the conformally flat limit of quadratic gravity (CFQG) and beyond

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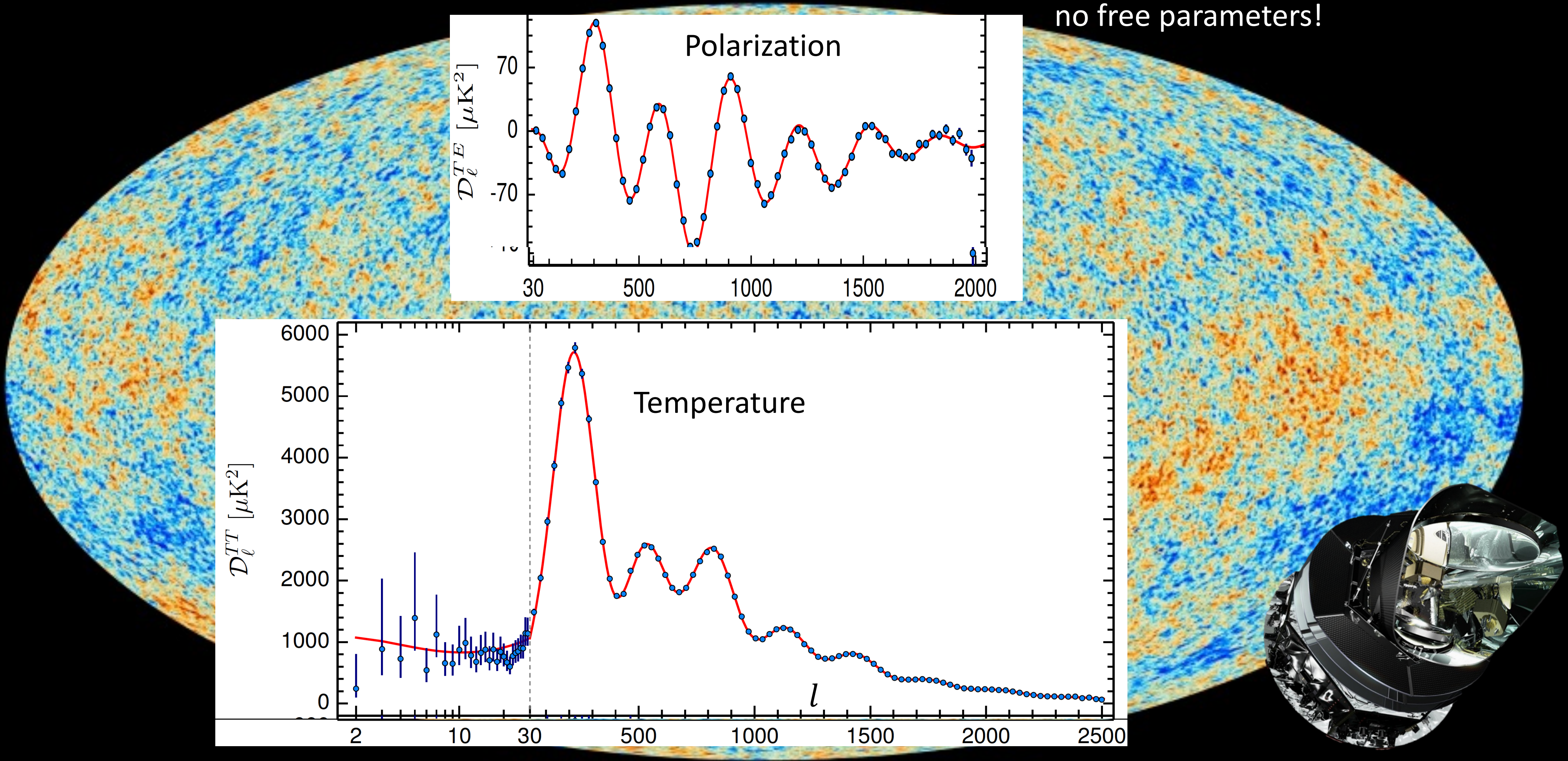
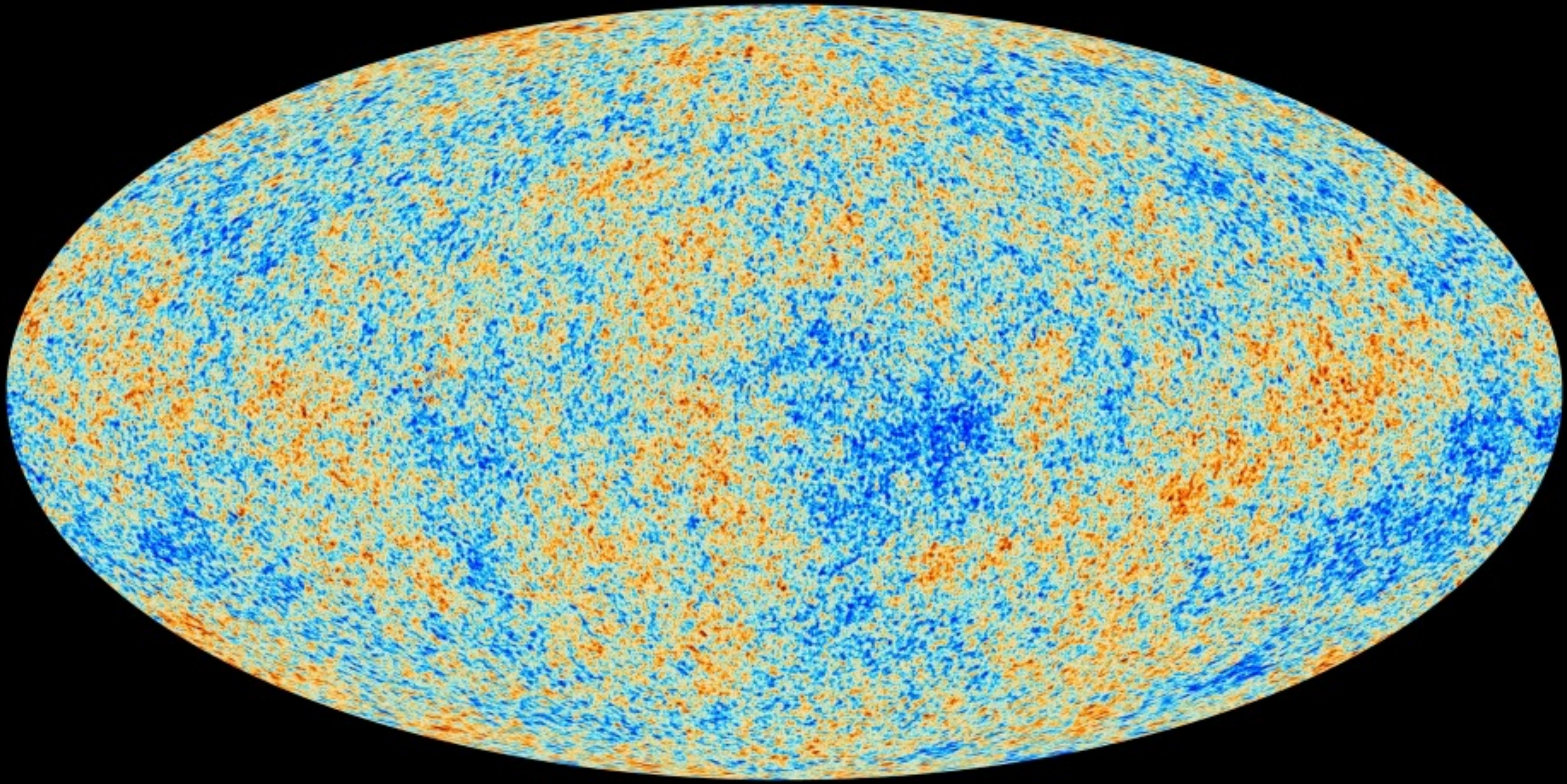
simple universe

