Existence of IR fixed points in the QCD chiral transition

Gergely Fejős

Eötvös University Budapest, Institute of Physics and Astronomy
&
Brookhaven National Laboratory, Physics Department

ERG 2026 conference
University of Sussex

2 September, 2026

GF and T. Hatsuda, Phys. Rev. D**110**, 016021 (2024)
GF and D. Suenaga, Phys. Rev. D**111**, 076018 (2025)
GF and B. Molnar, work in progress

Introduction

- QCD with N_f no. of quarks:

$$\mathcal{L}_{QCD} = -\frac{1}{4} G^a_{\mu\nu} G^{\mu\nu a} + \bar{q}^\alpha \left(i\gamma^\mu (D_\mu) \delta^{\alpha\beta} - m^{\alpha\beta} \right) q^\beta$$

- Chiral transformations:

$$q_L \to \exp(i\theta^a_L T^a) q_L \quad q_R \to \exp(i\theta^a_R T^a) q_R$$

$\longrightarrow$ $U(N_f) \times U(N_f)$ approximate symmetry
$\longrightarrow$ the symmetry becomes exact for $m = 0$

- Chiral symmetry breaking: $\langle \bar{q}^\alpha q^\beta \rangle \neq 0$
$\Rightarrow U(N_f) \times U(N_f) \to U(N_f)$

- Longstanding question(s):
What is the transition order for $m = 0$? Can it be 2nd order?

Introduction

- Axial anomaly:

$$Z = \int \mathcal{D}\bar{q}\mathcal{D}q\mathcal{D}G \exp\left[i\int \mathcal{L}_{QCD}\right]$$

$\longrightarrow$ the measure $\mathcal{D}\bar{q}\mathcal{D}q$ breaks $U_A(1) \subset U(N_f) \times U(N_f)$

$\longrightarrow$ $U_A(1)$: $\theta_R^a \equiv -\theta_L^a \equiv \theta\delta^{a0}$

- The anomaly reduces chiral symmetry:

$$\begin{aligned} U(N_f) \times U(N_f) &\simeq SU(N_f) \times SU(N_f) \times U_V(1) \times U_A(1) \\ &\rightarrow SU(N_f) \times SU(N_f) \times U_V(1) \end{aligned}$$

- The symmetry breaking pattern for $T < T_C$ becomes

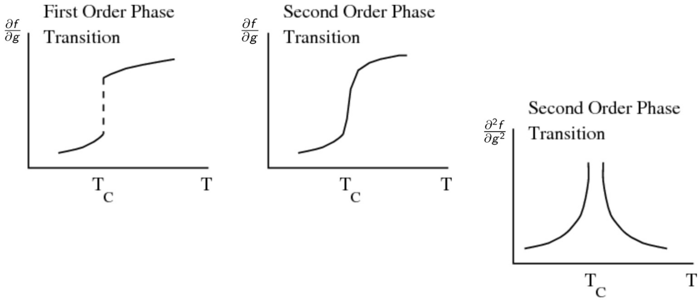
$$SU(N_f) \times SU(N_f) \rightarrow SU(N_f)$$

Introduction

- Classification of phase transitions:

  $\longrightarrow Z = \exp(-\mathcal{F}/T)$
  $\longrightarrow \mathcal{F} = \mathcal{F}(g_i)$, $f = \mathcal{F}/V$ (generalized free energy)

- First order phase transition: f smooth, $\partial f/\partial g_i$ not smooth
- Second order phase transition: f and $\partial f/\partial g_i$ smooth,
  $\partial^2 f/\partial g_i \partial g_j$ not smooth



Scaling

- Usual example:

$$\mathcal{F} = \mathcal{F}(T, V, \mu) \equiv V f(T, \mu)$$

$\longrightarrow$ particle number: $N = -V\partial f/\partial\mu$
$\longrightarrow$ number susceptibility: $\chi = -V\partial^2 f/\partial\mu^2$

- The susceptibility is related to fluctuations:

$$\chi/V = \frac{1}{T}\int d^d x \langle \delta n(x)\delta n(0)\rangle \quad \rightarrow \quad \infty$$

