@IRFP - pions
RZ	NPB'26 **2306.12914**	Dilaton improves Goldstones
Shifman RZ	PRD, **2310.16449**	β'_* in N=1 conformal window
RZ	PRD 2312.13761	broken χ-sym.@IRFP - pions & dilaton
Cresswell-Litim,RZ	2502.00190	Dilaton Physics from Asymptotic Freedom
Stegeman,RZ	JHEP 2508.18537	dilaton and gravitational FFs $m_\pi = 170$ MeV
Stegeman,RZ	JHEP 2508.18537	dilaton and gravitational FFs $m_\pi = 450$ MeV

- **PDG banned σ-meson (dilaton candidate 20 years)**

  Pelaez et al'96 good values Caprini, Colangelo, Leutwyler'06 good value, small error, Roy eq. with LHC

- **Pre&post LHC model building - (not attached to gauge theories)**

  Rattazi et al 1306.4601 Grinstein et al 0708.1463 Terning et al 1406.5192} ....

  Some pragmatic, some negative conclusions

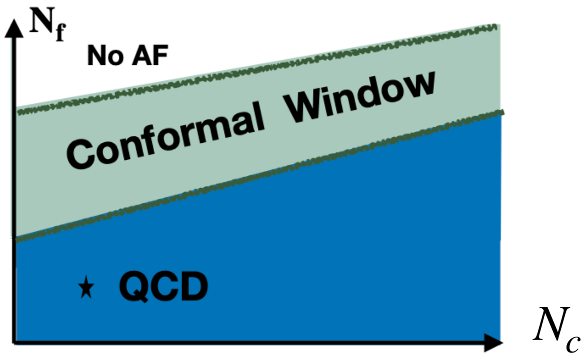
- **Dilaton (gravity explaining origin of Planck Mass)** **Wetterich, Zee, Shaposhnikov, Karananas..**

- **Dilaton-EFT'15 (understand lattice results)** **Appelquist, Piai, Ingolby, Golterman, Shamir, RZ**

BIG PICTURE

	no SSB $\langle \bar{q} q \rangle = 0$	SSB $\langle \bar{q} q \rangle \neq 0$	
no flow $\beta = 0$	**CFT** powerful tools	**CFT**$_{\text{SSB}}$ less known $m_{\rho,\omega,N} \neq 0$	$m_\sigma = 0$
flow $\beta \neq 0$	**IR-CFT** good understanding e.g. hyperscaling "conformal window"	**IR-CFT**$_{\text{SSB}}$ a) in progress … b) is QCD of this type? $m_{\rho,\omega,N} \neq 0$	$m_\sigma = 0?$

dilaton-candidate



The **dilaton-hypothesis** cannot be excluded: clear **remnant of it in data**
.Nucleon more solid w.r.t. background: i) better data ii) residue not artificially small

P.S.

1) **soft-pion theorem**, satisfied by all fits, **reassuring** of **data-quality**

$$\text{soft-pion theorem} \quad \Rightarrow \quad \hat{b} - \frac{r_{\rm eff}}{m^2_{\rm eff}} = 0 \pm 0.1 \;,$$

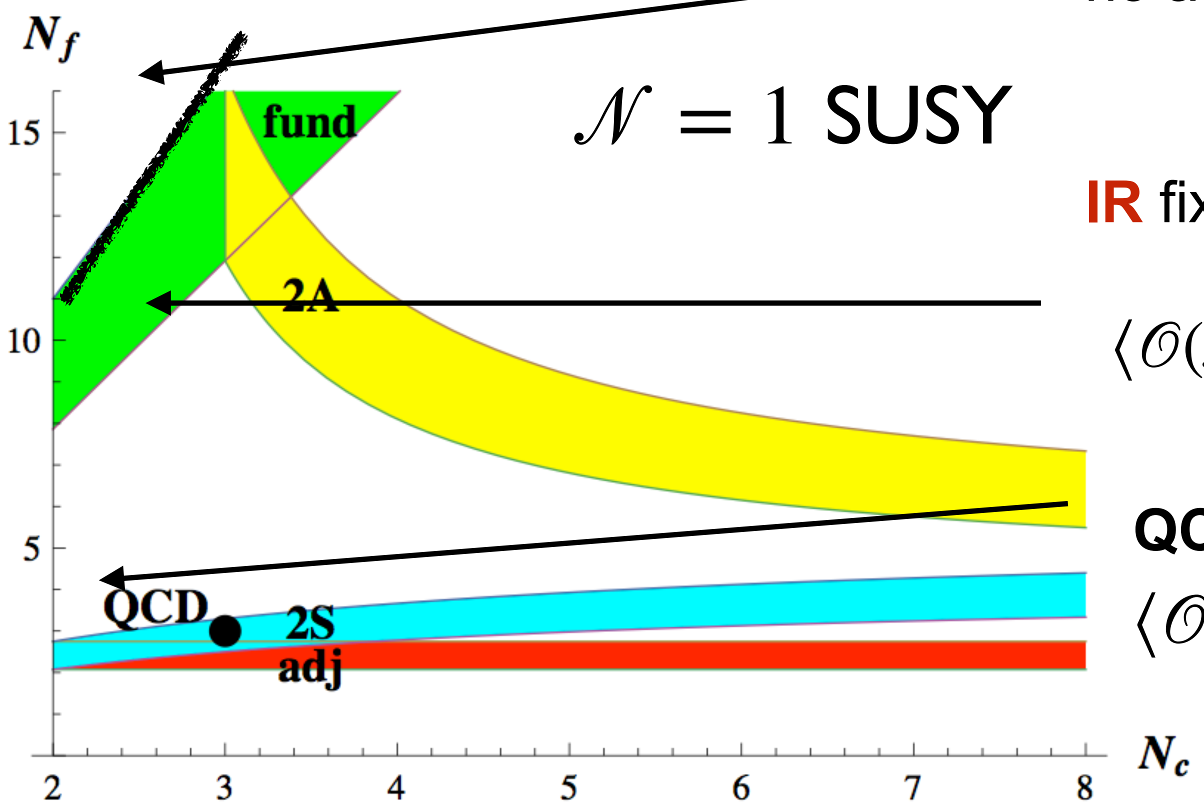
2) paper on $m_\pi \approx 450$ **MeV data** for gluonic gravitational form factors in prep.
similar results but also for spin-1 ρ-meson and spin-3/2 Δ-baryon ….

