Non-Gaussian statistics of the order parameter across a continuous phase transition

Adam Rançon

Laboratoire PhLAM – Université de Lille

ERG 2026, University of Sussex

02/09/2026

with I. Balog, B. Delamotte, F. Rose, L. Šaravanja and S. Sahu,
PRL ’22 , PRE ’25, SciPost Phys. ’25, PRE ’25, PRE ‘26

as well as N. Dupuis, T. Roscilde and the group of D. Clément
Allemand et al. Nature ‘26

Université de Lille

PhLAM
Physique des Lasers
Atomes et Molécules

Introduction

Introduction

Statistics/probability needed for complex systems (many degrees of freedom, correlations, etc.)

Ex: biology, demographics, engineering, finance, statistical physics, …

Introduction

Statistics/probability needed for complex systems (many degrees of freedom, correlations, etc.)

Ex: biology, demographics, engineering, finance, statistical physics, …

When d.o.f independent or weak correlations: Central Limit Theorem (CLT) and generalization (Levy distribution)

Introduction

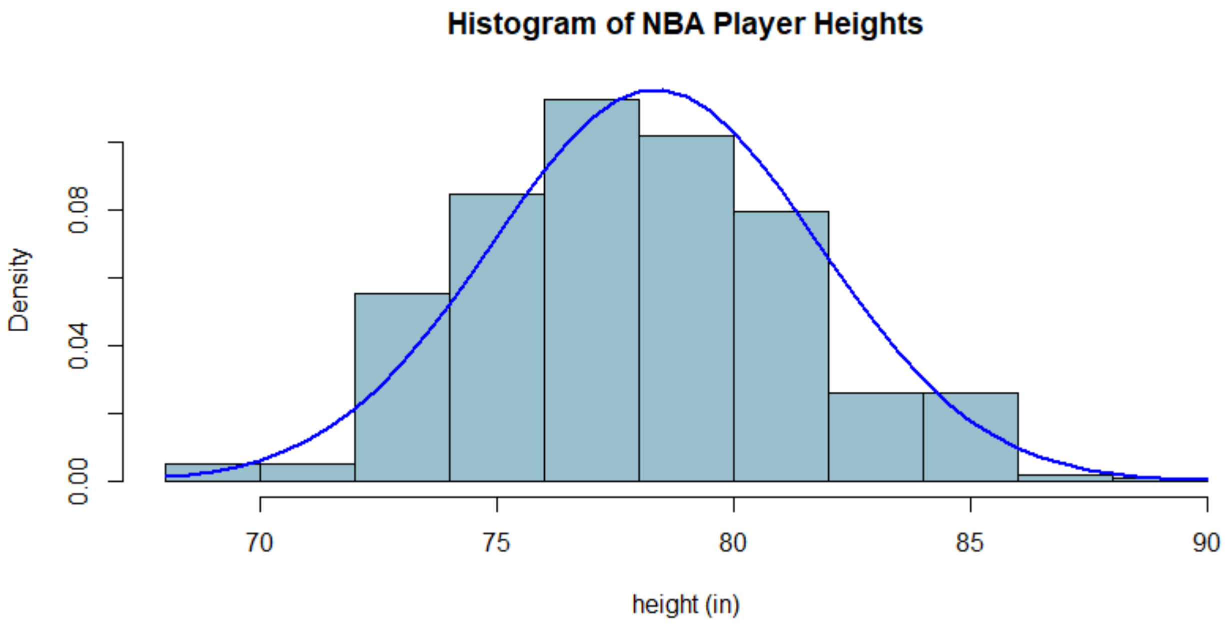
Statistics/probability needed for complex systems (many degrees of freedom, correlations, etc.)

Ex: biology, demographics, engineering, finance, statistical physics, …

When d.o.f independent or weak correlations: Central Limit Theorem (CLT) and generalization (Levy distribution)

Sum of independent random events

Normal distribution



Central Limit Theorem

Take N random i.i.d variables $\hat{s}_i = \pm 1$ (coin flips, spins, Brownian motion, Ising at T=∞)

Distribution of the sum for large N, $\hat{S} = \sum_i \hat{s}_i$

Central Limit Theorem

Take N random i.i.d variables $\hat{s}_i = \pm 1$ (coin flips, spins, Brownian motion, Ising at T=∞)

Distribution of the sum for large N, $\hat{S} = \sum_i \hat{s}_i$

Law of large number

$$P_N(\hat{S} = Ns) \to \delta(s)$$

$$s = \frac{1}{N}\sum_i s_i \qquad \langle s^2 \rangle = \frac{1}{N}$$

Central Limit Theorem

Take N random i.i.d variables $\hat{s}_i = \pm 1$ (coin flips, spins, Brownian motion, Ising at T=∞)

Distribution of the sum for large N, $\hat{S} = \sum_i \hat{s}_i$

Law of large number

$$P_N(\hat{S} = Ns) \to \delta(s)$$

$$s = \frac{1}{N}\sum_i s_i \qquad \langle s^2 \rangle = \frac{1}{N}$$

Central Limit Theorem (CLT)

$$P_N(\hat{S} = \sqrt{N}\tilde{s}) \to \frac{e^{-\frac{\tilde{s}^2}{2}}}{\sqrt{2\pi}}$$

$$\tilde{s} = \frac{1}{\sqrt{N}}\sum_i s_i \qquad \langle \tilde{s}^2 \rangle = 1$$

Strongly correlated variables ?

Strongly correlated variables ?

Generalization to strongly correlated variables ?

Generalization of the CLT?

Renormalization group !

NB: connection between RG and CLT undertood already in the 70's (Jona-Lasino)

Generalized CLT for strongly correlated variables

Correlations: $\langle s_i s_j \rangle = G(|i-j|)$

$$\tilde{s} = \frac{1}{\sqrt{N}} \sum_i s_i$$

Weak correlations: $\langle \tilde{s}^2 \rangle \propto N^0$ Similar to i.i.d.'s: standard CLT

Strong correlations: $\lim_{N \to \infty} \langle \tilde{s}^2 \rangle = \infty$

Example: Second order phase transitions

Generalized CLT for strongly correlated variables

Correlations: $\langle s_i s_j \rangle = G(|i-j|)$

$$\tilde{s} = \frac{1}{\sqrt{N}} \sum_i s_i$$

Weak correlations: $\langle \tilde{s}^2 \rangle \propto N^0$ Similar to i.i.d.'s: standard CLT

Strong correlations: $\lim\limits_{N\to\infty} \langle \tilde{s}^2 \rangle = \infty$

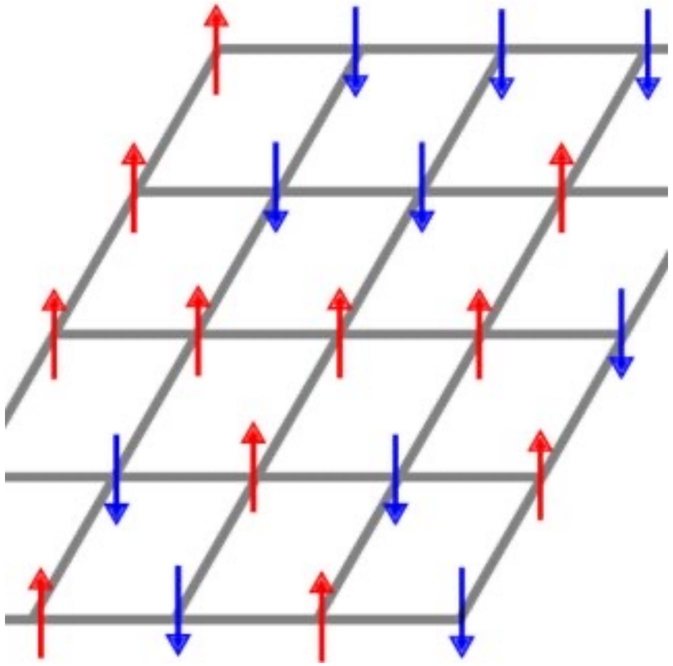
Example: Second order phase transitions

Standard model of statistical physics: Ising model

$$P(\{s_i\}) \propto e^{-H/T}$$

Energy of configuration: $H = -J \sum_{\langle i,j \rangle} s_i s_j$

Number of spins: $N = L^d$



Distribution of the magnetization

Distribution of the magnetization

High temperature phase: many independent blocks $\langle \tilde{s}^2 \rangle \propto \chi$

Weak correlations and gaussian distribution

$$\tilde{s} = \frac{1}{\sqrt{N}} \sum_i s_i$$

Distribution of the magnetization

High temperature phase: many independent blocks $\langle \tilde{s}^2 \rangle \propto \chi$

Weak correlations and gaussian distribution

$$\tilde{s} = \frac{1}{\sqrt{N}} \sum_i s_i$$

Exactly at T_c : $\langle \tilde{s}^2 \rangle \propto L^{2-\eta}$ Strong correlations and non-Gaussian distribution

$$\langle s^2 \rangle \propto L^{-(d-2+\eta)} = L^{-2\beta/\nu}$$

$$s = \frac{1}{N} \sum_i s_i$$

Distribution of the magnetization

High temperature phase: many independent blocks $\langle \tilde{s}^2 \rangle \propto \chi$

Weak correlations and gaussian distribution

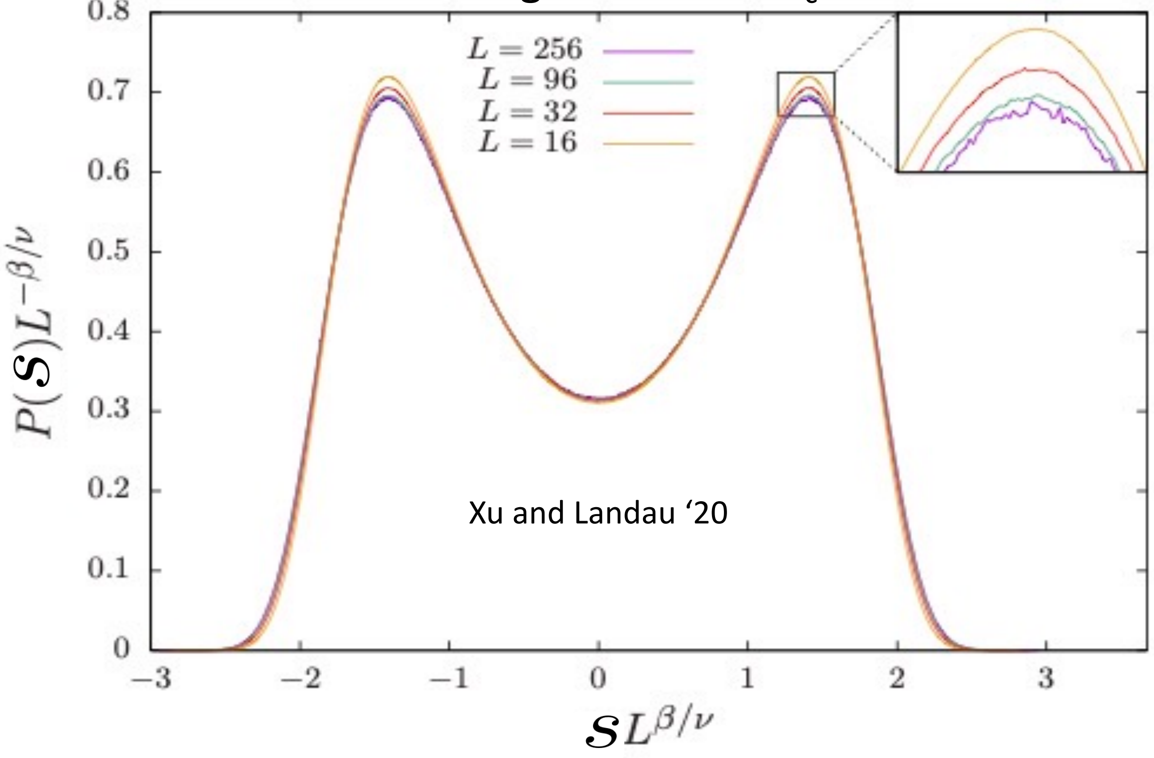
$$\tilde{s} = \frac{1}{\sqrt{N}} \sum_i s_i$$