$\longrightarrow$ the correlation length must diverge at T_C
$\longrightarrow$ fluctuations are important at all length scales
$\longrightarrow$ no characteristic length $\Rightarrow$ self-similarity

Scaling

- How does self-similarity manifests itself mathematically?
- Scale dependent free energy: $\mathcal{F}_k = V f_k$
  (modes with wavenumbers $|\vec{q}| > k$ are included)
- Self-similarity at a 2nd order transition point:

  $\longrightarrow$ $f_k = k^d \bar{f}_*$

  $\longrightarrow$ the free energy becomes a fixed point (FP) of the renormalization group flow

  $\longrightarrow$ finite T transition: FP with one relevant direction
- Construction of $\mathcal{F}_k$ close to T_C:

Ginzburg-Landau paradigm

1) Existence of a local order parameter Φ
2) Taylor expansion $\mathcal{F}_k$ in terms of Φ
3) Structure of $\mathcal{F}_k$ obeys symmetries

Scaling

- The order parameter for the chiral transition: $\Phi_{ij} \sim \bar{q}_i q_j$
  $\longrightarrow$ chiral transformation: $\Phi \to U_L \Phi U_R^\dagger$
- The most general free energy functional

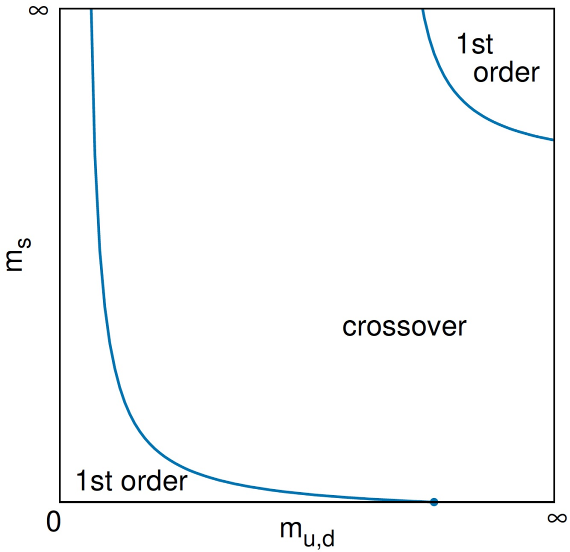
$$\mathcal{F}_k = \int_x \Big[\, m_k^2 \,\mathrm{Tr}\,(\Phi^\dagger\Phi) + g_{1,k}\big(\,\mathrm{Tr}\,(\Phi^\dagger\Phi)\big)^2 + g_{2,k}\,\mathrm{Tr}\,(\Phi^\dagger\Phi\Phi^\dagger\Phi) + \ldots$$
$$+ \,\mathrm{Tr}\,\big(\partial_i\Phi^\dagger\partial_i\Phi\big) + \ldots\Big]$$

- Scaling possible? FP with exactly one relevant coupling?
- Pisarski & Wilczek (1984)[1] [perturbative RG, ϵ expansion]:
  There is no fixed point with one relevant direction.
  $\longrightarrow$ 2nd order transition is impossible for any N_f!
  $\longrightarrow$ one exception: $N_f = 2$ with substantial $U_A(1)$ anomaly
  [anomalous term: $a(\det\Phi + \det\Phi^\dagger)$]

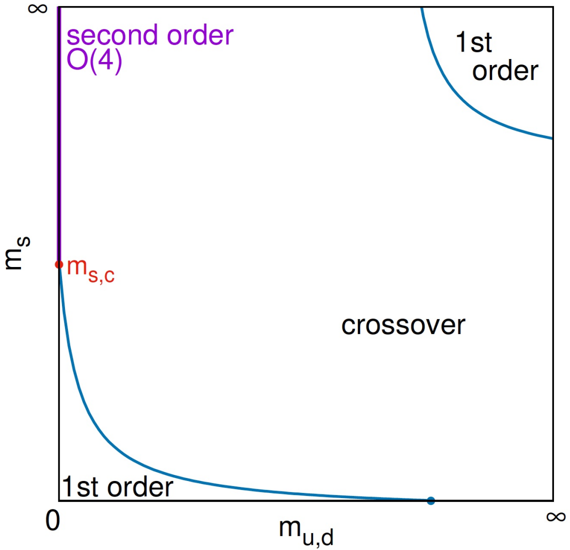
[1] R. D. Pisarski and F. Wilczek, Phys. Rev. D**29**, 338 (1984)