3) **D-term story**: D(0) < 0 reasonable conjecture Perevalova, Polyakov, Schweitzer\16
- Criticism: Ji, Yang'25 Counterexamples D > 0 Hydrogen Czerniaki eta l'21
- Here: $D^{\text{hadron}}(0)$ < 0 provided σ-meson dominates over background

Explanation based on σ-dominance (less fundamental)

Phases of gauge theories - **Conformal Window**

- gauge theory **massless quarks** in some **irrep** (e.g. fund. of say $SU(N_c)$)
- Focus on green = fund irrep



$\mathcal{N} = 1$ SUSY

no asymptotic freedom (ignore)

IR fixed point = **conformal window**

$$\langle \mathcal{O}(x)\mathcal{O}(0)\rangle_{CFT} \propto \frac{1}{(x^2)^{\Delta_{\mathcal{O}}}} \quad x^2 \to \infty$$

QCD: *chiral SSB* & *confinement*

$\langle \mathcal{O}(x)\mathcal{O}(0)\rangle_{QCD} \propto$ complicated

Dilaton Chiral Perturbation Theory

Building blocks are coset fields:

$U = \exp(i\pi/F_\pi)$ and $\chi = F_D \exp(-D/F_D)$

Building principle:

global chiral invariance & (local) Weyl invariance

$$g_{\mu\nu} \to e^{-2\alpha} g_{\mu\nu} \qquad \chi \to \chi e^{\alpha} \qquad U \to U$$

Leading order dilaton-χPT

1) pion-sector

$$\Delta_{\bar{q}q} = 3 - \boxed{\gamma_*}$$

$m_q \neq 0$ explicit breaking

$$\mathcal{L}_{\text{LO}}^{\text{d}\chi\text{PT}} = \frac{F_\pi^2}{4}\hat{\chi}^2 \text{Tr}[\partial^\mu U \partial_\mu U^\dagger] + \frac{B_0 F_\pi^2}{2}\boxed{\hat{\chi}^{\Delta_{\bar{q}q}}}(\text{Tr}[\mathcal{M}U^\dagger + U\mathcal{M}^\dagger] + \frac{1}{2}\boxed{(\partial\chi)^2}$$

$$- \frac{\Delta_{\bar{q}q}}{4}\text{Tr}[\mathcal{M} + \mathcal{M}^\dagger]\hat{\chi}^4) + \boxed{\frac{1}{12}\chi^2 R} + \mathcal{L}_{\text{anom}}\boxed{(\beta'_*)} + \boxed{V(\chi)} \,,$$

removes tadpole
(**Zumino-term**)

local Weyl invariance
Improvement term
4) gravitational form factors

matches trace anomaly
2) **Seiberg-dualities**

related to
$m_D \neq 0$
in chiral limit
3) **soft-thms**

An emerging picture

skip

- Message seems to be: integer γ_* is special

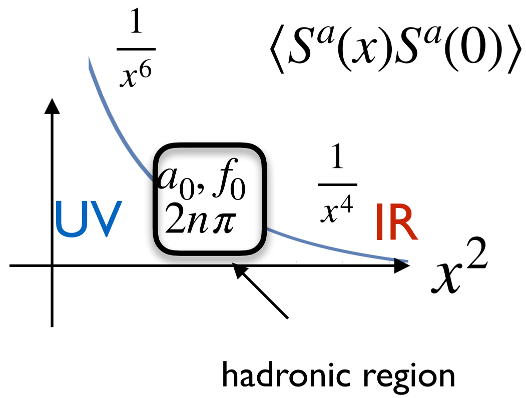
$\gamma_* = 2$ unitarity bound (Mack'77) = 1 free scalar

$\gamma_* = 1$ lower end of CW = 2 free scalars $\Delta^{UV}_{S^a} = 2$

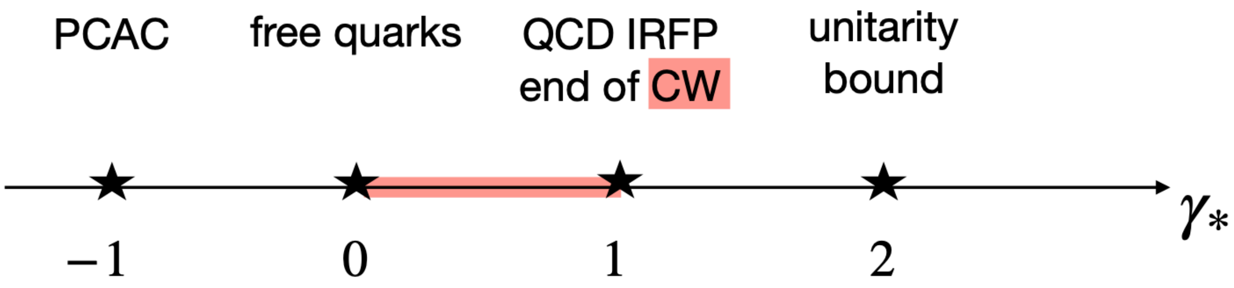
$\Big\}$ degenerate $\mathcal{N} = 1$ SUSY