Coulson, Crittenden, NT (1994)
no free parameters!



acoustic peaks (sound waves in plasma)

ESA Planck satellite



is this pattern a direct signature of quantum gravity?

nearly scale-invariant primordial spectrum

cf. vac fluctuations of a *four*-derivative free quantum field

$$S \sim \int d^4x \, (\Box\, \phi)^2$$

$$\Rightarrow \; \langle \phi(\vec{x})\phi(0)\rangle \sim \int d^3\vec{k}\, \langle \phi_{\boldsymbol{k}} \phi_{-\boldsymbol{k}} \rangle \boldsymbol{e}^{\boldsymbol{i}\vec{k}.\vec{x}} \sim \int d^3\vec{k}\, \frac{\boldsymbol{e}^{\boldsymbol{i}\vec{k}.\vec{x}}}{\boldsymbol{k}^{\boldsymbol{3}}}$$

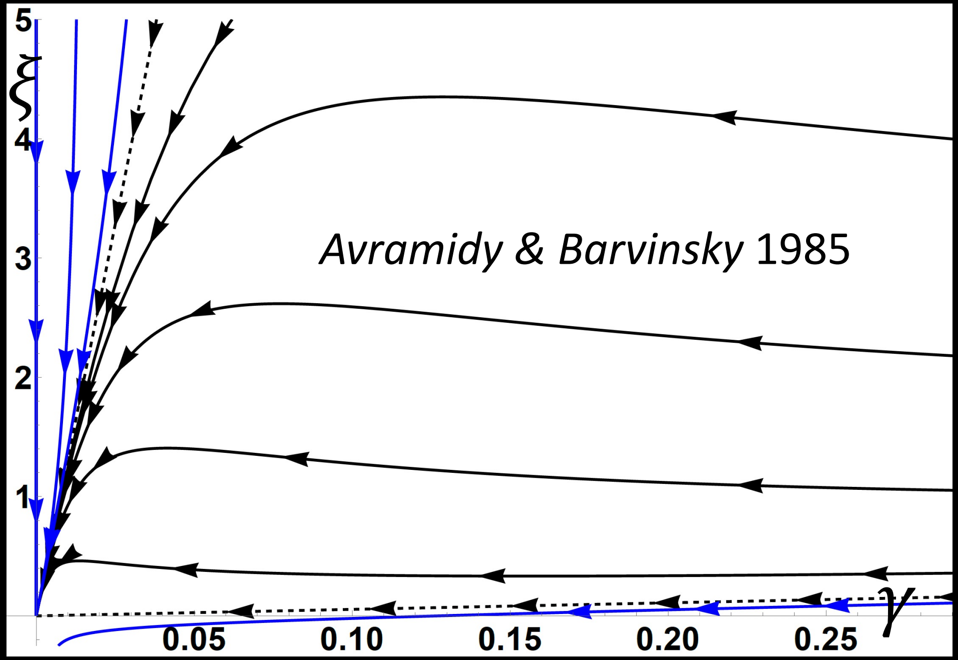
minimal approach to quantum gravity

Most general renormalizable diff invariant action (quadratic gravity) *K. Stelle 1976*

Ricci scalar Weyl tensor

Euclidean action $$\int d^4x\sqrt{g}\left(\Lambda - \frac{1}{2}M_{Pl}^2 R + \frac{1}{\xi}R^2 + \frac{1}{2\gamma}C^2\right) + \text{total div}$$

Asymptotically free!



Problems arise in the continuation to Minkowski

Notably, Ostrogradsky's 1848 theorem:
the classical Hamiltonian is unbounded below.

Choose $\xi, \gamma > 0$ so the 4-derivative terms are positive. These dominate in the UV.

At high energies, we have "pure" QG: $S_E \sim \int d^4x\sqrt{g}\left(\frac{1}{\xi}R^2+\frac{1}{2\gamma}C^2\right)$

Consider the limit $\gamma \downarrow 0$, $C^{\mu}{}_{\nu\lambda\rho} \to 0 \Rightarrow g_{\mu\nu} = \Omega^2(x)\delta_{\mu\nu}$

Anderson, Bateman, Herzog & NT 2026 & in prep

Assume the metric is nonsingular $\Rightarrow$ $\Omega = \lambda^{-1}e^{\lambda\sigma} > 0, -\infty < \sigma < \infty.$

$$\Rightarrow S_E = \frac{36}{\xi}\int d^4x \left(\frac{\Box\Omega}{\Omega}\right)^2 \underset{\xi\,=\,72\,\lambda^2}{=} \frac{1}{2}\int d^4x\,(\Box\,\sigma + \lambda(\partial\sigma)^2)^2$$

cf. Yang-Mills

Interacting, 4-derivative, UV-complete: the "perfect square" (PS) theory

(special case of Holdom's class of higher derivative, shift-invariant theories)

Diff invariance of QG and its conformally flat limit yields a Ward identity which protects the perfect square form hence there is only one coupling constant.

A remarkable symmetry

$$g_{\mu\nu} = \Omega^2 \eta_{\mu\nu} \Longrightarrow R = -6\frac{\Box\Omega}{\Omega^3}\ ; \qquad S \sim \int R^2 = 36 \int \left[\frac{\Box\Omega}{\Omega}\right]^2 = 36 \int \frac{\Box\Omega}{\Omega^2}\Omega$$

This action has a discrete on-shell classical symmetry: $\Omega \leftrightarrow \dfrac{L^2\Box\Omega}{\Omega^2}$.