Exactly at T_c : $\langle \tilde{s}^2 \rangle \propto L^{2-\eta}$ Strong correlations and non-Gaussian distribution

$$\langle s^2 \rangle \propto L^{-(d-2+\eta)} = L^{-2\beta/\nu}$$

$$s = \frac{1}{N} \sum_i s_i$$

3D Ising model at $T=T_c$



Stable law
for strongly correlated variables

$$P_L(s) \sim e^{-L^d I(s)} = e^{-\tilde{I}(sL^{\beta/\nu})}$$

standard CLT $\quad P_N(\tilde{s}) \sim e^{-\frac{\tilde{s}^2}{2}}$

Scaling form of the PDF

Scaling form of the PDF

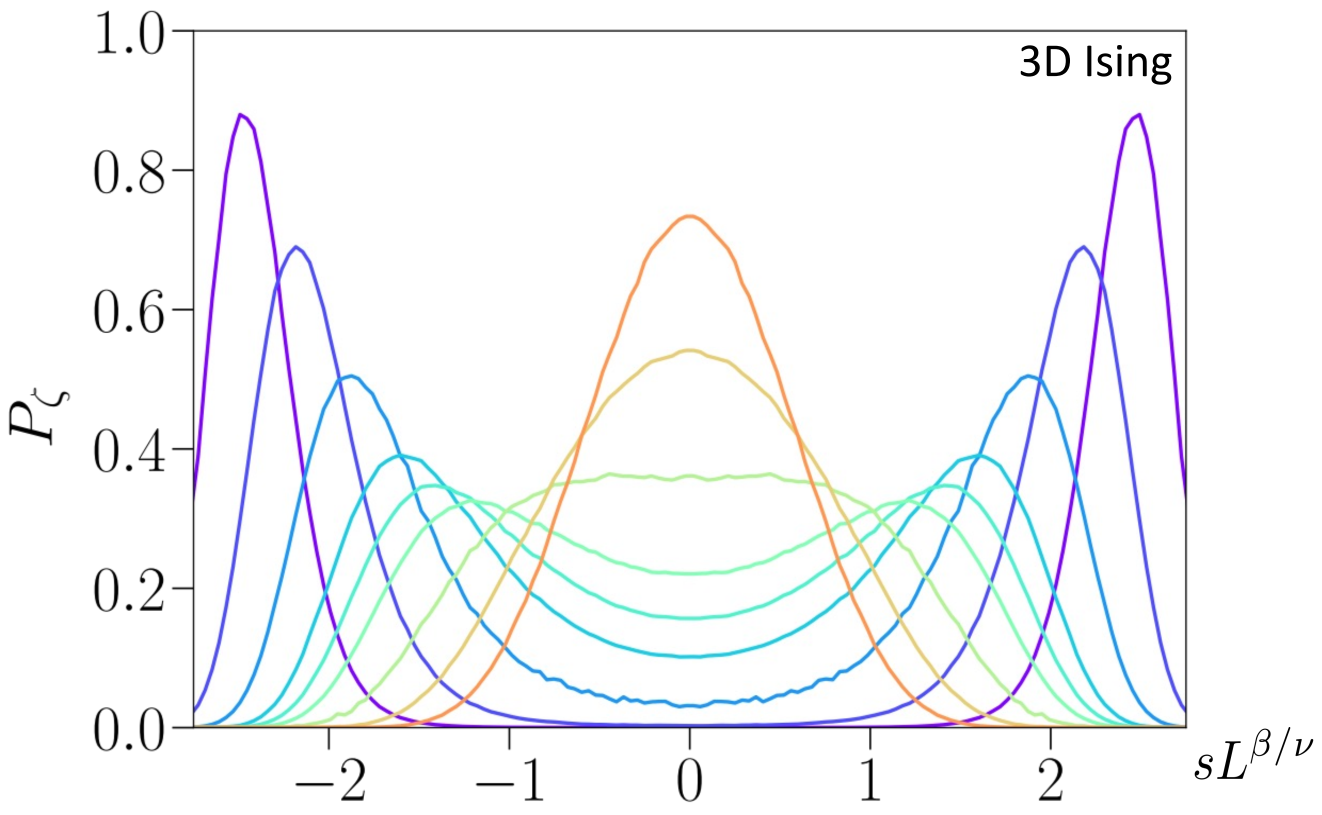
In the limit $L, \xi \to \infty$, the probability obeys scaling form

$$P_L(s,T) = L^{\beta/\nu} P(sL^{\beta/\nu}, L/\xi)$$

Scaling form of the PDF

In the limit $L, \xi \to \infty$, the probability obeys scaling form

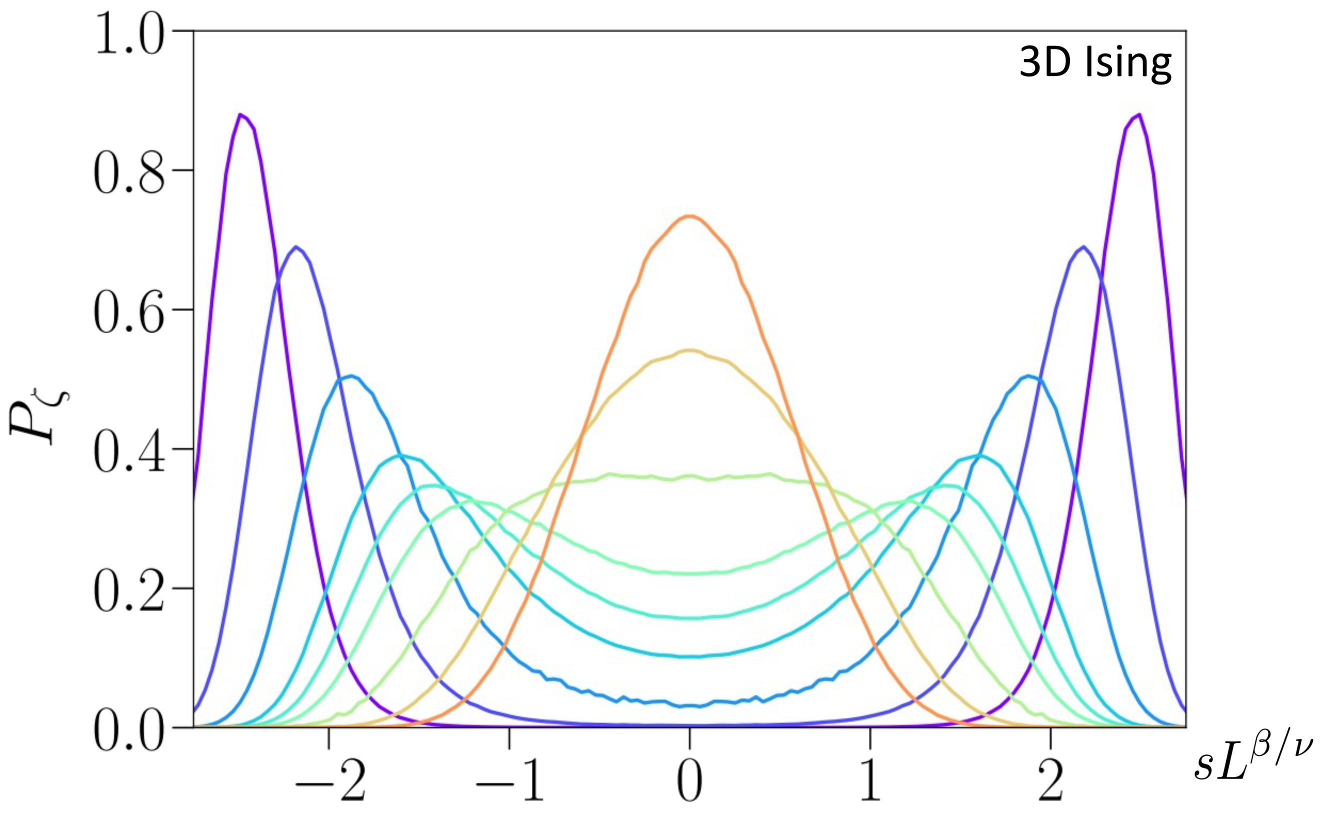
$$P_L(s,T) = L^{\beta/\nu} P(sL^{\beta/\nu}, L/\xi)$$



Scaling form of the PDF

In the limit $L, \xi \to \infty$, the probability obeys scaling form

$$P_L(s,T) = L^{\beta/\nu} P(sL^{\beta/\nu}, L/\xi)$$



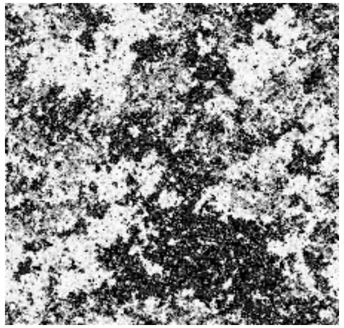
Key points:

- Depends on the ratio $\zeta = L/\xi$
- Universal form (but depends on BC)
- Becomes Gaussian for $\xi \ll L$

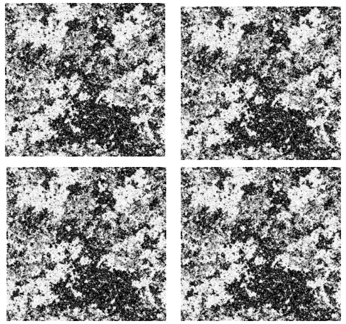
Understanding the scaling of typical magnetization fluctuations

Understanding the scaling of typical magnetization fluctuations

Block-spins are not independent, or we would recover the standard CLT



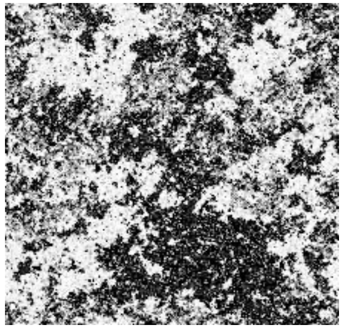
$\neq$



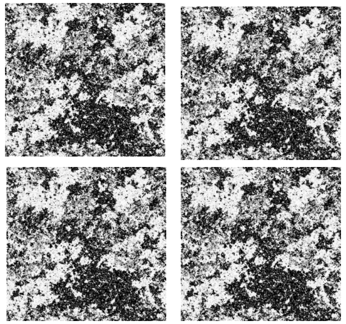
We can’t glue systems of size L/2 to get a system of size L because of long-range correlations

Understanding the scaling of typical magnetization fluctuations

Block-spins are not independent, or we would recover the standard CLT

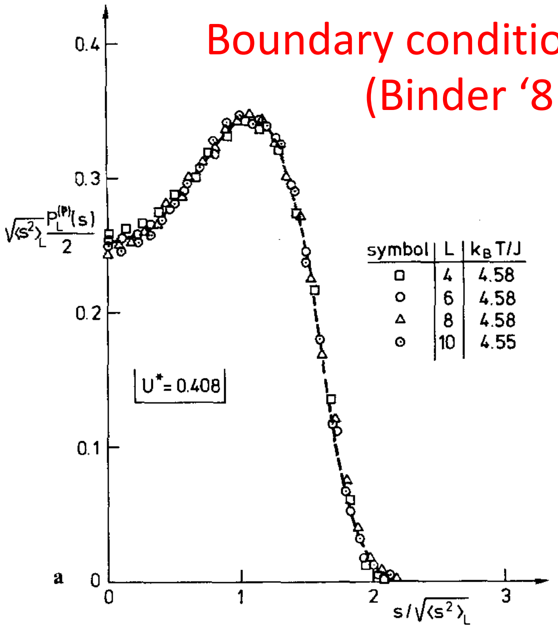


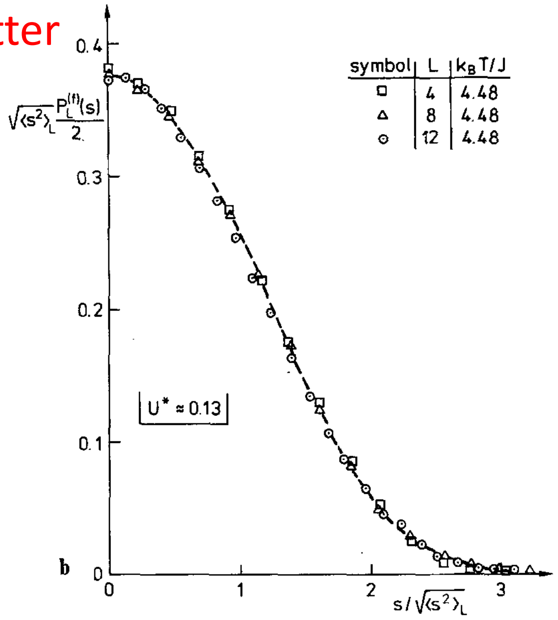
≠



We can't glue systems of size L/2 to get a system of size L because of long-range correlations