$\gamma_* = 0$ upper end of CW = 2 free quarks $\Delta^{UV}_{S^a} = 3$

$\gamma_* = -1$ PCAC bound (Wilson'69)



QCD-like theories (no scalars) $\gamma_m = -\gamma_{\bar{q}q}|_{\mu=0} = \gamma_*$



- Conformal window only uses 1/3 of allowed γ_*-range

2) $\beta'_* = 0$ from Seiberg Duality

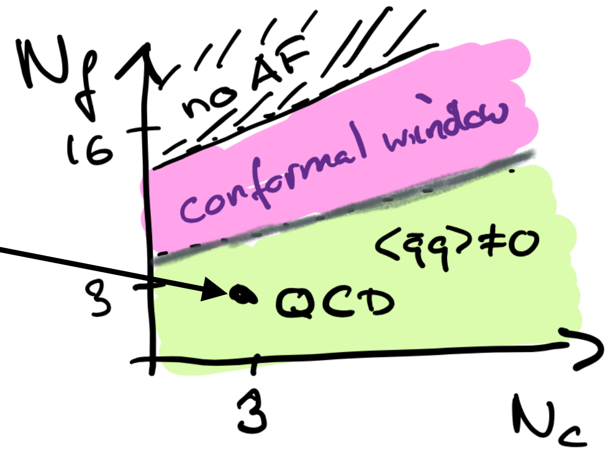
Also works for non-SUSY case using ``involved" RG-arguments

Gauge theory:

$$T^\rho{}_\rho|_{\rm phys} = \frac{\beta}{2g} G^2 \quad \Rightarrow \quad \gamma^*_{G^2} = \beta'_* \quad \Rightarrow \Delta_{T^\rho{}_\rho} = \Delta_{G^2} = 4 + \beta'_*$$

based on Shifman RZ '23, *earlier (complicated)* Anselmi, Grisaru, Johanson 97'

Does it really **make sense** to extend IRFP-scenario **below boundary** of the conformal **window?**



$\mathcal{N} = 1$ SUSY gauge theories (Seiberg duality)

electric theory

$SU(N)$ & $2N_f$ chiral matter fields

magnetic theory

$SU(N_D)$ & $N_D = N_f - N$ .. matter &

$M_i^{\bar{j}}$ colour neutral **meson field** (adjoint)

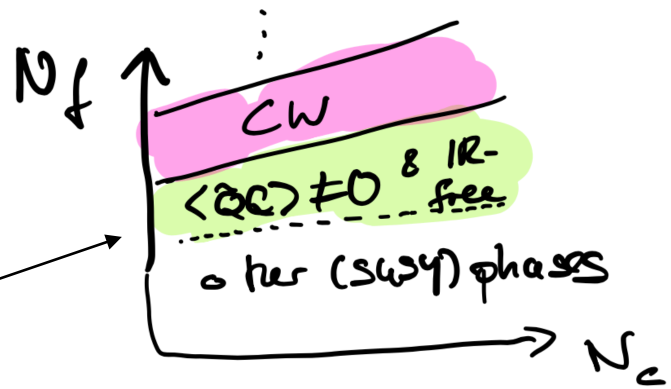
Dual IR? (a) some operators known to match (b) **global symmetries match IR**

(a) e.g. $\tilde{Q}^{\bar{j}} Q_i \quad \leftrightarrow \quad M_i^{\bar{j}}$

(b) e.g. $\langle T^{\rho}{}_{\rho}(x) T^{\alpha}{}_{\alpha}(0) \rangle_{\rm el} \overset{\rm IR}{\longleftrightarrow} \langle T^{\rho}{}_{\rho}(x) T^{\alpha}{}_{\alpha}(0) \rangle_{\rm mag}$

below CW
(R-sym. broken by VEV)

$$N + 1 < N_f < \frac{3}{2} N$$



IR-free
magnetic phase

… applying Seiberg duality

(a) $$2-\gamma_* = \Delta_{\tilde{Q}Q} = \Delta_M = 1 \Leftrightarrow \quad \gamma_* = 1$$

$\Rightarrow$ It does really **make sense** to extend IRFP-scenario in N=1 SUSY **below boundary** of the **conformal window.**

(b) matching correlator: $$\langle T^\rho{}_\rho(x) T^\rho{}_\rho(0)\rangle \propto \frac{1}{(x^2)^{4+\beta'_*}}$$

$$\beta'_*|_{\rm el} = \beta'_*|_{\rm mag} \quad \Leftrightarrow \quad \langle T^\rho{}_\rho(x) T^\alpha{}_\alpha(0)\rangle_{\rm mag} \overset{\rm IR}{\longleftrightarrow} \langle T^\rho{}_\rho(x) T^\alpha{}_\alpha(0)\rangle_{\rm el}$$

$\beta'_*|_{\rm mag} = 0$ as weakly coupled $\Rightarrow$ $\beta'_* \to 0$ by continuous **matching** to **N=1 SUSY** conformal window @boundary, again can be extended below!

Anselmi, Grisaru, Johanson 97' Shifman RZ '23

2nd result

3) $\Delta_{\mathcal{O}} = d - 2$ **double-soft dilaton thms**

$\mathcal{O} =$ operator generating dilaton mass:

$$2m_D^2 = \langle D \,|\, T^\rho_\rho \,|\, D \rangle$$

(Double) soft theorems for Goldstones

Idea: light-particle X (mass gap above): $q_X \to 0$ (soft) matrix element simplifies!