Columbia-plot

- Weak $U_A(1)$ anomaly at the critical point:



- Strong $U_A(1)$ anomaly at the critical point:



- Earlier FRG studies agree with the P&W argument[2]
- Recently: series of lattice simulations claim that the transition is 2nd order (up to the conformal window)[3]
- Other recent lattice simulations do not seem to observe a first order transition towards the chiral limit[4]
- Dyson-Schwinger approach[5]
- Conformal bootstrap approach[6]

Where is the IR fixed point?

[2] K. Fukushima, K. Kamikado and B. Klein, Phys. Rev. D**83**, 116005 (2011)
G. Fejos, Phys. Rev. D**90**, 096011 (2014)
S. Resch, F. Rennecke and B.-J. Schaefer, Phys. Rev. D**99**, 076005 (2019)

[3] F. Cuteri, O. Philipsen, and A. Sciarra, JHEP **11**, 141 (2021)
J. P. Klinger, R. Kaiser and O. Philipsen, PoS LATTICE2024, 172 (2025)
J. P. Klinger, R. Kaiser, O. Philipsen and J. Schaible, PoS LATTICE2025, 119 (2026)

[4] L. Dini et al., Phys. Rev. D**105**, 034510 (2022)
Y. Zhang et al., arXiv:2606.28086

[5] J. Bernhardt and C.-S. Fischer, Phys. Rev. D**108**,114018 (2023)

[6] S. R. Kousvos and A. Stergiou, SciPost Phys. **15**, 075 (2023)

Fixed points of the free energy

- Potential problems with the perturbative analysis:
  $\longrightarrow$ it assumes that $\epsilon = 4 - d$ is small
- Improving the truncation does not help
  $\longrightarrow$ ϵ expansion at LO is insensitive to going beyond $\mathcal{O}(\phi^4)$
  $\longrightarrow$ λ_6, λ_8, ... do not matter close to $d = 4$

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  $\longrightarrow$ λ_6, λ_8, ... do not matter close to $d = 4$
- An inherently $d = 3$ approach is important!
- FRG LPA: $\mathcal{O}(\phi^4)$ truncation is equivalent to LO in ϵ expansion
- FRG LPA: $\mathcal{O}(\phi^6)$ truncation:

$$
\begin{aligned}
\mathcal{F} &= \int \Big[\mathrm{Tr}\,[\partial_i \Phi^\dagger \partial_i \Phi] + V_{ch}[\Phi] \Big] \\
V_{ch}[\Phi] &= m^2\, \mathrm{Tr}\,[\Phi^\dagger \Phi] + g_1 \big(\mathrm{Tr}\,[\Phi^\dagger \Phi] \big)^2 + g_2\, \mathrm{Tr}\,[\Phi^\dagger \Phi \Phi^\dagger \Phi] \\
&+ \lambda_1 \big(\mathrm{Tr}\,[\Phi^\dagger \Phi] \big)^3 + \lambda_2\, \mathrm{Tr}\,[\Phi^\dagger \Phi] \cdot \mathrm{Tr}\,[\Phi^\dagger \Phi \Phi^\dagger \Phi] \\
&+ g_3\, \mathrm{Tr}\,[\Phi^\dagger \Phi \Phi^\dagger \Phi \Phi^\dagger \Phi] \\
&+ V_A (\rightarrow \text{ vanishing for } N_f > 6\,)
\end{aligned}
$$

Fixed points of the free energy

- $\mathbf{N_f = 5, 6}$:

$$V_A = a \cdot (\det \Phi^\dagger + \det \Phi)$$

- $\mathbf{N_f = 4}$:

$$V_A = a \cdot (\det \Phi^\dagger + \det \Phi) + b \cdot \operatorname{Tr}[\Phi^\dagger \Phi](\det \Phi^\dagger + \det \Phi)$$