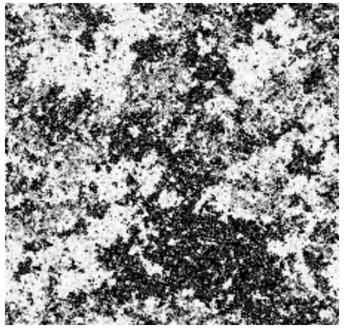
Boundary conditions matter
(Binder '81)



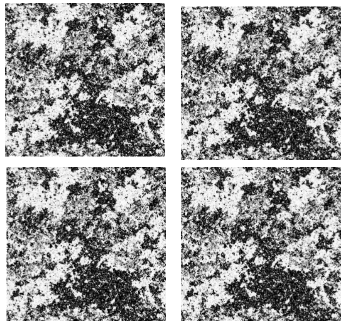


Understanding the scaling of typical magnetization fluctuations

Block-spins are not independent, or we would recover the standard CLT

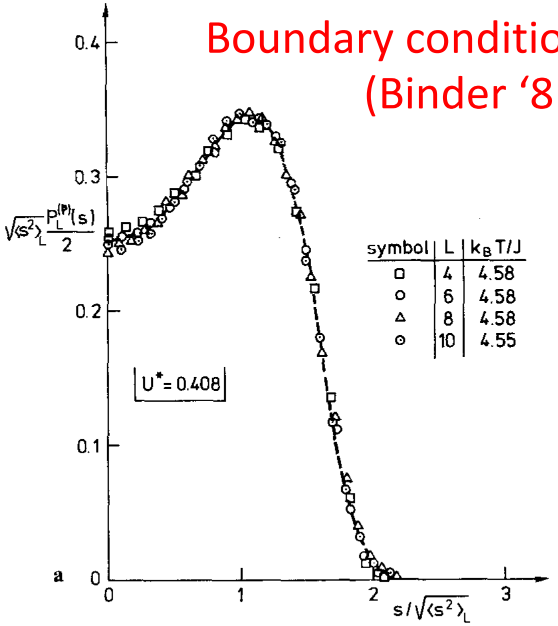


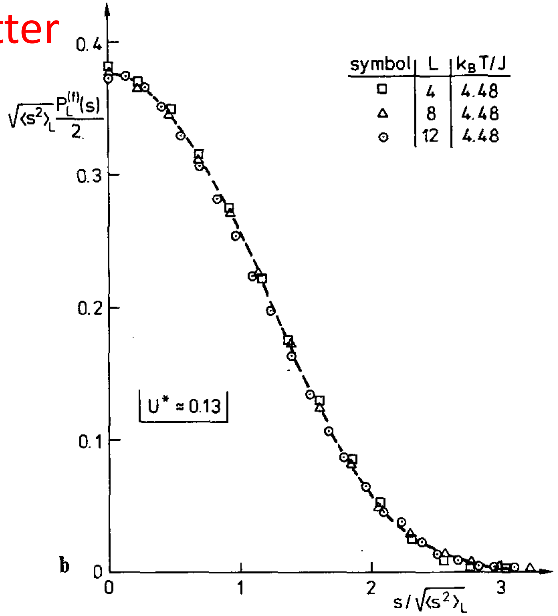
≠



We can't glue systems of size L/2 to get a system of size L because of long-range correlations

Boundary conditions matter
(Binder '81)





Can we still compute the stable law?

Functional RG

Functional RG

Scale-dependent Gibbs energy
(Wetterich '93)

$$e^{-\Gamma_k[\phi]} = \int \mathcal{D}\varphi e^{-H[\varphi] - \frac{1}{2}(\phi-\varphi).R_k.(\phi-\varphi) + \frac{\delta\Gamma_k}{\delta\phi}.(\varphi-\phi)}$$

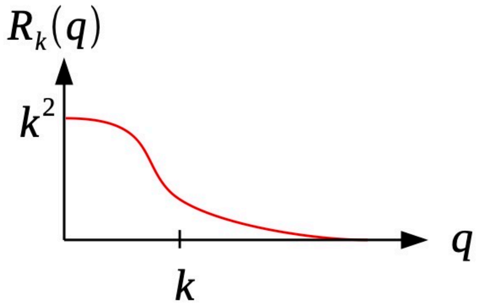
Functional RG

Scale-dependent Gibbs energy

(Wetterich ‘93)

$$e^{-\Gamma_k[\phi]} = \int \mathcal{D}\varphi e^{-H[\varphi]-\frac{1}{2}(\phi-\varphi).R_k.(\phi-\varphi)+\frac{\delta\Gamma_k}{\delta\phi}.(\varphi-\phi)}$$

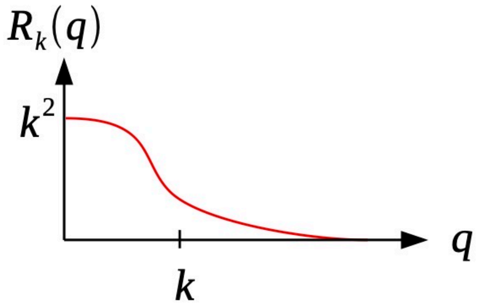
$$\partial_k\Gamma_k = \frac{1}{2}\mathrm{Tr}\left(\partial_k R_k(\Gamma_k^{(2)}+R_k)^{-1}\right)$$

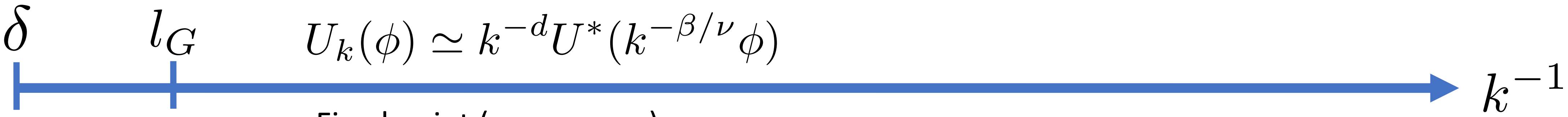


Scale-dependent Gibbs energy (Wetterich ‘93)

$$e^{-\Gamma_k[\phi]} = \int \mathcal{D}\varphi e^{-H[\varphi]-\frac{1}{2}(\phi-\varphi).R_k.(\phi-\varphi)+\frac{\delta\Gamma_k}{\delta\phi}.(\varphi-\phi)}$$

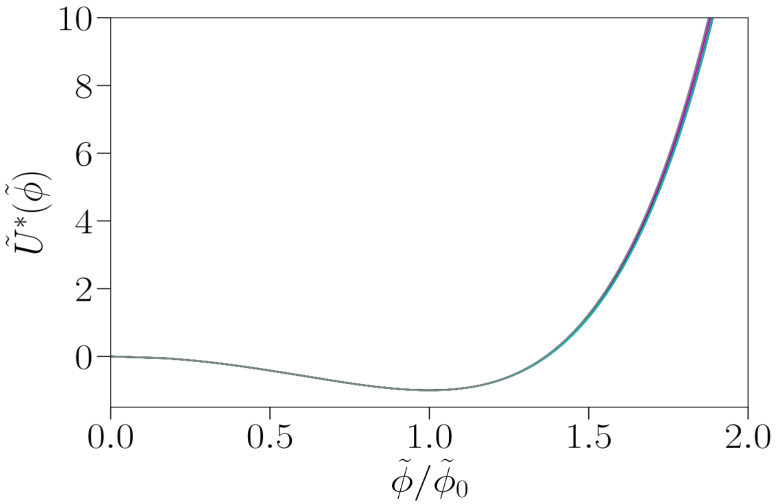
$$\partial_k\Gamma_k = \frac{1}{2}\mathrm{Tr}\left(\partial_k R_k(\Gamma_k^{(2)}+R_k)^{-1}\right)$$





$$U_k(\phi) \simeq k^{-d}U^*(k^{-\beta/\nu}\phi)$$

Fixed point (non convex)

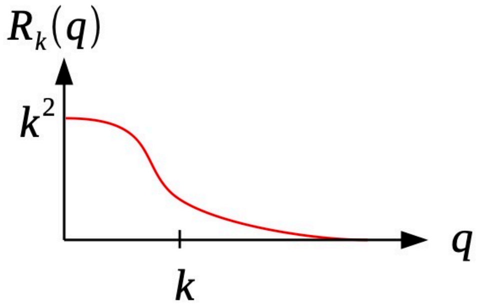


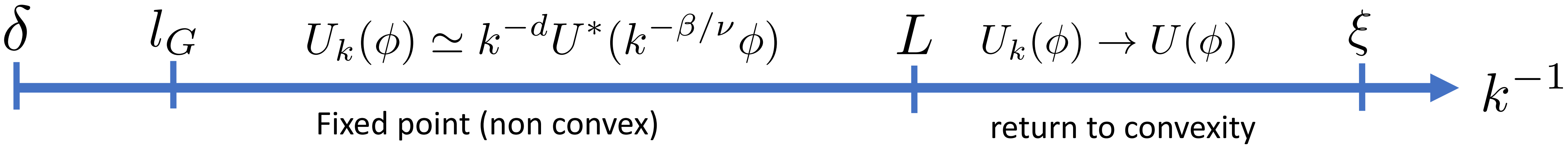
Functional RG

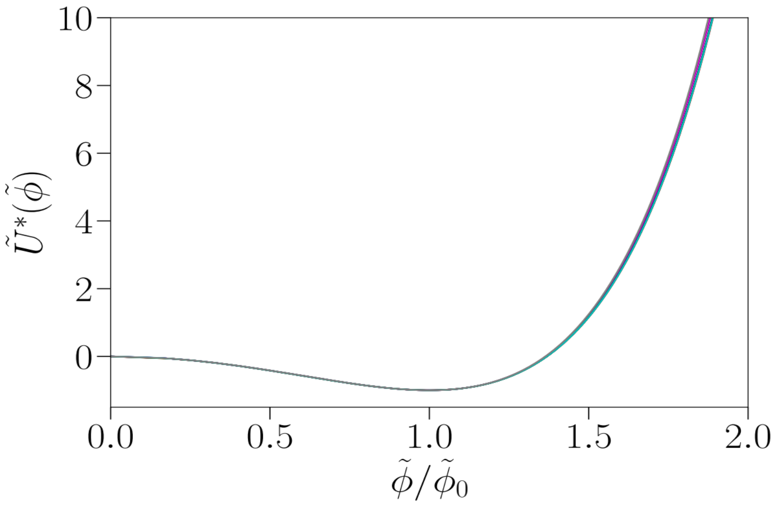
Scale-dependent Gibbs energy
(Wetterich '93)

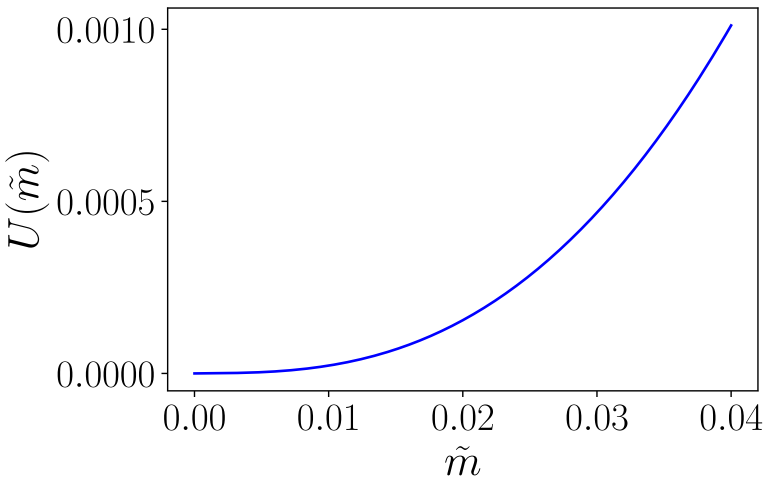
$$e^{-\Gamma_k[\phi]} = \int \mathcal{D}\varphi e^{-H[\varphi]-\frac{1}{2}(\phi-\varphi).R_k.(\phi-\varphi)+\frac{\delta\Gamma_k}{\delta\phi}.(\varphi-\phi)}$$

$$\partial_k \Gamma_k = \frac{1}{2}\mathrm{Tr}\left(\partial_k R_k (\Gamma_k^{(2)} + R_k)^{-1}\right)$$









Functional for the rate function

Balog et al ‘22

Functional for the rate function

Define a "flowing" probability distribution and rate function

$$P_k(\boldsymbol{S}) = \exp(-L^d I_k(\boldsymbol{S})) = \int D\varphi\, \delta\left(\boldsymbol{S} - L^{-d}\int_x \varphi\right) \exp(-H(\varphi) - 1/2\varphi.R_k.\varphi)$$