- celebrated **pion soft thm**

$$\langle \pi^a(q)\beta|\mathcal{O}(0)|\alpha\rangle = -\frac{i}{F_\pi}\langle\beta|[Q_5^a,\mathcal{O}(0)]|\alpha\rangle + \lim_{q\to 0} iq\cdot R^a$$

remainder, almost always zero $\quad R^a_\mu = -\frac{i}{F_\pi}\int d^dx e^{iq\cdot x}\langle\beta|TJ^a_{5\mu}(x)\mathcal{O}(0)|\alpha\rangle$

- less known **dilaton soft thm**

$$\langle D(q)\beta|\mathcal{O}(0)|\alpha\rangle = -\frac{1}{F_D}\langle\beta|i[Q_D,\mathcal{O}(0)]|\alpha\rangle + \lim_{q\to 0} iq\cdot R$$

$$i[Q_D,\mathcal{O}(x)] = (\Delta_{\mathcal{O}} + x\cdot\partial)\mathcal{O}(x) \qquad R_\mu = -\frac{i}{F_D}\int d^dx e^{iq\cdot x}\langle\beta|TJ^D_\mu(x)\mathcal{O}(0)|\alpha\rangle$$

Simple applications

- double-soft pion: $T^\rho{}_\rho = \mathcal{O}_{\bar{q}q} + \mathcal{O}(\delta g)$

$$\boxed{\mathcal{O}_{\bar{q}q} = (1+\gamma_*)\sum_q m_q \bar{q}q}$$

$$2m_\pi^2 = \langle \pi^a | T^\rho{}_\rho | \pi^a \rangle = \langle \pi^a | \mathcal{O}_{\bar{q}q} | \pi^a \rangle = \frac{-(1+\gamma_*)m_q}{F_\pi} \langle 0 | i[Q_5^a, \bar{q}\mathbb{1}_{N_f} q] | \pi^a \rangle$$

$$= \frac{2m_q(1+\gamma_*)}{F_\pi} \langle 0 | P^a | \pi^a \rangle = \frac{-2m_q(1+\gamma_*)}{F_\pi^2} \langle \bar{q}q \rangle \ ,$$

$$\boxed{m_\pi^2 F_\pi^2 = -(1+\gamma_*) m_q \langle \bar{q}q \rangle = -2m_q \langle \bar{q}q \rangle}$$

Gell-Mann-Oakes-Renner'67

(if $\gamma_* = 1$ once more)!

- single-soft dilaton: $F_D m_D^2 = \langle D | T^\rho{}_\rho | 0 \rangle = \Delta_{\mathcal{O}_{\bar{q}q}} \langle \mathcal{O}_{\bar{q}q} \rangle$

using $\langle D(q) | T_{\mu\nu} | 0 \rangle = \frac{F_D}{d-1}(m_D^2 \eta_{\mu\nu} - q_\mu q_\nu)$ $\quad i[Q_D, \mathcal{O}(x)] = (\Delta_\mathcal{O} + x \cdot \partial)\mathcal{O}(x)$

derivative plays no role.. unlike for the double-soft thm

Double-soft dilaton theorem

RZ, 2312.13761

$$2m_D^2 = \langle D|\mathcal{O}(x)|D\rangle = -\frac{1}{F_D}\langle 0|i[Q_D,\mathcal{O}(x)]|D\rangle = -(\Delta_{\mathcal{O}} + x\cdot\partial)\langle 0|\mathcal{O}(x)|D\rangle$$

- There is x-dependence in matrix element: $\langle 0|\mathcal{O}(x)|D(p)\rangle = F_{\mathcal{O}}e^{-ipx}$
- Interpret as distribution to be smeared out*, $\mathbf{1}_V = \frac{1}{V}\int d^dx$

$$\mathbb{1}_V[x\cdot\partial\langle 0|\mathcal{O}(x)|D\rangle] = -d\frac{1}{V}\int_V d^dx\langle 0|\mathcal{O}(x)|D\rangle$$

- Formula with **positive mass,** as $d > \Delta_{\mathcal{O}}$ and $\langle\mathcal{O}\rangle < 0$ to be lower than pert. vac.

$$m_D^2F_D^2 = \frac{1}{2}(\Delta_{\mathcal{O}} - d)\Delta_{\mathcal{O}}\langle\mathcal{O}\rangle > 0 \quad **$$

$\Rightarrow$ d-term essential as otherwise mass would be negative!

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* $\mathcal{O}_h = \int d^dx\mathcal{O}(x)h(x)$ where $h \in \mathcal{S}(\mathbb{R}^d)$ ** **d-term correspond to Zumino-term in EFT!**

A theorem on mass-operator scaling dimension

- Assume $T^{\rho}{}_{\rho} = \mathcal{O} + \dots$ with dots not contributing in matrix elements below.
- Intermediate result from before:

$$\boxed{m_D^2 F_D = \frac{1}{2}(d - \Delta_{\mathcal{O}})\langle 0|\mathcal{O}|D\rangle}$$

- Definition of decay constant:

$$\langle D(q)|T_{\mu\nu}|0\rangle = \frac{F_D}{d-1}(m_D^2 \eta_{\mu\nu} - q_\mu q_\nu)$$

$$\boxed{m_D^2 F_D = \langle 0|T^{\rho}{}_{\rho}|D\rangle = \langle 0|\mathcal{O}|D\rangle}$$

- Comparing, one concludes:

$$\boxed{\Delta_{\mathcal{O}} = d - 2}$$

3rd main result

consequences

- $m_q = 0,$ only $T^{\rho}_{\rho} \supset G^2$ with $\Delta_{G^2} = 4 + \beta'_* = 4 \neq 2$ cannot provide mass

sets **question mark** on PCDC-type formula $\bar{m}_D^2 \propto \beta\langle G^2\rangle$ (widely assumed)

- $m_q \neq 0,$ only $T^{\rho}_{\rho} \supset m_q\bar{q}q$ with $\Delta_{\bar{q}q} = 3 - \gamma_* = 2$ exactly as it should!