- $\mathbf{N_f = 3}$:

$$\begin{aligned} V_A &= a \cdot (\det \Phi^\dagger + \det \Phi) + b \cdot \operatorname{Tr}[\Phi^\dagger \Phi](\det \Phi^\dagger + \det \Phi) \\ &+ a_2 \cdot (\det \Phi^\dagger + \det \Phi)^2 \end{aligned}$$

- $\mathbf{N_f = 2}$:

$$\begin{aligned} V_A = {} & a \cdot (\det \Phi^\dagger + \det \Phi) + b_1 \cdot \operatorname{Tr}[\Phi^\dagger \Phi](\det \Phi^\dagger + \det \Phi) \\ & + a_2 \cdot (\det \Phi^\dagger + \det \Phi)^2 + a_3 \cdot (\det \Phi^\dagger + \det \Phi)^3 \\ & + b_2 \cdot \left(\operatorname{Tr}[\Phi^\dagger \Phi]\right)^2 (\det \Phi^\dagger + \det \Phi) \\ & + b_3 \cdot \left(\operatorname{Tr}[\Phi^\dagger \Phi]\right)^3 (\det \Phi^\dagger + \det \Phi) + b_4 \cdot \operatorname{Tr}(\Phi^\dagger \Phi)^2 (\det \Phi^\dagger + \det \Phi) \end{aligned}$$

Fixed points of the free energy

- LPA with optimal regulator:

$$
\begin{aligned}
V_k &= \sum_n g_k^{(n)} \mathcal{O}_n \\
k\partial_k V_k &= \sum_n k\partial_k g_k^{(n)} \mathcal{O}_n = \frac{k^5}{6\pi^2} \mathrm{Tr}\,[k^2 + V_k'']^{-1} \\
&= \sum_n \frac{k^5}{6\pi^2} [\ldots]\, \mathcal{O}_n
\end{aligned}
$$

- β functions: $[g_n = k^{(6-n)/2} \bar{g}_n]$

$$
\beta_n \equiv k\partial_k \bar{g}_n = -\frac{1}{2}(6-n)\bar{g}_n + k\partial_k g_n / k^{(6-n)/2}
$$

- Fixed points: $\beta_i = 0 \;\forall\; i$
  $\longrightarrow$ find solutions, check stability matrix $\partial\beta_i/\partial g_j$
  $\longrightarrow$ seek fixed points with 1 relevant direction
  (i.e. positive eigenvalue) $\Rightarrow$ thermal transition

Fixed points of the free energy

- There exists a class of non-anomalous fixed points spanning the entire flavor range

  $\rightarrow$ for $N_f > 6$: no anomaly, FP has one relevant direction
  $\rightarrow$ for $N_f = 5, 6$: anomaly, FP has one relevant direction
  $\rightarrow$ for $N_f = 2, 3, 4$: anomaly, FP has two relevant directions
- For the phenomenologically important $(N_f = 2, 3)$ cases, the transition is first order
- If the $U_A(1)$ anomaly is restored at T_C, then no anomalous directions exist!
  $\rightarrow$ for $N_f = 2, 3, 4$: FP has one relevant direction

Fixed points of the free energy

- For $N_f = 2$ and $N_f = 3$ the situation is more subtle
- $\underline{N_f = 3}$:
  $\longrightarrow$ there exist a fixed point with nonzero anomaly with one relevant direction
  $\longrightarrow$ however: the stability matrix has complex eigenvalues
- $\underline{N_f = 2}$:
  $\longrightarrow$ there also exist a fixed point with large (infinite) anomaly coupling with one relevant direction [$O(4)$ FP]

Fixed points of the free energy (case of QC_2D)