But this is $\frac{L^2 R}{6} = -\dfrac{L^2\Box\Omega}{\Omega^3} = -\dfrac{L^2\Box\Omega}{\Omega^2}\dfrac{1}{\Omega} \rightarrow -\Omega\,\frac{\Omega^2}{L^2\Box\Omega} = \frac{6}{L^2 R}$

This non-perturbative symmetry will turn out to be Hidden Ghost Parity.

As we shall see, it ensures that transition probabilities are positive.

Sam Bateman + NT 2607.00096

Consider free, 4-derivative scalar field satisfying $\Box^2\,\sigma = 0$ (a Green hyperbolic equation)
Feynman propagator $\widetilde{\Delta}_F = -i/(p^2 + i\epsilon)^2$: *double* poles at $p^0 = \pm(|\vec{p}| - i\epsilon)$
Positive energy modes propagate forward in time.
Ensures causality and the optical theorem ('t Hooft-Veltman).
Free quantum Hamiltonian $\widehat{H}$ has *eigenvalues* ≥ 0.

How is this consistent with Ostrogradsky? The classical Hamiltonian is unbounded below.
Resolution: *coherent states* give arbitrarily negative *expectation values* if $|f\rangle = e^{i\int f\sigma}|0\rangle$,
then $\langle f|\sigma\,|f\rangle = \int \Delta\, f \equiv \sigma_f$ where Δ is the advanced minus retarded Green's functions (see over).
Thus $\langle f|\widehat{H}\,|f\rangle = H_{cl}(\sigma_f)$ is arbitrarily negative while evalues of $\widehat{H} \geq 0 \Rightarrow$ inner product is indefinite.

Instead of a Hilbert space, we must work in a Krein space $\mathcal{K}$, the indefinite analogue
$\mathcal{K} = \mathcal{K}_+ + \mathcal{K}_-$, $\hat{\kappa}\,\mathcal{K}_\pm = \pm\mathcal{K}_\pm$; $\mathcal{K}_\pm$ is the positive/negative definite subspace of $\mathcal{K}$; $\hat{\kappa}$ is *ghost parity*

Covariant quantization (Algebraic QFT)

Bogoliubov *et al.*

Start from commutator function (*algebra*) – advanced minus retarded GFs

$$[\sigma(x),\sigma(0)] = i\Delta(x); \quad \Delta(x) = -\epsilon(x^0)\theta(x^2) \;\Leftrightarrow\; \widetilde{\Delta} = \epsilon(p^0)2\pi\delta'(p^2)$$

Spectral assumption $p^0 \geq 0$ fixes vacuum (*representation*)

$$W(x) = \langle 0|\sigma(x)\sigma(y)|0\rangle \quad \Leftrightarrow \quad \widetilde{W} = \theta(p^0)2\pi\delta'(p^2) \quad \text{(indefinite!)}$$

cf. Holdom 2023, 2024

Then use $\delta'(p^2) = -\partial_{m^2}\delta(p^2 - m^2)|_{m^2=0}$ when computing cross sections

the Born rule generalized to a Krein space

Born rule: consider a physical process $A = P_{out}\, S\, P_{in}$ and $\text{Prob}(A) = \mathrm{Tr}\big(A^\dagger A\big)$.

(standard density matrix formalism: probabilities are positive in a Hilbert space)

Now generalize to Krein space:

If A is ghost-parity symmetric,

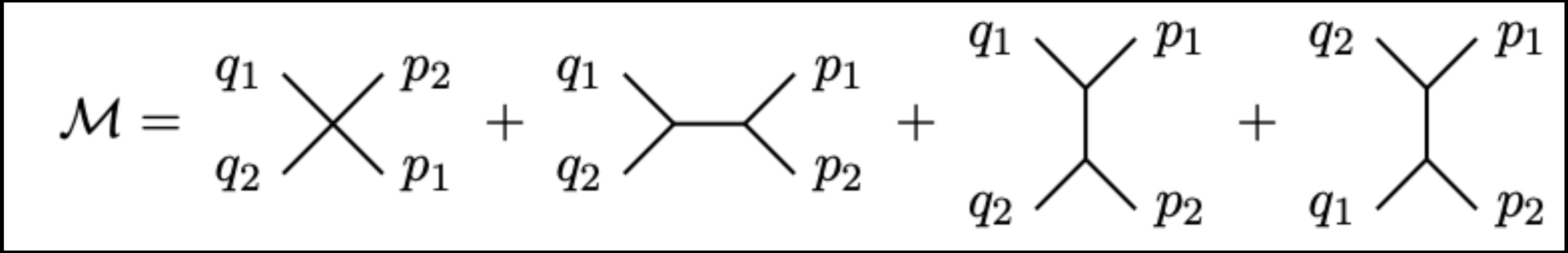
$$\exists\ \widehat{\kappa}\ \text{s.t.}\ \hat{\kappa}\, A\hat{\kappa} = A,\ \ \text{Prob}(A) = \mathrm{Tr}\big(\hat{\kappa}A^\dagger\hat{\kappa}A\big) \geq 0 \text{ because } \langle\psi\hat{\kappa}\psi\rangle \geq 0.$$

(Subtle: any *particular* κ breaks covariance but the result is covariant).