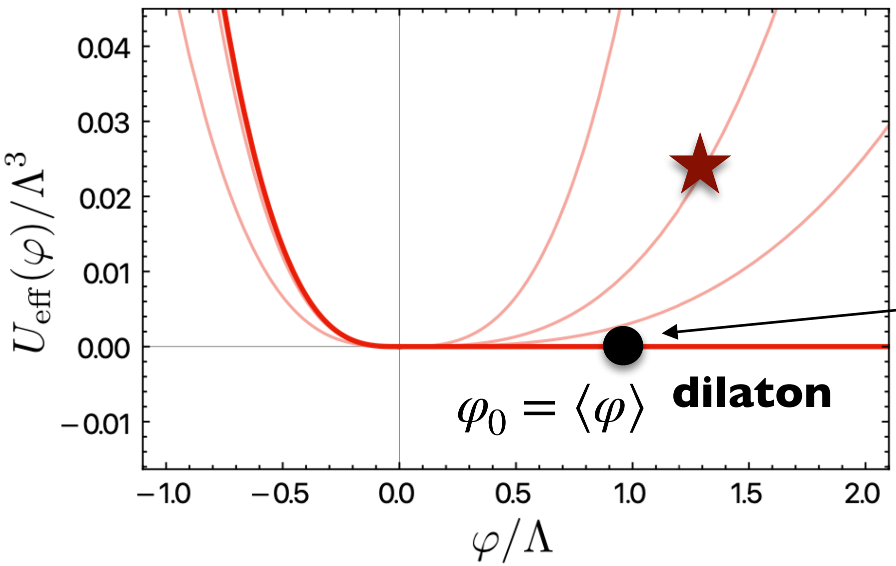
Gross-Neveu-Yukawa theory

refs: Cresswell-Litim'JHEP'24 (model),
Semenof-Steward PLB'24 (massless dilaton)
Cresswell-Litim-RZ 2502.00190 (dilaton physics)

$$S = \int_x d^3x \left(\bar\psi i \gamma_\mu \partial^\mu \psi + \frac{1}{2H^2} (\partial\varphi)^2 - H\varphi\bar\psi\psi - \frac{1}{2}\lambda_2\varphi^2 - \frac{1}{3\,!}\lambda_3\varphi^3 \right)$$

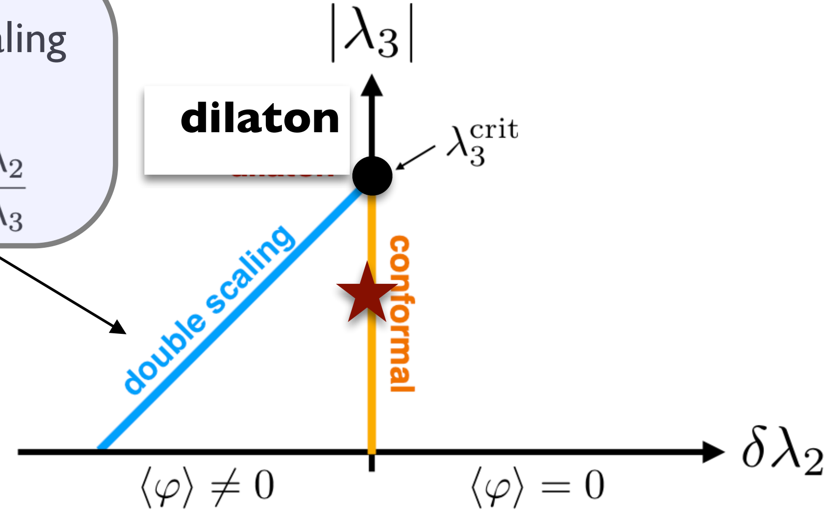
- asymptotically free, Yukawa [H]=1/2, rescaled scalar $[\varphi] = [H\phi] = 1$
- number of fermion $N \to \infty$, only fermion loops relevant, get quantum potential.

$$U_{\text{eff}}(\varphi) = \frac{1}{2}\delta\lambda_2\,\varphi^2 + \frac{1}{3!}\delta\lambda_3\,\varphi^3 + \frac{1}{3!}\lambda_3^{\text{crit}}(|\varphi|^3 - \varphi^3)$$