- The analysis can also be done for two color QCD with $N_f = 2$
- Instead of $SU(2) \times SU(2)$ chiral symmetry, there is an emergent $SU(4)$ Pauli-Gürsey symmetry
- Ginzburg-Landau potential with $SU(4)$ symmetry
  $\longrightarrow$ results of the ϵ expansion differs from that of the FRG
  $\longrightarrow$ FRG reveals the existence of an anomaly free fixed point with one relevant direction
  $\longrightarrow$ second order transition is also likely for QC_2D

$$
\begin{aligned}
\mathcal{F}_k &= \int_x \Big(\mathrm{Tr}\,[\partial_i \Phi^\dagger \partial_i \Phi] + V_k(\Phi) \Big) \\
V_k &= m_k^2 I_1 + g_{1,k} I_1^2 + g_{2,k} I_2 + b_{1,k} I_1^3 + c_{1,k} I_1 I_2 \\
&+ a_{2,k} I_A + a_{4,k} I_A^2 + a_{22,k} I_1 I_A \\
&+ b_{2,k} I_1^2 I_A + b_{3,k} I_1 I_A^2 + b_{4,k} I_A^3 + c_{2,k} I_2 I_A
\end{aligned}
$$

Stability analysis of the fixed points

- Can we trust the stability analysis? $\rightarrow$ not quite
- $O(N)$ model[7]: phase transition governed by Wilson-Fisher FP

  $\longrightarrow$ ϵ expansion predicts one relevant direction

  $\longrightarrow$ global FP potential confirms even at low orders

- $O(N) \times O(2)$ model[8]: WF FP becomes unstable, new fixed point governs the transition

  $\longrightarrow$ the stability matrix is very sensitive to the truncation

  $\longrightarrow$ grid method: unphysical unstable direction for $N_p \lesssim 81$

  $\phi \in [0, 1.5\,\phi_{\text{min}}]$

  $\longrightarrow$ N_p can be thought of as the number of couplings

[7] A. Jüttner, D. F. Litim and E. Marchais, Nucl. Phys. B 921 (2017), 769

[8] S. Yabunaka and B. Delamotte, J. Stat. Mech. 2025(2), 023204

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  $\longrightarrow$ grid method: unphysical unstable direction for $N_p \lesssim 81$
  $\phi \in [0, 1.5\,\phi_{\text{min}}]$
  $\longrightarrow$ N_p can be thought of as the number of couplings
- $SU(2) \times SU(2)$ model is somewhat similar to $O(4) \times O(2)$
- A $\mathcal{O}(\phi^6)$ truncation contains only a few couplings
  $\longrightarrow$ the stability analysis is questionable

[7] A. Jüttner, D. F. Litim and E. Marchais, Nucl. Phys. B 921 (2017), 769
[8] S. Yabunaka and B. Delamotte, J. Stat. Mech. 2025(2), 023204

Stability analysis of the fixed points

- Global fixed point potentials in the $SU(N_f) \times SU(N_f)$ model?
- Example: $N_f = 2$ without $U_A(1)$ anomaly

$$I_1 = \mathrm{Tr}\,(\Phi^\dagger\Phi), \quad I_2 = \mathrm{Tr}\,(\Phi^\dagger\Phi\Phi^\dagger\Phi)$$
$$k\partial_k V_k(I_1, I_2) = \frac{k^5}{6\pi^2}\,\mathrm{Tr}\,[k^2 + V_k'']^{-1}$$

- Problem: I_1 and I_2 needs to be identified in the RHS

  $\longrightarrow$ V_k'' depends on the fields, not invariants

  $\longrightarrow$ Taylor expanded potentials: straightforward

  $\longrightarrow$ global potentials: seems difficult to derive a non-perturbative rhs in terms of I_1, I_2

Stability analysis of the fixed points

- A possible way to derive a non-perturbative flow equation in terms of the invariants:

$$k\partial_k V_k(I_1, I_2) = \frac{k^4}{12\pi^2}\tilde{\partial}_k \,\mathrm{Log}\,\mathrm{Det}\,(k^2 + V_k'')$$

$\longrightarrow$ the mass structure depends on the background employed
$\longrightarrow$ V_k'' is typically block diagonal
$\longrightarrow$ the determinant becomes the product of subdeterminants
$\longrightarrow$ each subdeterminant depends on the invariants separately

- Convenient choice for background: $\langle\Phi\rangle = v_0 T_0 + v_3 T_3$

$\longrightarrow$ subdet.'s 2×2, 1×1, 1×1 in both s and π sectors

$\longrightarrow$ $\mathrm{Det}\,(k^2 + V_k'') \longrightarrow (\sum_{i,j} A_{ij}\; I_1^i\; I_2^j)(\sum_{i,j} B_{ij}\; I_1^i\; I_2^j)\cdots$