A theory with (weak) κ-symmetry thereby yields positive transition probabilities

(consistent with Kolmogorov axioms) although it is pseudo-unitary in the mathematical sense.

Example: PS theory



$$\frac{d\sigma}{d\Omega} = \frac{\partial^4}{\partial m_1^2 \partial m_2^2 \partial m_3^2 \partial m_4^2} \left(\frac{|\mathbf{p}_1|\ |\mathcal{M}|^2}{(16\pi)^2 |\mathbf{q}_1| s} \right) \Bigg|_{m^2=0} = \frac{3\lambda^4}{32\pi^2 s},$$

agrees with Holdom
good UV behaviour

We generalized this to prove that all tree level transition probabilities are positive.
With Anderson and Herzog, we proved that there are no IR loop divergences.
From the optical theorem, we expect sufficiently inclusive physical processes are IR finite.

Note: i) only transition probabilities, NOT amplitudes, are placed on shell
ii) there is no projection onto a ghost-free sector: all states are traced over

Hidden Ghost Parity

Becomes manifest when we embed theory into a 2-derivative, $O(1,1)$-symmetric “host”:

$$S_{\Omega\Upsilon} = \int \partial\Omega\partial\Upsilon - \frac{\lambda^2}{2}(\Omega\Upsilon)^2 \underset{\text{integrate out }\Upsilon}{\rightarrow} \frac{1}{2\lambda^2}\int \left(\frac{\Box\Omega}{\Omega}\right)^2 ; \qquad \Omega = \lambda^{-1}e^{\lambda\sigma},\ \Upsilon = \lambda^{-1}e^{-2\lambda\sigma}\,\Box\, e^{\lambda\sigma}$$

(see also papers of Salvio et. al)

$SO(1,1)$ symmetry yields conserved “ghost charge”

Ghost parity is $\Omega \leftrightarrow \Upsilon$, *i.e.,* charge conjugation, or $\sigma \rightarrow -\sigma + \lambda^{-1}\ln(\Box\,\sigma + \lambda(\partial\sigma)^2)$

The “host theory” has an indefinite Euclidean action, but a well-defined perturbative expansion, identical to that of standard 2-deriv, $O(2)$-symmetric $\frac{g}{4!}(\vec{\phi}^2)^2$ theory, with $g = -3\lambda^2$.
Long known to be asymptotically free at negative g.

Remarkable correspondences:

Anderson, Bateman, Herzog, NT
arXiv: 2608.12210

$$\mathrm{QG} \xrightarrow{\gamma \downarrow 0} \mathrm{CFQG}$$

$$\frac{1}{2}(\partial\vec{\phi})^2 + \frac{g}{4!}(\vec{\phi}^2)^2$$

diff. invariance
restricts Lagrangian
to perfect square

$\xi = 72\lambda^2$

$g = -3\lambda^2$

standard O(2) quartic scalar
at negative coupling

PS

$$\frac{1}{2}(\Box\,\sigma + \lambda(\partial\sigma)^2)^2$$

Explicit checks

All three beta functions agree at two loops; (using Salvio *et al.'s* two loop result for QG)

PS and quartic scalar agree at 3 loops;

Using known results for quartic scalar theory, we predict CFQG's β function to 6 loops!

An unbounded-below Hamiltonian

Zero modes: spatially homogeneous line element can be written

$-N(t)^2dt^2 + h_{ij}(t)(dx^i+N^i(t)dt)\left(dx^j + N^j(t)dt\right) \implies \quad C^2 = \frac{1}{2}(\ddot{h}_{ij}-\frac{1}{3}\delta_{ij}\ddot{h}_{kk})^2, R^2 = (\ddot{h}_{kk})^2$

Sending γ to zero eliminates (more accurately, decouples) all dynamical modes of the spatial metric except the trace (or scale factor σ). The lapse $N(t)$ and the shift $N^i(t)$ have no kinetic terms. They are Lagrange multipliers which fix the total energy-momentum P^μ to zero.

Particle excitations have positive energy. So the background *must* have negative energy.

To describe a (spatially compact) background as well as radiation, with arbitrarily positive density, within it, it is *necessary* that the Hamiltonian density be unbounded below.