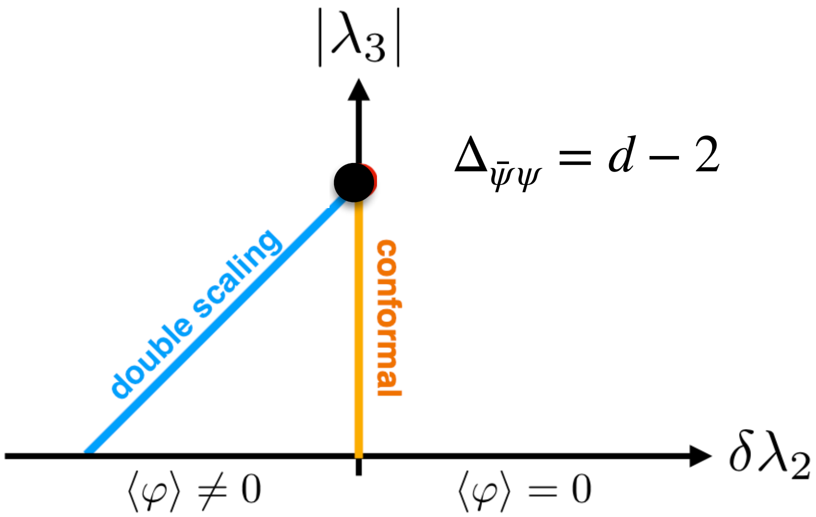
double scaling limit

$$\varphi_0 = -2\frac{\delta\lambda_2}{\delta\lambda_3}$$



- Result a) line of fixed points end in **dilaton phase** with **one-sided flat potential**
  (N.B since coset field is positive, $\chi = \exp(-D/F_D) \geq 0$, this has to be the case)

- Result b) **VEV** φ_0 can be **tuned** by deformation in **double-scaling limit**
  (N.B meaningful since there is another scale H - "plays the role of Λ_{QCD}")



not so simple
"discontinuity of limit"
(awaits publication)

- Result c) **fermion scaling dimensions** $\Delta_{\bar{\psi}\psi} = d - 2$,
  consistent with **soft theorem** prediction and $(\gamma_* = 1)$

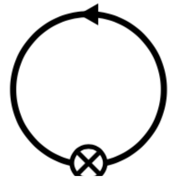
Soft theorems @ work

1. Let us consider fermion mass perturbation: $\delta H = \delta m \bar{\psi}\psi + \frac{N}{3}\delta\lambda_3 \varphi^3$
   Applying Feynman-Hellmann and double-soft thm:

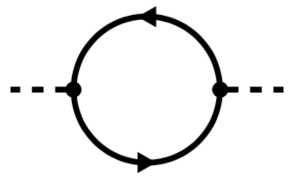
$$\boxed{m_D^2 F_D^2 = \Delta_{\bar{\psi}\psi}(\Delta_{\bar{\psi}\psi} - d)\delta m \langle \bar{\psi}\psi \rangle + \Delta_{\varphi^3}(\Delta_{\varphi^2} - d)\delta\lambda_3 \langle \varphi^3 \rangle}$$

$$\langle \varphi^3 \rangle = \varphi_0^3 \,, \quad \langle \bar{\psi}\psi \rangle = \frac{N}{2\pi}\varphi_0^2$$

from OPE can
subtract divergent part



2. From resumed correlation function get: $\boxed{m_D^2 F_D^2 = \frac{N}{2}\delta\lambda_3 \varphi_0^3}$



3. Requires double-scaling limit (stability) $\delta\lambda_3 = -\frac{2}{\pi\varphi_0}\delta m$

- Follows result d): exact $m_D^2 F_D^2$ matches soft-theorem, full consistency
  and we have expressed all "hadronic parameters" in terms of microscopic ones!

The essence of QCD and the dilaton

- **A dilaton in QCD?** Who? Consensus it would be the $\sigma \equiv f_0(500)$-meson

$$\sqrt{s_\sigma} = m_\sigma - \frac{i}{2}\Gamma_\sigma = (441^{+16}_{-8} - i272^{+9}_{-12.5})\,\mathrm{MeV}\;,$$

Caprini, Colangelo, Leutwyler'06
Roy-equations+input

- **Question**: does m_σ become massless or nearly massless in chiral limit?
  Fact: ***nobody knows*** *(more below)*

- using **dilaton-χPT:**
  1) can reproduce width ($SU(3)_F$-analysis): $\Gamma_\sigma = 616\,{}^{-108}_{+146} \pm \text{syst}^*$ MeV
  2) soft-mass even too large (EFT-convergence broken)

- What can we say about **lowering quark masses** ?

assuming dilaton (soft thm formula), $m_{u,d} \to 0$ not enough as $m_\sigma \to 0.95 m_\sigma$

$$m_\sigma^2 F_\sigma^2 = 4(m_u + m_d + m_s)\langle \bar{q}q\rangle + \ldots$$

⇒ focus on degenerate quark masses

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* notion of σ decay constant F_σ not well-defined, took model-value needs further thought

The higgs boson as a dilaton

Attention: different ways to implement ... some universal and some not.

- If **v = 0, SM conformal** (up to log-running), Higgs like a dilaton

$$(1+\frac{h}{v}) \to \chi = e^{-\frac{D}{F_D}} \to (1+\frac{h}{F_D})$$

universal

If number of **doublets = 1** $\Rightarrow$ $\boxed{v = F_\pi}$ and $\boxed{r = \frac{F_\pi}{F_D}}$ determines diff. to SM

- One can deduce indirectly: $r_{QCD} = 1.0(2) \pm$ syst, **intriguing!**
  a) **no symmetry reason** for this to happen (however, systematics…)
  b) closeness to unity, **LO-invisible @ LHC**