Problem: flow of the rate function is not closed

Balog et al '22

Functional for the rate function

Define a “flowing” probability distribution and rate function

$$P_k(\boldsymbol{S}) = \exp(-L^d I_k(\boldsymbol{S})) = \int D\varphi\, \delta\left(\boldsymbol{S} - L^{-d}\int_x \varphi\right) \exp(-H(\varphi) - 1/2\varphi.R_k.\varphi)$$

Problem: flow of the rate function is not closed

Define a (scale-dependent) “constraint effective action”

$$\exp(-\hat{\Gamma}_k[\phi]) = \int D\varphi \exp\left(-H(\varphi) - 1/2(\phi-\varphi).\hat{R}_k.(\phi-\varphi) + \frac{\delta\hat{\Gamma}_k}{\delta\phi}.(\varphi-\phi)\right)$$

with $\hat{R}_k(q) = \begin{cases} \infty, & \text{if } q = 0, \\ R_k(q), & \text{if } q > 0. \end{cases}$

Flow equation of $\hat{\Gamma}_k[\phi]$ closed and $I_k(\boldsymbol{S}) = \hat{\Gamma}_k[\phi = \boldsymbol{S}]$

Balog et al ‘22

Flow equation for the rate function

Balog et al ‘22

Flow equation for the rate function

Simple approximation: “Local Potential Approximation”

$$\partial_k I_k(\boldsymbol{s}) = \frac{1}{2L^d} \sum_{q\neq 0} \frac{\partial_k \hat{R}_k(q)}{q^2 + \hat{R}_k(q) + I_k''(\boldsymbol{s})}$$

Balog et al ‘22

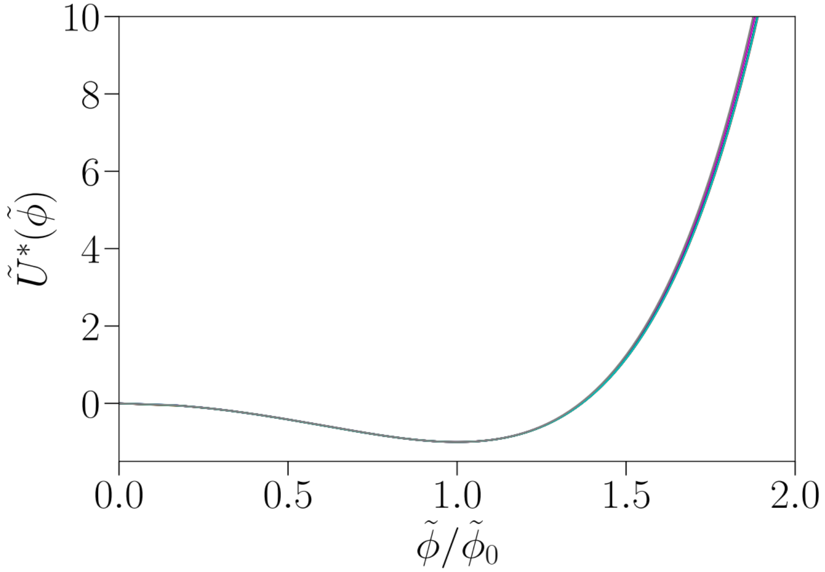
Flow equation for the rate function

Simple approximation: “Local Potential Approximation”

$$\partial_k I_k(\boldsymbol{s}) = \frac{1}{2L^d} \sum_{q\neq 0} \frac{\partial_k \hat{R}_k(q)}{q^2 + \hat{R}_k(q) + I_k''(\boldsymbol{s})}$$

$$I_k(\boldsymbol{s}) \propto k^{-d} U^*(k^{-\beta/\nu} \boldsymbol{s})$$

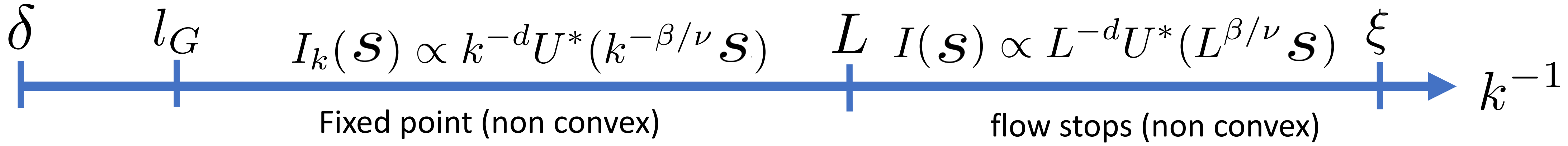
Fixed point (non convex)

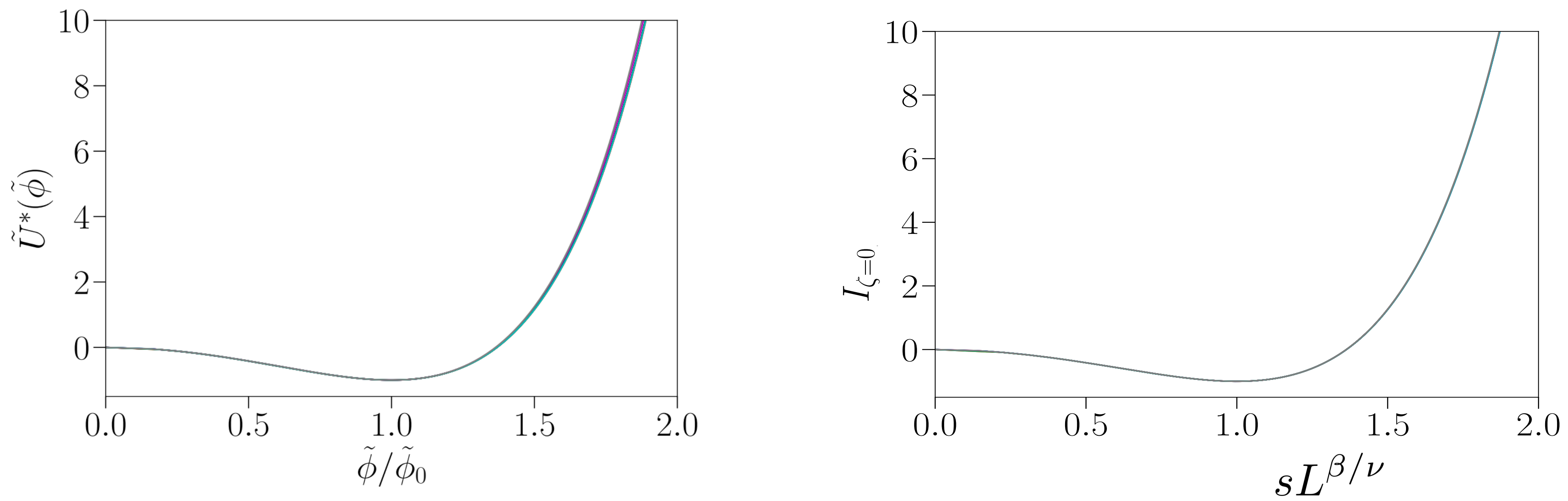


Balog et al ‘22

Simple approximation: “Local Potential Approximation”

$$\partial_k I_k(\boldsymbol{s}) = \frac{1}{2L^d} \sum_{q\neq 0} \frac{\partial_k \hat{R}_k(q)}{q^2 + \hat{R}_k(q) + I_k''(\boldsymbol{s})}$$





Compares well with Monte Carlo simulations

Balog et al ‘22

Cummulants and non-Gaussianity

Scaling of PDF $P_L(s,T) = L^{\beta/\nu} P(sL^{\beta/\nu}, L/\xi)$

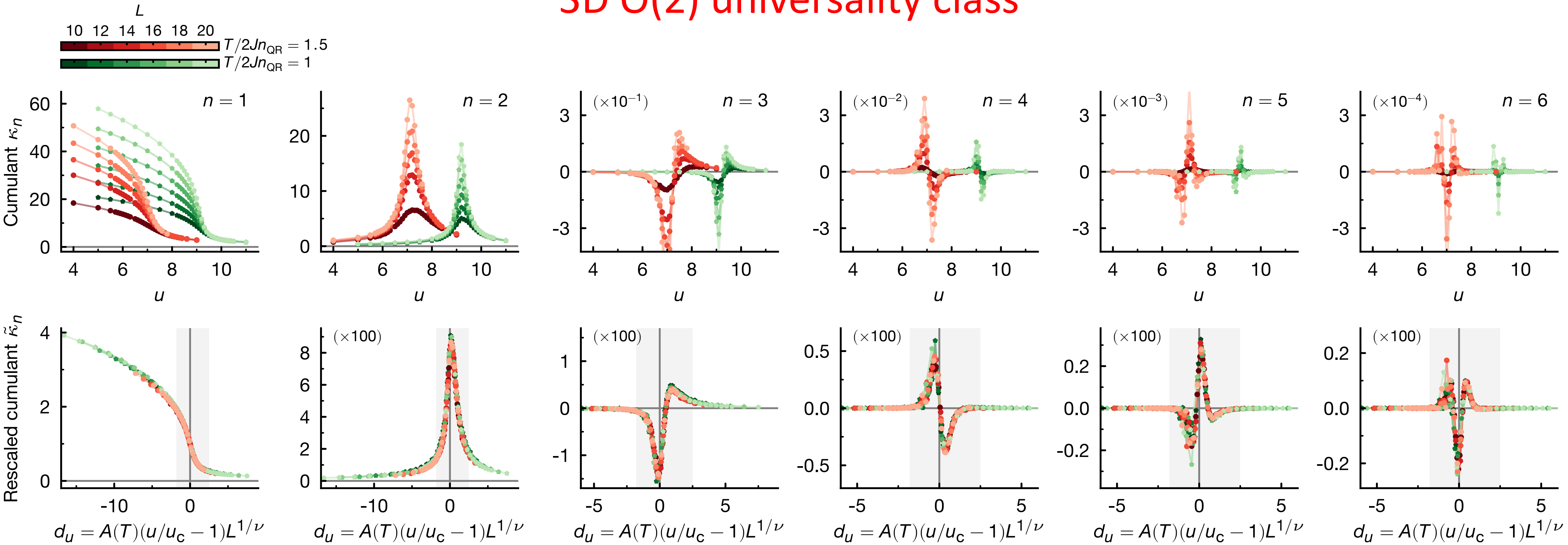
Scaling of the cumulants $\kappa_n(L,T) = \langle |s|^n \rangle_c = L^{-n\beta/\nu} f_n(L/\xi)$

Cummulants and non-Gaussianity

Scaling of PDF $P_L(s,T) = L^{\beta/\nu} P(sL^{\beta/\nu}, L/\xi)$

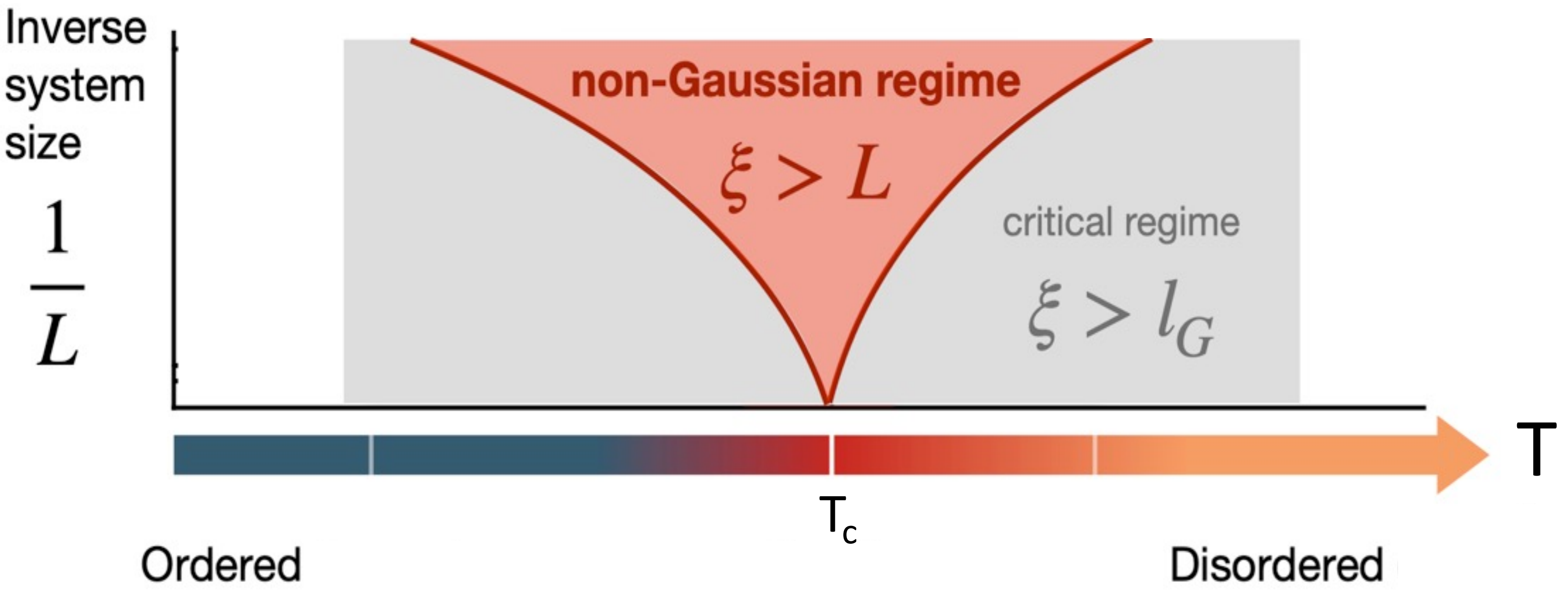
Scaling of the cumulants $\kappa_n(L,T) = \langle |s|^n \rangle_c = L^{-n\beta/\nu} f_n(L/\xi)$