Stability analysis of the fixed points

- No. of independent invariants for N_f flavors:

$$I_1 = \mathrm{Tr}\,(\Phi^\dagger\Phi), \quad I_2 = \mathrm{Tr}\,(\Phi^\dagger\Phi)^2, \quad \cdots \quad I_{N_f} = \mathrm{Tr}\,(\Phi^\dagger\Phi)^{N_f}$$

- For calculating $\mathrm{Det}\,(k^2 + V_k'')$:

$\longrightarrow$ background: $\langle\Phi\rangle = v_0 T_0 + \cdots v_{N_f} T_{N_f}$

$\longrightarrow$ V_k'' becomes block diagonal, subdet's are chirally invariant

$\longrightarrow$ $N_f = 2, 3$ can be done using symbolic programming

$\longrightarrow$ $N_f \geq 4$ becomes increasingly involved

- The dimensionless flow equation is expressed by a β functional:

$$k\partial_k \bar{V}_k(\bar{I}_1, \cdots \bar{I}_{N_f}) = \beta[\bar{V}_k(\bar{I}_1, \cdots \bar{I}_{N_f})]$$

$\longrightarrow$ global fixed point(s): $\beta[\bar{V}_*] \stackrel{!}{=} 0$

Stability analysis of the fixed points

- Solution of the $\beta[\bar{V}_*] \overset{!}{=} 0$ equation:

1. Taylor method: $\bar{V}_* = \sum_{\{n_i\}} \bar{c}_{n_1,n_2,...,n_{N_f}} \bar{I}_1^{n_1} \bar{I}_2^{n_2} ... \bar{I}_{N_f}^{n_{N_f}}$

$\longrightarrow$ V_*' and V_*'' derivatives are calculated analytically

$\longrightarrow$ pick a truncation and obtain equations for all $c_{\{n_i\}}$ (derivatives at zero field) $\Rightarrow$ difficult

$\longrightarrow$ higher order truncations are analytically demanding

2. Grid method: $\bar{V}_*$ is established in a hypercubic lattice

$$\bar{V}_* \longrightarrow \bar{V}_{*,p} \ (p = 1, ...N_p)$$

$\longrightarrow$ $\bar{V}_*'$ and $\bar{V}_*''$ derivatives are calculated numerically

$\longrightarrow$ obtain algebraic equations for all $\bar{V}_{*,p}$

$\longrightarrow$ numerically demanding, $N_f \geq 3$ is hopeless

Stability analysis of the fixed points

3. Collocation grid method: $\bar{V}_* = \sum_{\{n_i\}} \bar{c}_{n_1,n_2,\dots,n_{N_f}} \bar{I}_1^{n_1} \bar{I}_2^{n_2} \dots \bar{I}_{N_f}^{n_{N_f}}$

$\longrightarrow$ pick a truncation but do not derive and solve FP eq. for derivatives (couplings)

$\longrightarrow$ consider a collocation grid $\bar{V}_* \to \bar{V}_{*,p}$ and solve $\beta[\bar{V}_{*,p}] = 0$

$\longrightarrow$ $\bar{V}_*'$ and $\bar{V}_*''$ derivatives are calculated analytically (fast)

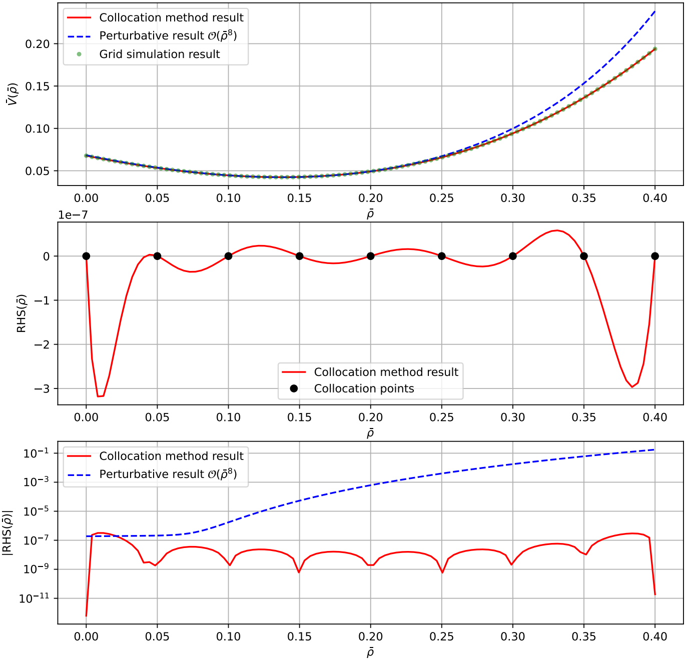
$\longrightarrow$ no need to derive FP eq's for couplings (fast)