- An idea for model: **new gauge sector IRFP**,
  EWSB as in technicolor and **dilaton** as **naturally light Higgs**

$$\mathcal{L} \supset \frac{1}{4} v^2 e^{-2D/F_D} \mathrm{Tr}[D^\mu U D_\mu U^\dagger] - v e^{-D/F_D} \bar{q}_L Y_d U \mathcal{D}_R + \ldots$$

non-universal

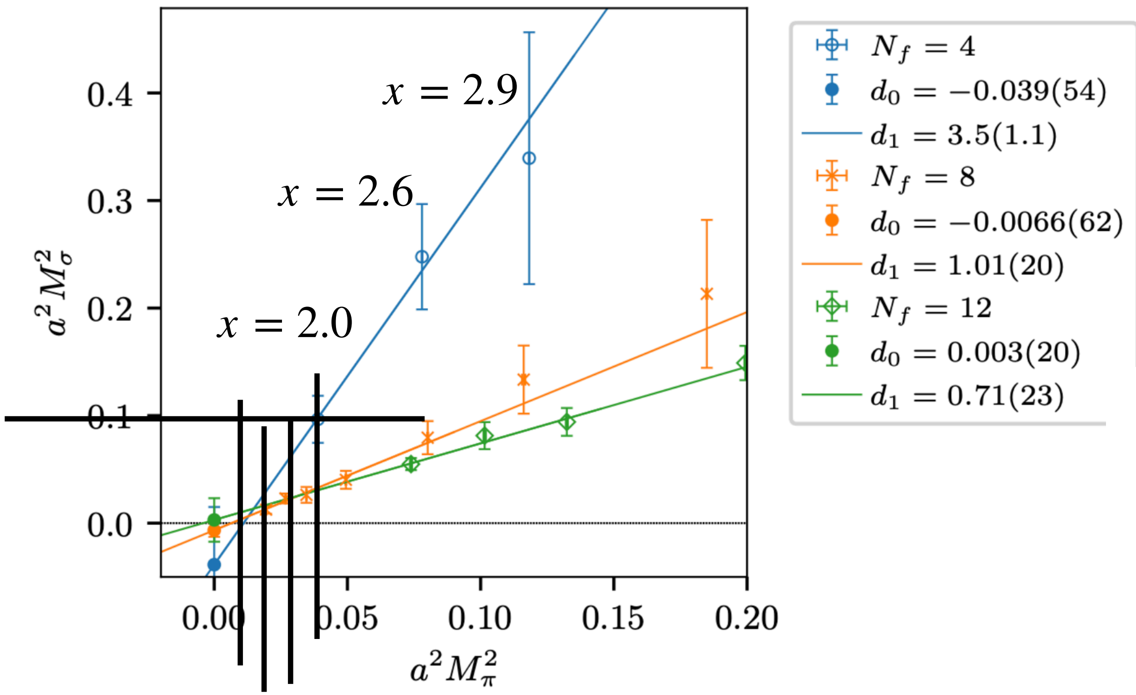
Like SM@LO but **why** coupled in this way?
Suspect, if there is a symmetry reason for $r \approx 1$,
then same reason enforces Lagrangian as above.

to be continued …

Degenerate light quarks and m_σ

$$x \equiv \frac{m_\pi}{F_\pi} / \left[\frac{m_\pi}{F_\pi}\right]_{QCD}$$

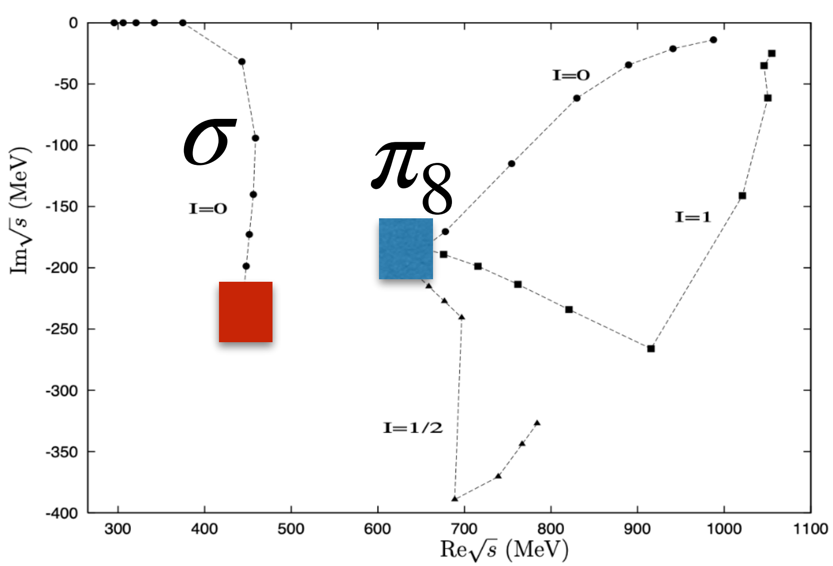
- LatKMI'25: Nf = 4,8,12 lattice



linear extrapolation (strong assumption)
not incompatible with massless dilaton

- Oller'03: Nf = 3 N/D-method (no reliable error)

$x = 2.5$



SU(3)-symmetric, $m_{\pi_8} = 350 MeV$

QCD - nature quark masses

The dilaton improves Goldstones

based on

2306.12914 RZ

The standard improved scalar field

- Two terms curved space, no dim. couplings* $\mathcal{L} = \frac{1}{2}\left((\partial\varphi)^2 - \xi R \varphi^2\right)$

$$\boxed{T^{\rho}{}_{\rho} = -d_{\varphi}(\partial\varphi)^2 + \xi(d-1)\partial^2\varphi^2 \underset{\text{eom}}{=} (d-1)(\xi - \xi_d)\partial^2\varphi^2}$$

- Conformal $T^{\rho}_{\rho} = 0$, only for $\xi = \boxed{\xi_d \equiv \frac{(d-2)}{4(d-1)}} \to \frac{1}{6}$ (d=4)
  - improved EMT Callan, Coleman, Jackiw'70, finite EMT (necessary as observable)
  - earlier in GR: Penrose'64 required by weak equivalence principle Chernikov&Tagirov'68
  - finite integrated Casimir-effect deWitt'75
- Heuristically, $\mathscr{L} \propto R\phi^2$, not possible to write with coset field $U = e^{i\frac{\pi^a T^a}{F_\pi}}$