(This is the case for example in GR where the Friedmann equation reads $\int_\Sigma \left(T_0^0 - \frac{1}{8\pi G}G_0^0\right) = 0.$)

(Hamiltonian constraint)

Due to Ostrogradsky, this scalar theory of gravity can describe realistic cosmological backgrounds with high energy particles and fluids within them.

Four-derivative theory ⇒ more solutions than Einstein.

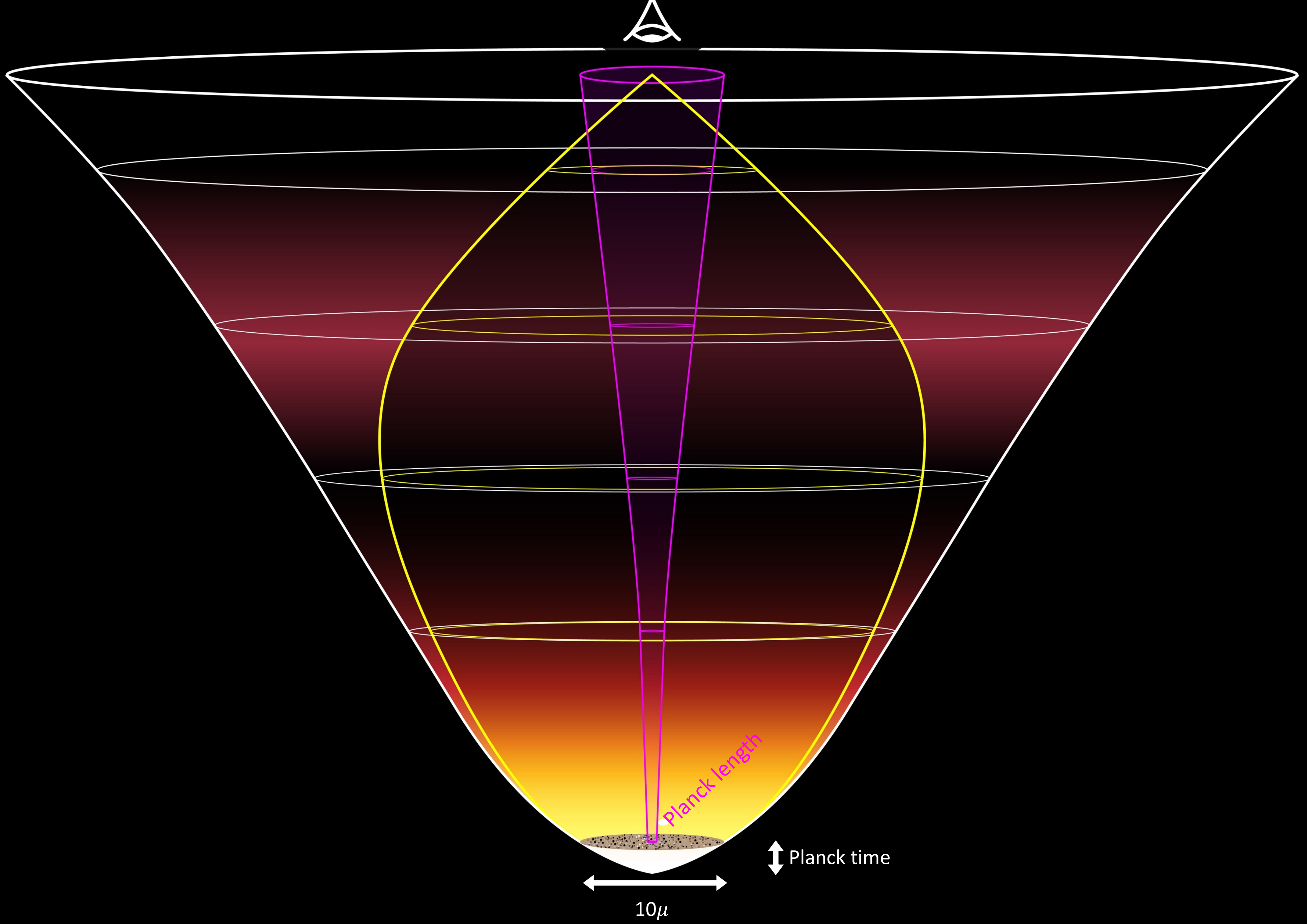
For FRW backgrounds, the first integral of the eom is the $O(1,1)$ ghost charge.

Setting this to zero, the second integral is the cosmological constant.

The third integral is the Hamiltonian which is set zero (and can include radiation).