3D O(2) universality class



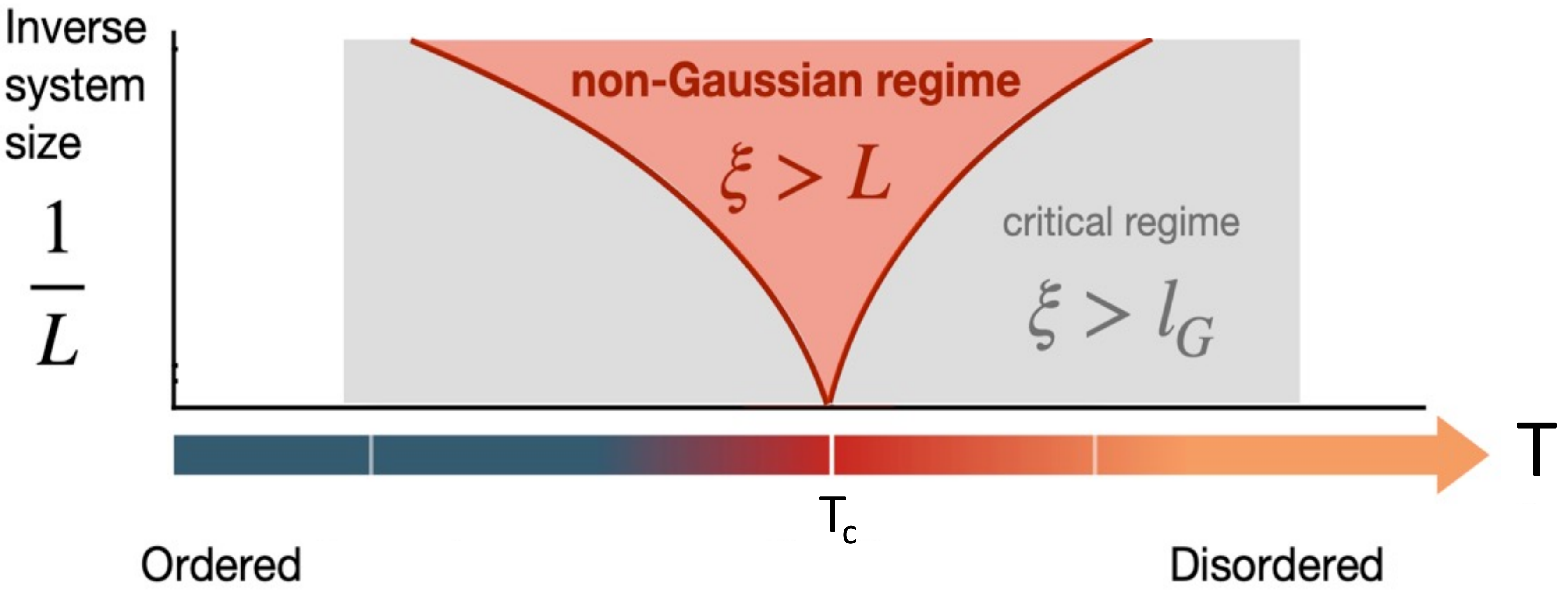
How to see the non-Gaussianity?

How to see the non-Gaussianity?



Need system large enough to be critical,
but small enough not to be in the Gaussian regime !

How to see the non-Gaussianity?

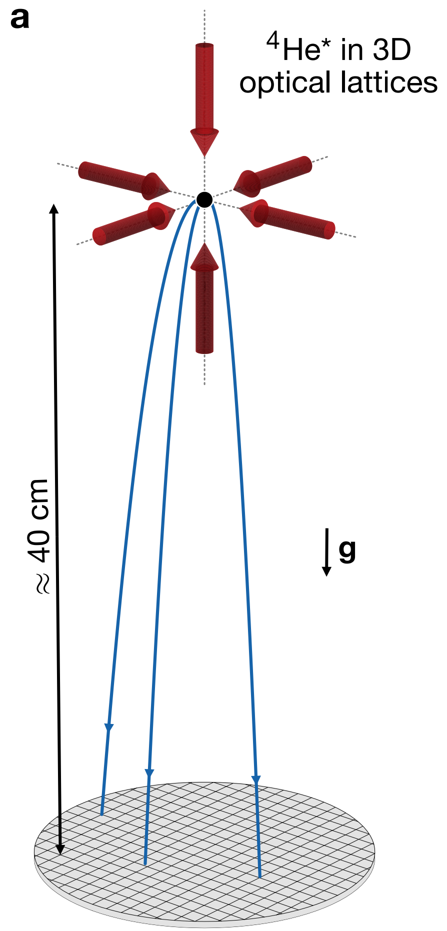


Need system large enough to be critical,
but small enough not to be in the Gaussian regime !

Cold atom experiments can be in this regime!