Dolgov & Voloshin'82 Leutwyler-Shifman '89, Donoghue-Leutwyler' 91

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* may also work in flat space from start, but less elegant

Intermezzo on relevance for flow theorems

- Focus d=2 for simplicity, Weyl anomaly $T^\rho_\rho = cR$ reveals central charge of CFT.

  c-theorem (Zamalodchikov'86).: $\Delta c = c_{UV} - c_{IR} \geq 0$

  Cardy'88.: $\Delta c \propto \int d^2x\, x^2 \langle T^\rho_\rho(x) T^\rho_\rho(0)\rangle \quad \Rightarrow \quad T^\rho_\rho \to 0$ in UV and IR fast enough

  d=2 ok, Goldstone special anyway

- d=4, if **Goldstones not improvable** $T^\rho_\rho = -\frac{1}{2}\partial^2\pi^2$, then **log-IR divergence**

  a-thm* & $\Box R$-flow analogue formula IR-divergent

$\Rightarrow$ Goldstone improvement desirable

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*for a-thm, Luty, Polchiniski, Rattazzi'12 ' provide argument formula is IR-onvergent as inclusive enough

The Goldstone improvement proposal

- dilaton-pion system improvement

$$\mathcal{L}_{\rm LO} = \mathcal{L}_{\rm kin,4} + \boxed{\mathcal{L}_4^R} - V_4(\chi)$$

$$\mathcal{L}_{\rm kin,d} = \frac{F_\pi^2}{4}\hat{\chi}^{d-2}{\rm Tr}[\partial^\mu U \partial_\mu U^\dagger] + \frac{1}{2}\chi^{d-4}(\partial\chi)^2$$

standard Lag.

$$\boxed{\mathcal{L}_d^R = \frac{\kappa}{4} R\, \chi^{d-2}}$$

0, no mass (later..)

improvement term, κ to be **determined**

- **locally Weyl invariant $\Rightarrow$** conformal invariance.

$$\boxed{\kappa = \kappa_d \equiv \frac{2}{(d-1)(d-2)} \stackrel{d\to 4}{\to} \frac{1}{3}}$$

Compared to $\xi_4 = 1/6$ like a ``double improvement" (more to say)

- realises decay constant in EFT

$$\langle 0|T_{\mu\nu}|D(q)\rangle \stackrel{\rm def}{=} \frac{F_D}{d-1}(m_D^2\eta_{\mu\nu} - q_\mu q_\nu) \quad = \quad \langle 0|\boxed{T_{\mu\nu}^R}|D(q)\rangle = \langle 0|\frac{1}{6}(\eta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)\chi^2|D(q)\rangle$$

3a. **Improvement** $T^{\rho}_{\rho} = 0$ **use of equation of motion**

- dilaton eom: $\chi \partial^2 \chi = 2\mathcal{L}^{\pi}_{\text{kin},4} - \partial_{\ln \chi} V_4$

$$T_{\mu\nu} = \frac{F_\pi^2}{2} \hat{\chi}^2 \text{Tr}[\partial_\mu U \partial_\nu U^\dagger] + \partial_\mu \chi \partial_\nu \chi - \eta_{\mu\nu}(\mathcal{L}_{\text{kin},4} - V_4) + T^R_{\mu\nu}$$

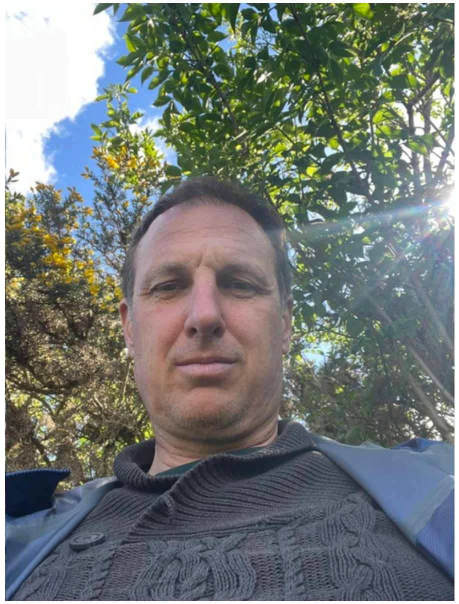
$$\boxed{T^R_{\mu\nu} = \frac{\kappa}{2}(g_{\mu\nu}\partial^2 - \partial_\mu \partial_\nu)\chi^2}$$

$$\begin{aligned} T^{\rho}{}_{\rho}|_{V=0} &= \frac{3}{2}\kappa \partial^2 \chi^2 - 2\mathcal{L}^{\pi}_{\text{kin},4} - 2\mathcal{L}^{D}_{\text{kin},4} \\ &\overset{eom}{=} \frac{3}{2}\kappa \partial^2 \chi^2 - (\partial \chi)^2 - \chi \partial^2 \chi \\ &= (3\kappa - 1)\{\chi \partial^2 \chi + (\partial \chi)^2\} = 0 \end{aligned}$$

$$\boxed{\kappa = \kappa_4 = \frac{1}{3}}$$

- **works** as expected from **local Weyl invariance**, also works d-dim curved space

Roman Zwicky
(Edinburgh University)



Past and current research:

non-pert. hadronic observables (B-physics)
QED applied to hadronic decays
some (exotic) BSM
conformal window
infrared interpretation of QCD with a dilaton

Interested to learn:

generalised symmetries
classical solutions & their dynamics
N=1 SUSY dynamics in conf. window
Hamiltonian truncation … and more

I enjoy:

hiking, table tennis, swimming comedy