We recover all radiation + Lambda FRW (Einstein) cosmologies (and only them).

For positive Λ, the Ostrogradsky instability is *reinterpreted* as de Sitter expansion. Accounting for their zero modes, asymp. de Sitter backgrounds are all stable.


Planck length
Planck time
10μ

Boyle, NT, Vaibhav to appear

Can we trace the universe back to temperatures above T_{Pl}?

1) In Einstein gravity, cross sections violate perturbative unitarity at hi T : in quadratic gravity they do not.
2) The background remains classical for $T \gg T_{Pl}$
3) The fluid approximation remains valid since $\mathcal{N}_{eff} \gg 1$
4) A new mechanism for dark matter production.

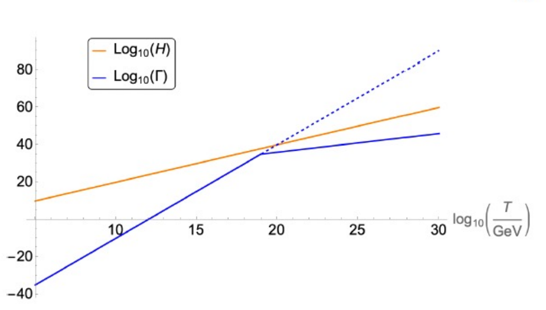


FIG. 2. Comparison of the gravitational annihilation rate $\Gamma(T)$ gravity with the Hubble rate $H(T)$. In Einstein gravity (dashed blue line), $\Gamma(T) > H(T)$ (and unitarity is violated) when $T \gtrsim M_{pl}$. In quadratic gravity (solid blue line), the unitarity bound $f_2 \lesssim \mathcal{O}(1)$ implies $\Gamma(T) \lesssim H(T)$ for all T, and $f_2 \ll 1$ implies $\Gamma(T) \ll H(T)$, so that dark matter never thermalizes.

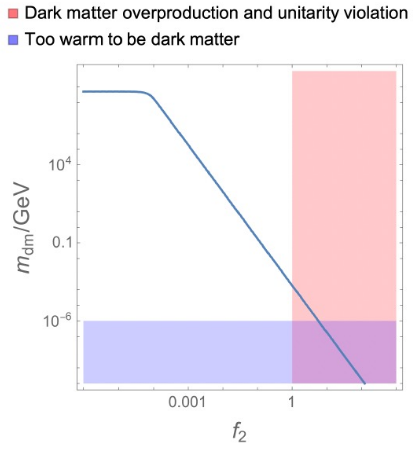


FIG. 3. Constraint on the Weyl coupling f_2 and the dark matter mass m_{dm} coming from conservation of dark matter yield.

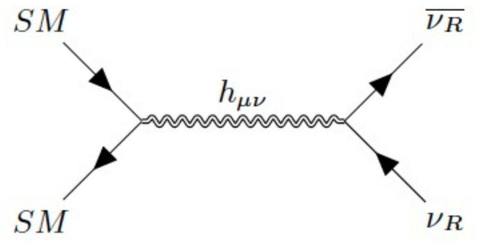


FIG. 1. Standard Model particles gravitationally annihilating into right-handed neutrino dark matter.