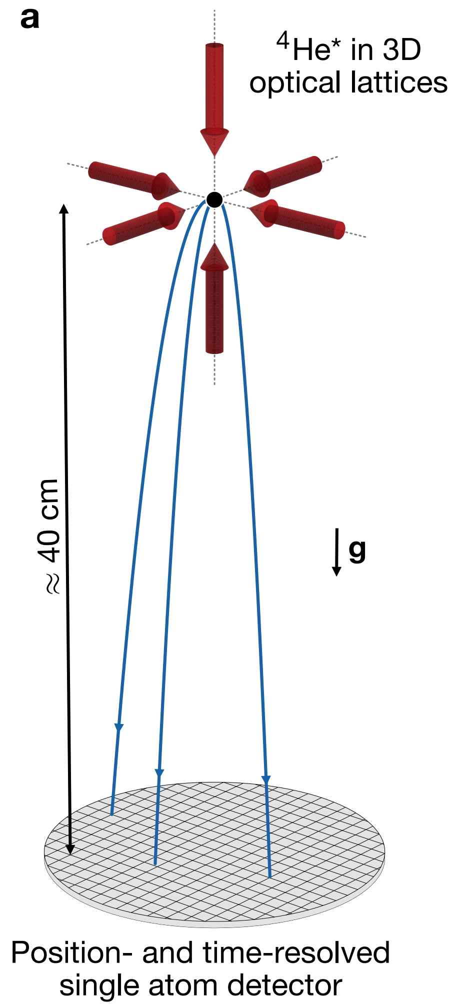
Metastable $^4He^*$ experiment

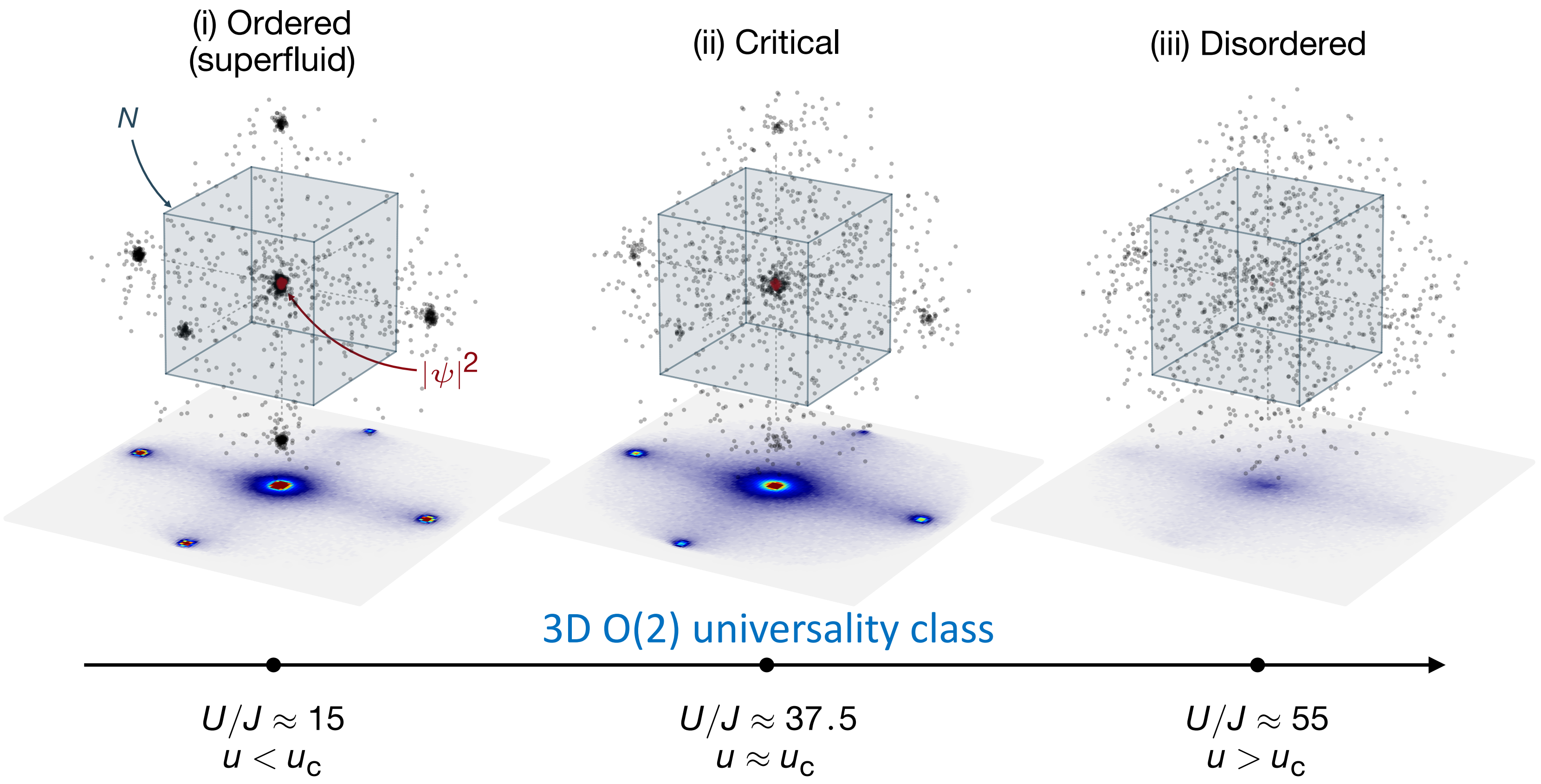
Metastable $^4He^*$ experiment



~ 4000 atoms

Metastable ^{4}He* experiment

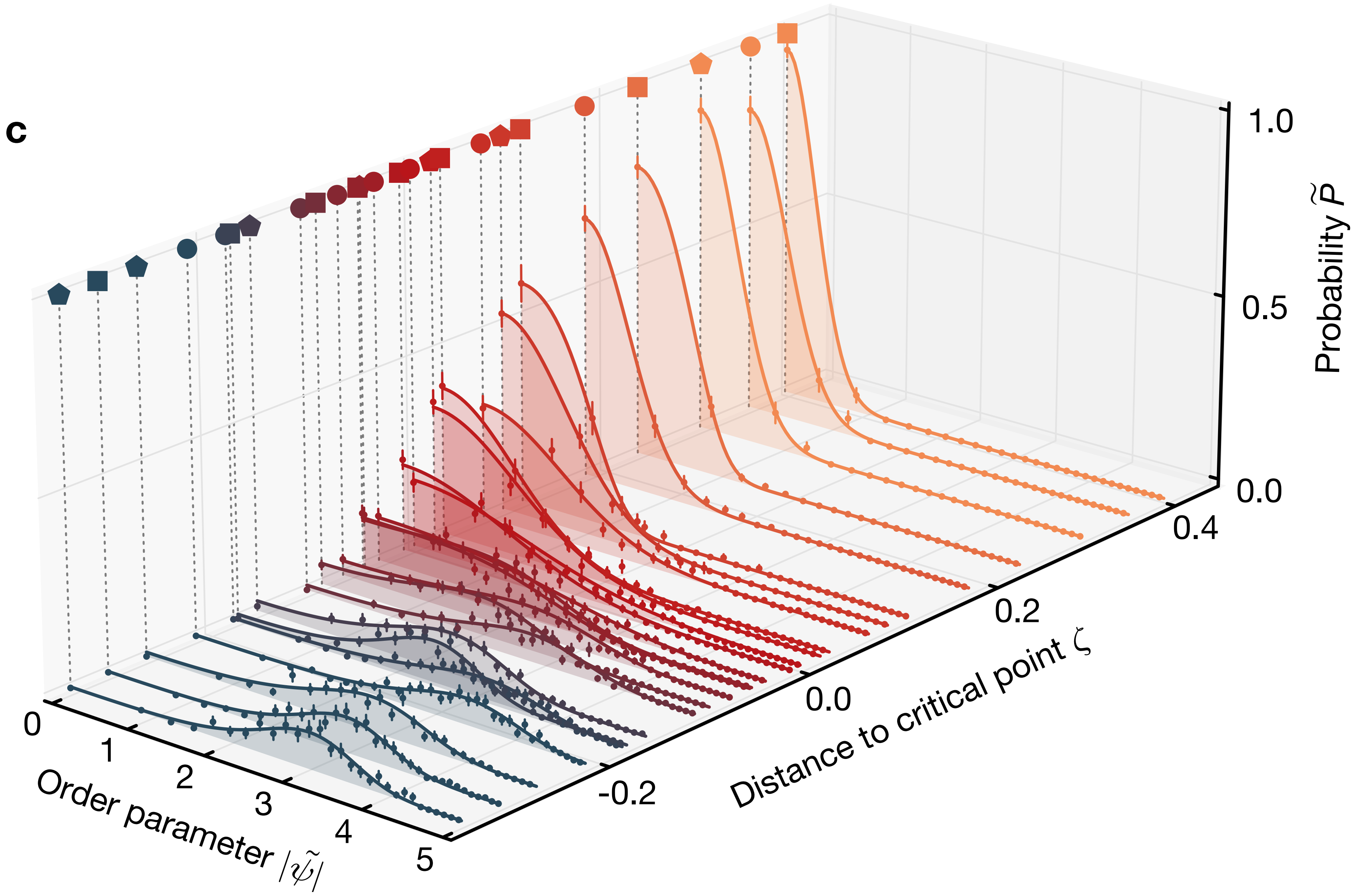




~ 4000 atoms

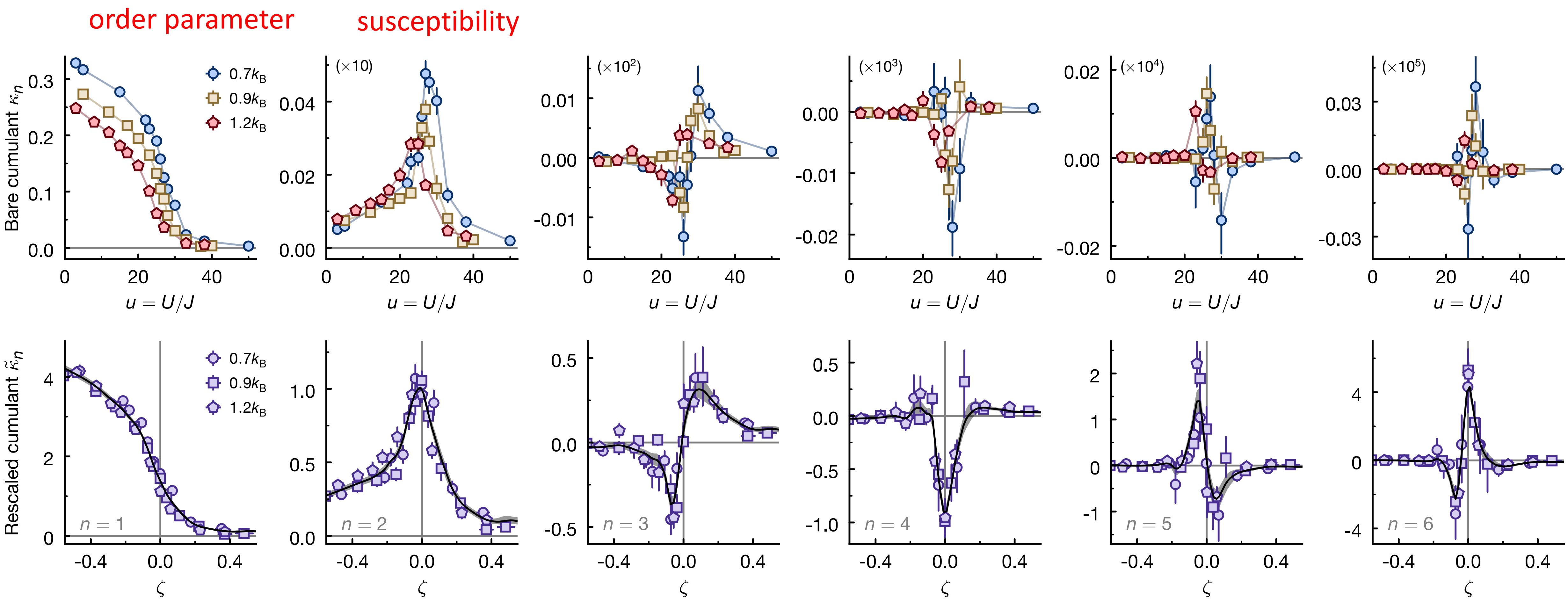
Finite temperature: classical phase transition!!

Measuring the PDF of the order parameter in cold atoms



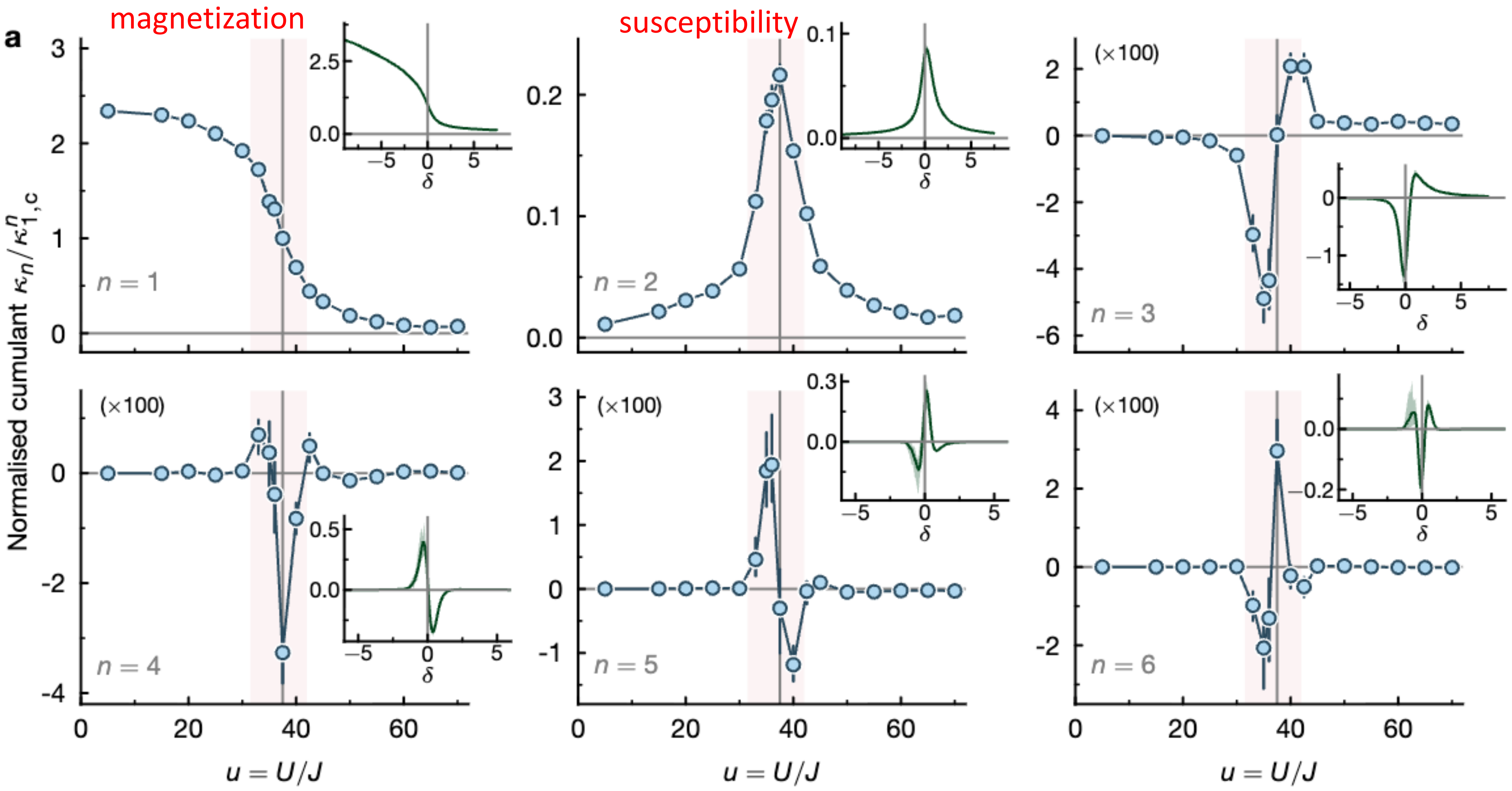
Paquiez et al., to be published

Non-Gaussianity of the PDF



Paquiez et al., to be published

Experiments vs numerics



Origin of the difference and challenges for the FRG community

Origin of the difference and challenges for the FRG community

- Quantum system at finite temperature : imaginary time direction of size β + projective measurement

$$P(s) \propto \mathrm{Tr}\left(e^{-\beta\hat{H}}\Big|\sum_i s_i\Big\rangle\Big\langle\sum_i s_i\Big|\right)$$

Constraint on the zero-momentum mode at the boundary in time

Quantum fluctuations irrelevant at finite temperature… but only for large enough systems!

Origin of the difference and challenges for the FRG community

- Quantum system at finite temperature : imaginary time direction of size β + projective measurement

$$P(s) \propto \mathrm{Tr}\left(e^{-\beta \hat{H}}\Big|\sum_i s_i\Big\rangle\Big\langle\sum_i s_i\Big|\right)$$

Constraint on the zero-momentum mode at the boundary in time

Quantum fluctuations irrelevant at finite temperature… but only for large enough systems!

- Spatial boundary conditions different from periodic boundary conditions: atoms in harmonic trap…

(Numerical simulations do not change qualitatively the variation of the cumulants)

Origin of the difference and challenges for the FRG community

- Quantum system at finite temperature : imaginary time direction of size β + projective measurement

$$P(s) \propto \mathrm{Tr}\left(e^{-\beta\hat{H}}\Big|\sum_i s_i\Big\rangle\Big\langle\sum_i s_i\Big|\right)$$

Constraint on the zero-momentum mode at the boundary in time

Quantum fluctuations irrelevant at finite temperature… but only for large enough systems!

- Spatial boundary conditions different from periodic boundary conditions: atoms in harmonic trap…

(Numerical simulations do not change qualitatively the variation of the cumulants)

Challenges for the community

How to describe projective measurements in FRG?