$\longrightarrow$ the procedure is not limited by multivariable $\bar{V}_*$

- Test for the O(8) model:

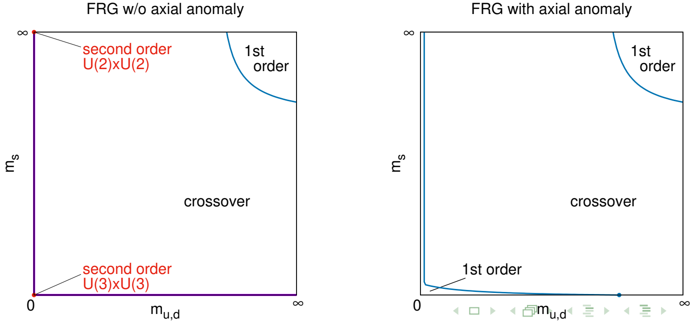
  $\bar{V}_*(\bar{\rho}) = \sum_{n=1}^{10} c_n \bar{\rho}^n$

  $\beta[\bar{V}_*] = -3\bar{V}_* + \bar{\rho}\bar{V}_*' + \frac{1}{6\pi^2}\left[\frac{15}{1+\bar{V}_*'(\bar{\rho})} + \frac{1}{1+\bar{V}_*'(\bar{\rho})+2\bar{\rho}\bar{V}_*''}\right]$


Collocation method result
Perturbative result $\mathcal{O}(\bar{\rho}^8)$
Grid simulation result
$\bar{V}(\bar{\rho})$
0.20
0.15
0.10
0.05
0.00 0.05 0.10 0.15 0.20 0.25 0.30 0.35 0.40
$\bar{\rho}$
1e−7
RHS($\bar{\rho}$)
0
−1
−2
−3
Collocation method result
Collocation points
|RHS($\bar{\rho}$)|
10^{-1}
10^{-3}
10^{-5}
10^{-7}
10^{-9}
10^{-11}
Collocation method result
Perturbative result $\mathcal{O}(\bar{\rho}^8)$

Summary

- Re-analysis of the Ginzburg-Landau free energy of the chiral transition
  $\longrightarrow$ ϵ expansion around $d = 4$ gives no IR stable fixed point
  $\longrightarrow$ FRG directly in $d = 3$ with $\mathcal{O}(\phi^6)$ truncation:
  new class of fixed points (all N_f!)
- The new fixed points are critical for $N_f \geq 5$
- The new fixed points are bicritical for $N_f = 2, 3, 4$
  $\longrightarrow$ without $U_A(1)$ anomaly all fixed points are critical



Summary

- Stability analysis is questionable $\Rightarrow$ establishing non-perturbative fixed point potentials is necessary
- Collocation grid method:
  $\longrightarrow$ keep the Taylor ansatz up to some order
  $\longrightarrow$ set up a collocation grid and require $\beta[V_{*,p}] = 0$
  $\longrightarrow$ algebraic equations for Taylor coeff.'s
- The method works well for the $O(N)$ model
  $\longrightarrow$ expectation: multi variable potentials can be handled
  $\longrightarrow$ stability analysis shows convergence at finer grids
- Future plan:
  $\longrightarrow$ $SU(2) \times SU(2)$ model: 2 variables
  $\longrightarrow$ inclusion of $U_A(1)$ anomaly: 3 variables
  $\longrightarrow$ $SU(N_f) \times SU(N_f)$ model: $N_f + 1$ variables (at least for $N_f = 3$)