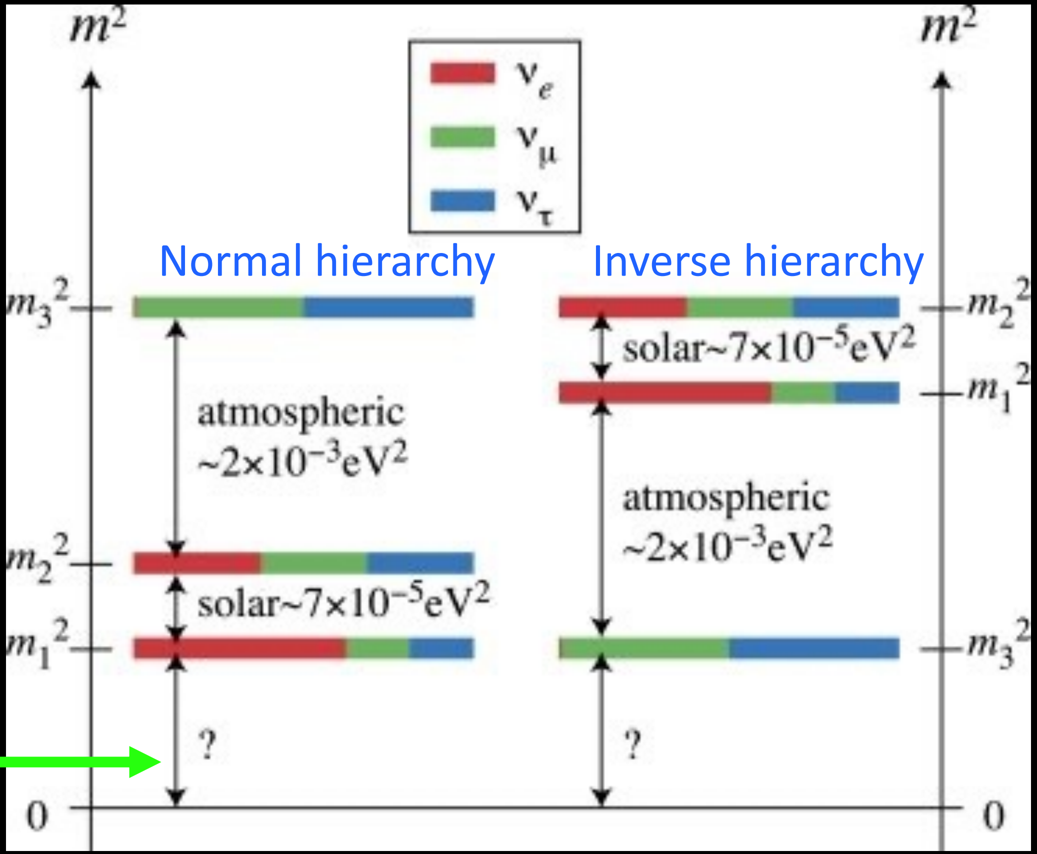
thank you for listening!

Light neutrinos: observations

current data

DESI DR2 BAO arXiv:2503.1473

we predict zero



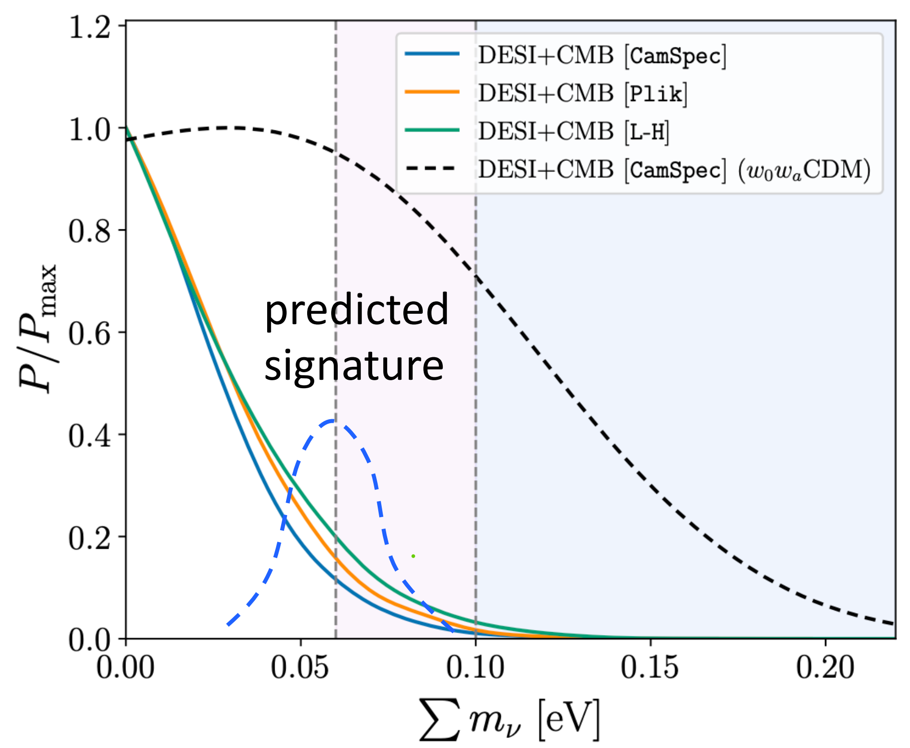


FIG. 15. 1D marginalized posterior constraints on $\sum m_\nu$ from DESI DR2 BAO measurements combined with different CMB likelihoods, assuming the ΛCDM+$\sum m_\nu$ model.

Normal hierarchy: $M_\nu \equiv \sum m_\nu \approx 0.06\ eV$

Inverted hierarchy: $M_\nu \approx 0.1\ eV$

(Ivanov *et al.* 2601.16165 claim M_ν <0.057 eV in ΛCDM)

cross-checks coming from JUNO, DUNE, Hyper K…

Longer term: ν-less 2β decay rate predicted exactly