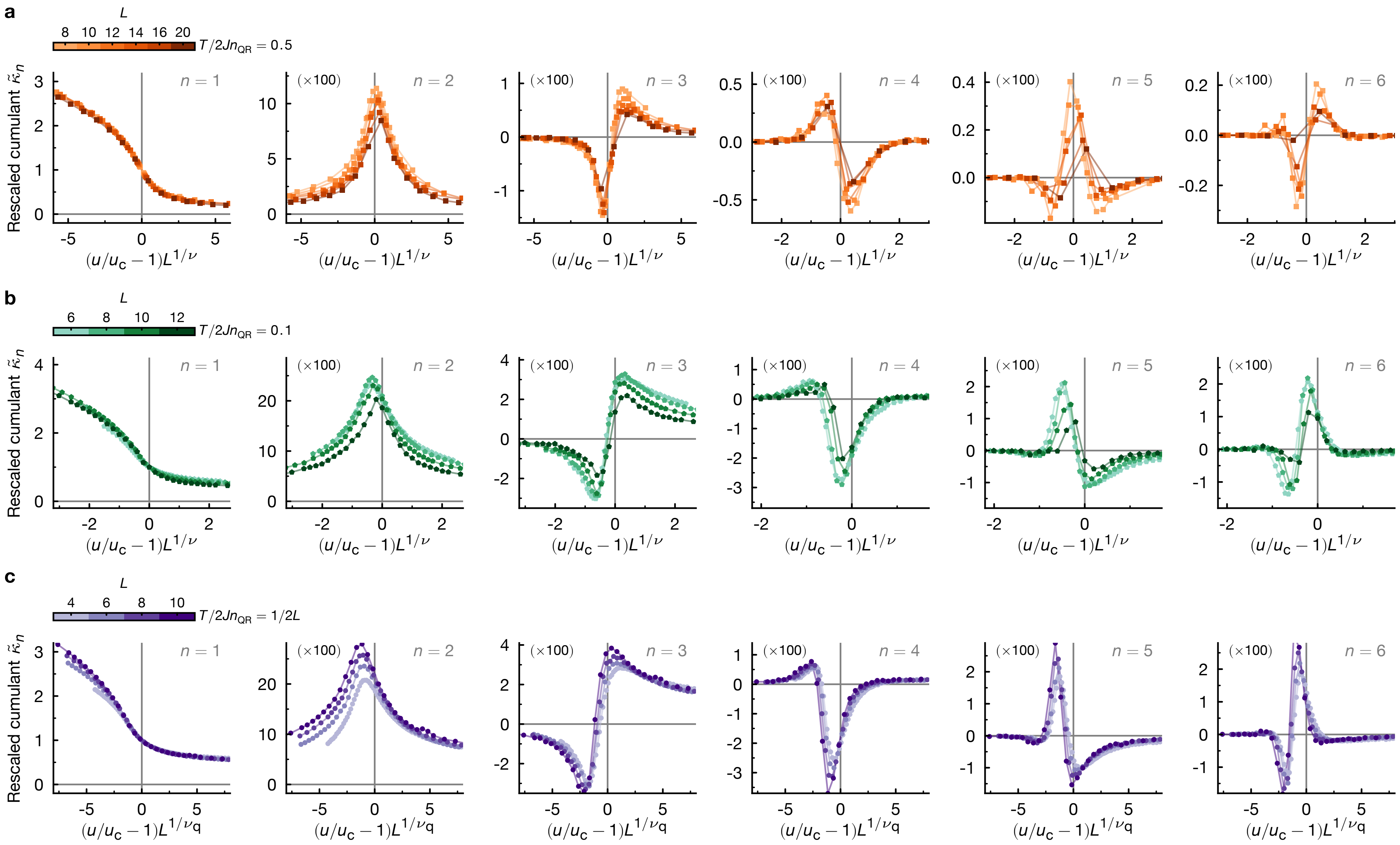
How to implement better boundary conditions (e.g. free/Dirichlet BC)
[Already messy in large N]

Conclusion

- CLT is the rule, but strong correlations (e.g. criticality) can give rise to universal non-Gaussian PDFs
- This non-Gaussianity can be observed in mesoscopic systems (not in the thermodynamic limit!)
- Cold atoms experiments offer a unique platform for this
- Observation of universal non-Gaussian cumulants, but different from expected XY universality class
- Universal effect of finite size corrections induced by quantumness… Compute with FRG???

Template

Quantum corrections to scaling

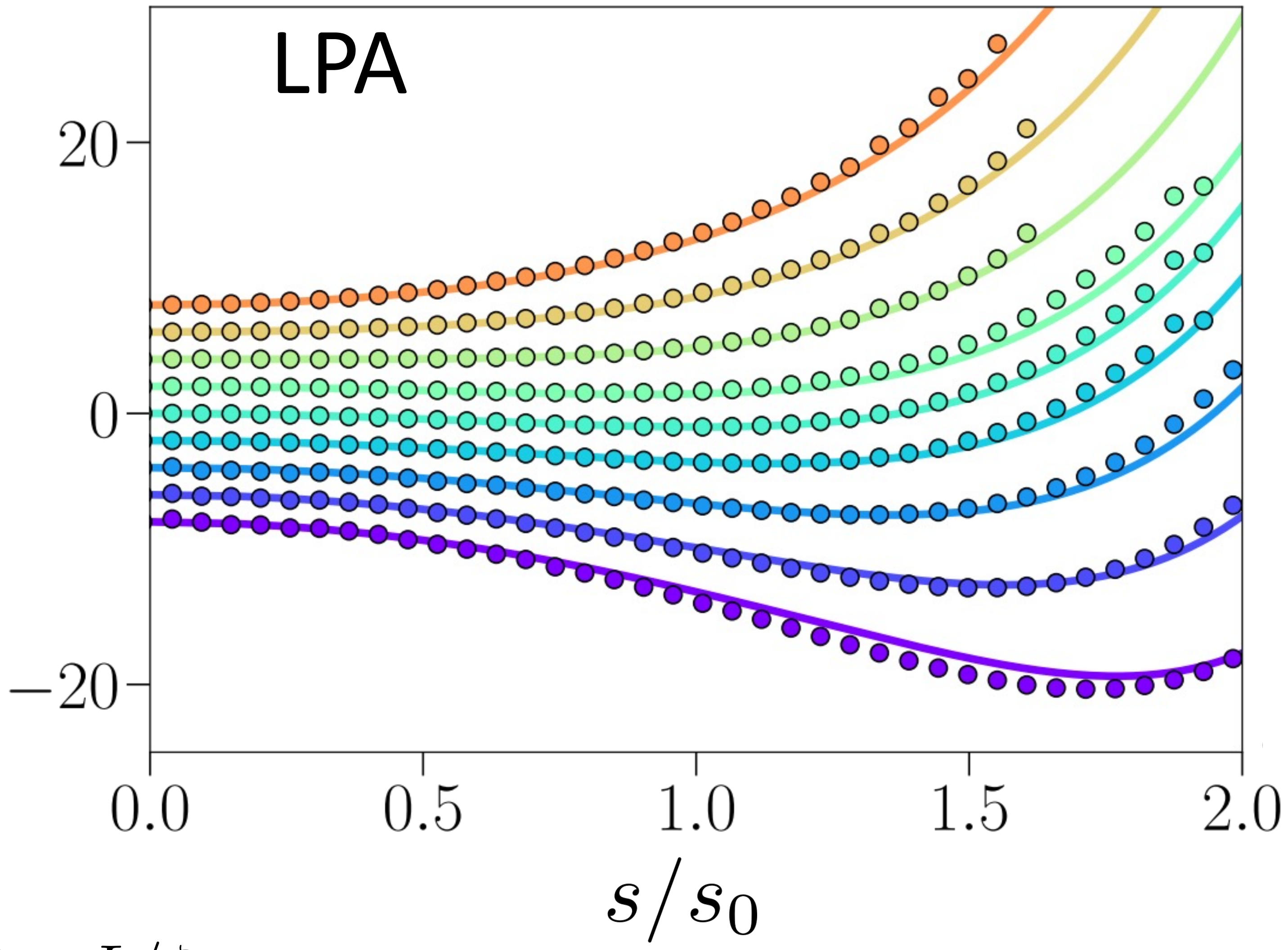


Family of rate functions

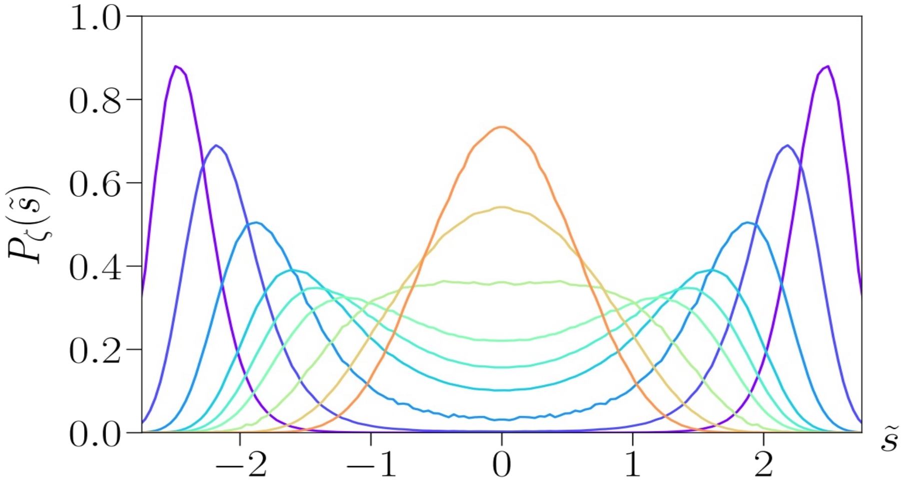
Balog et al '22

Family of rate functions

$$I_\zeta(L^{\beta/\nu}s)/I_{\zeta=0}(L^{\beta/\nu}s_0)$$



$\zeta = L/\xi$



Shape of rate function at LPA correct for all ζ up to global constant
(true also at 1-loop Sahu et al '24)

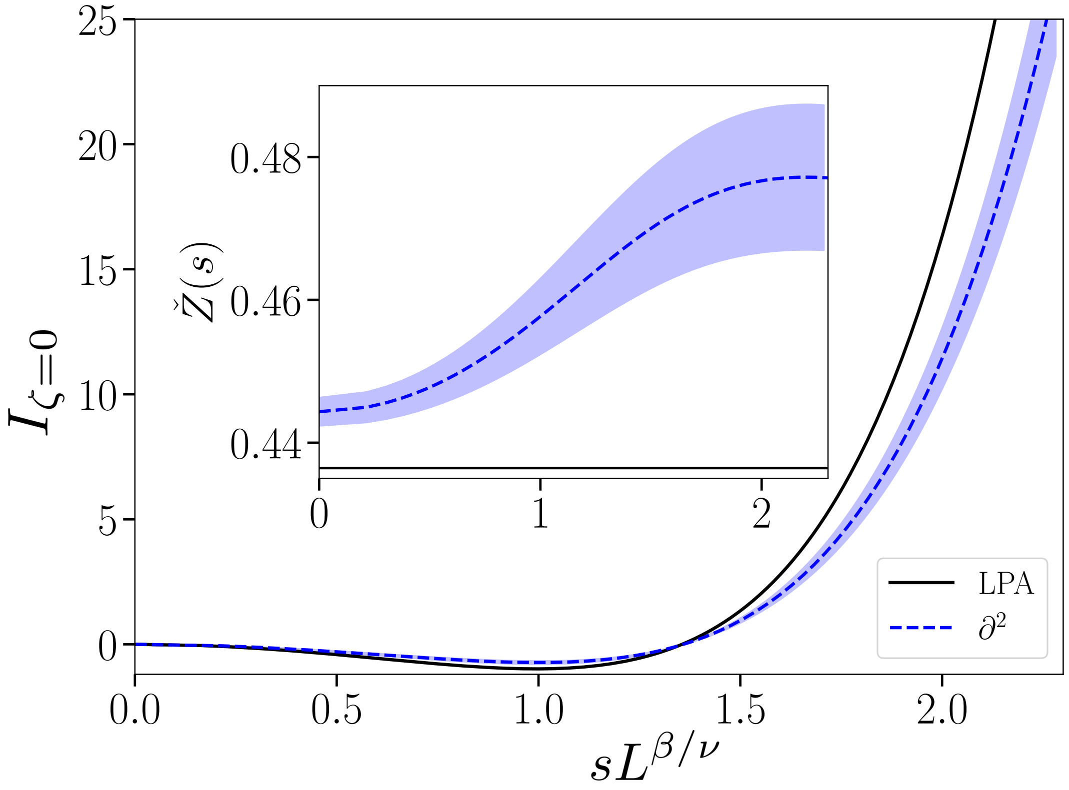
Balog et al '22

Convergence of the method

Rose et al ‘25

Convergence of the method

Possibility to do Derivative Expansion

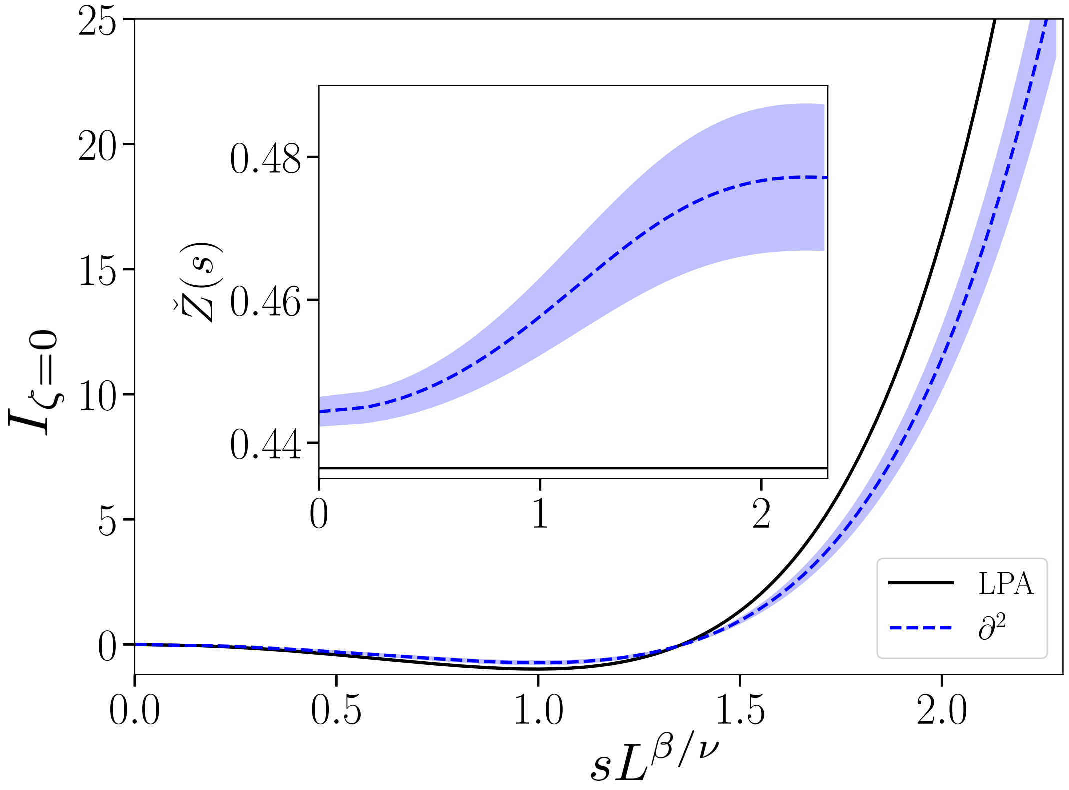


Error bar estimated following Balog et al '19 & De Polsi et al '20

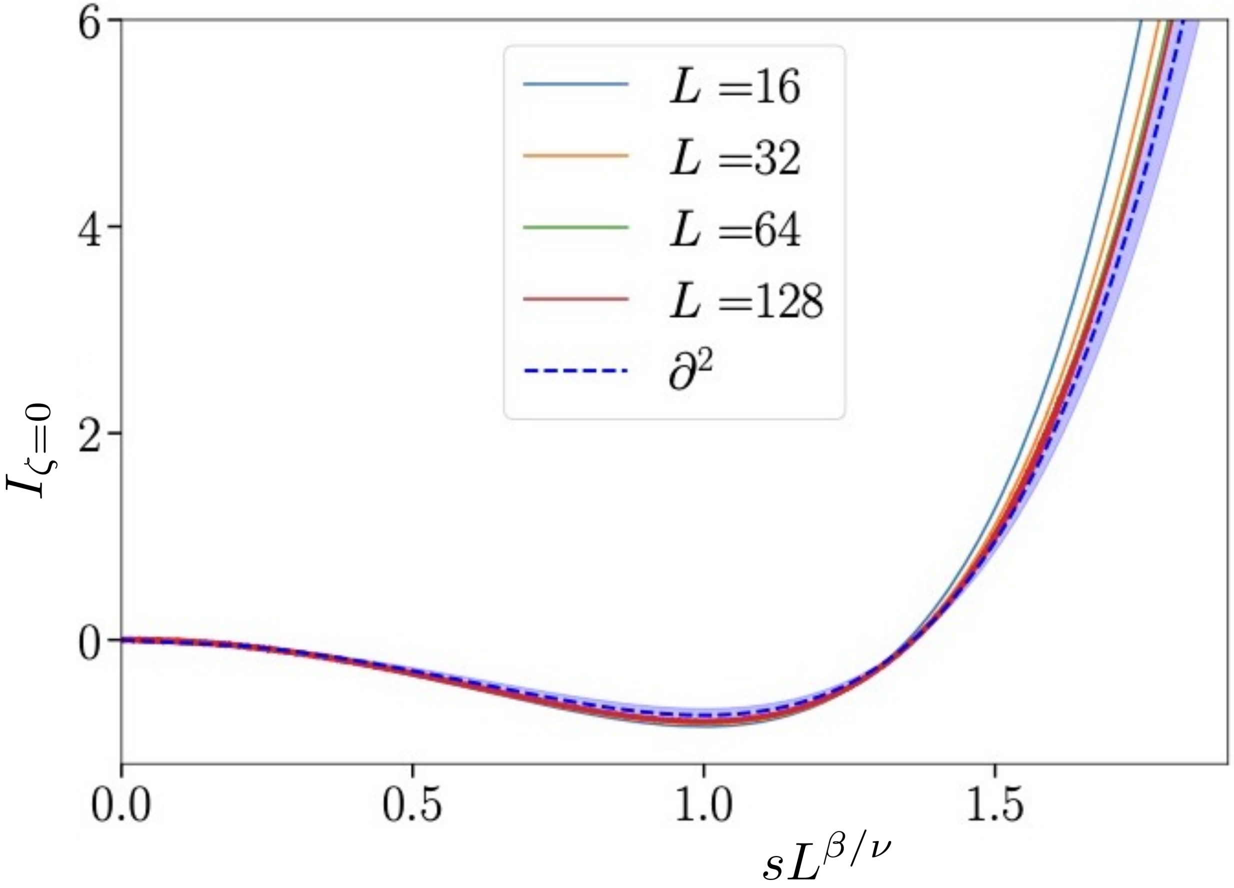
Rose et al '25

Convergence of the method

Possibility to do Derivative Expansion



Comparison to MC (no fitting parameter)



Error bar estimated following Balog et al ‘19 & De Polsi et al ‘20

Rose et al ‘25

Critical rare event

Balog et al ‘25

Critical rare event

Tail of distribution given by tail of rate function $I(s) \sim U(m = s) \sim s^{\delta+1}$

Balog et al '25

Critical rare event

Tail of distribution given by tail of rate function $I(s) \sim U(m = s) \sim s^{\delta+1}$

$$P_N(\hat{S} = Ns) \simeq \sqrt{\frac{NI''(s)}{2\pi}} \exp\left(-NI(s)\right) \quad \Longrightarrow \quad P_L(s) \sim s^{\frac{\delta-1}{2}} e^{-L^d a\, s^{\delta+1}}$$

Fisher '66, Tsypin '94, Bruce '95

Balog et al '25

Critical rare event

Tail of distribution given by tail of rate function $I(s) \sim U(m = s) \sim s^{\delta+1}$

$$P_N(\hat{S} = Ns) \simeq \sqrt{\frac{NI''(s)}{2\pi}} \exp\left(-NI(s)\right) \quad \Longrightarrow \quad P_L(s) \sim s^{\frac{\delta-1}{2}} e^{-L^d a\, s^{\delta+1}}$$

Fisher ‘66, Tsypin ’94, Bruce ‘95

Universal Large Deviations in the regime $L^{-\beta/\nu} \ll s \ll 1$

Balog et al ‘25

Critical rare event

Tail of distribution given by tail of rate function $I(s) \sim U(m = s) \sim s^{\delta+1}$

$$P_N(\hat{S} = Ns) \simeq \sqrt{\frac{NI''(s)}{2\pi}} \exp\left(-NI(s)\right) \quad \Longrightarrow \quad P_L(s) \sim s^{\frac{\delta-1}{2}} e^{-L^d a\, s^{\delta+1}}$$

Fisher '66, Tsypin '94, Bruce '95

Universal Large Deviations in the regime $L^{-\beta/\nu} \ll s \ll 1$

Can be generalized to $O(n)$ model $P_L(\mathbf{s}) \sim |\mathbf{s}|^{\psi} e^{-L^d a\, |\mathbf{s}|^{\delta+1}}$ $\quad \psi = n\frac{\delta - 1}{2}$

Power law prefactor observed: MC Ising 3D, hierarchical model, large n, FRG LPA (n=1,2,3)

Balog et al '25

Non-universal large deviations and finite size correction

Non-universal large deviations and finite size correction

For $s \sim 1$, proba has to be non-universal: transition from universal to non-universal rare events?

Non-universal large deviations and finite size correction

For $s \sim 1$, proba has to be non-universal: transition from universal to non-universal rare events?

Correction to scaling induced by finite size (established explicitly in large n)

Non-universal large deviations and finite size correction

For $s \sim 1$, proba has to be non-universal: transition from universal to non-universal rare events?

Correction to scaling induced by finite size (established explicitely in large n)

$$I(s) = L^{-d}\tilde{I}\left(sL^{\beta/\nu}\right) + \sum_k a_k L^{-d-\omega_k}\delta\tilde{I}_k\left(sL^{\beta/\nu}\right)$$

non-universal amplitudes

universal critical exponents (irrelevant)

universal irrelevant perturbations

Non-universal large deviations and finite size correction

For $s \sim 1$, proba has to be non-universal: transition from universal to non-universal rare events?

Correction to scaling induced by finite size (established explicitely in large n)

$$I(s) = L^{-d}\tilde{I}\left(sL^{\beta/\nu}\right) + \sum_k a_k L^{-d-\omega_k}\,\delta\tilde{I}_k\left(sL^{\beta/\nu}\right)$$

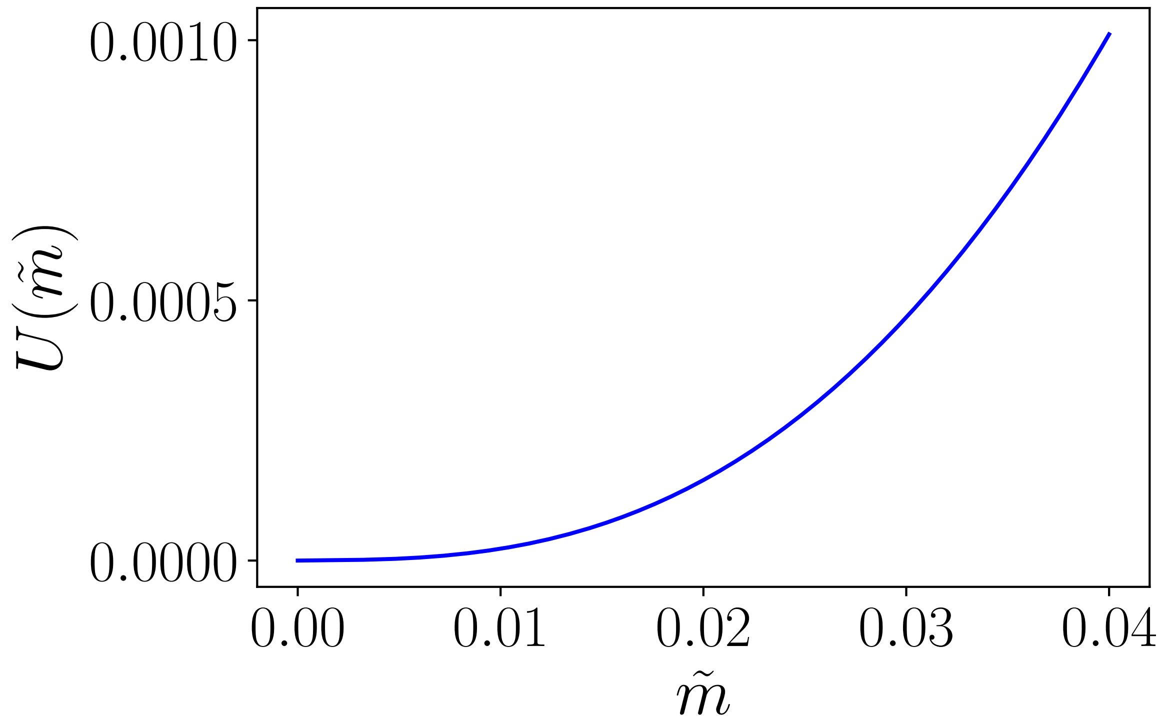
non-universal amplitudes

universal critical exponents (irrelevant)

universal irrelevant perturbations

Finite-size corrections-to-scaling generalizes Cramer's series!

Return to convexity

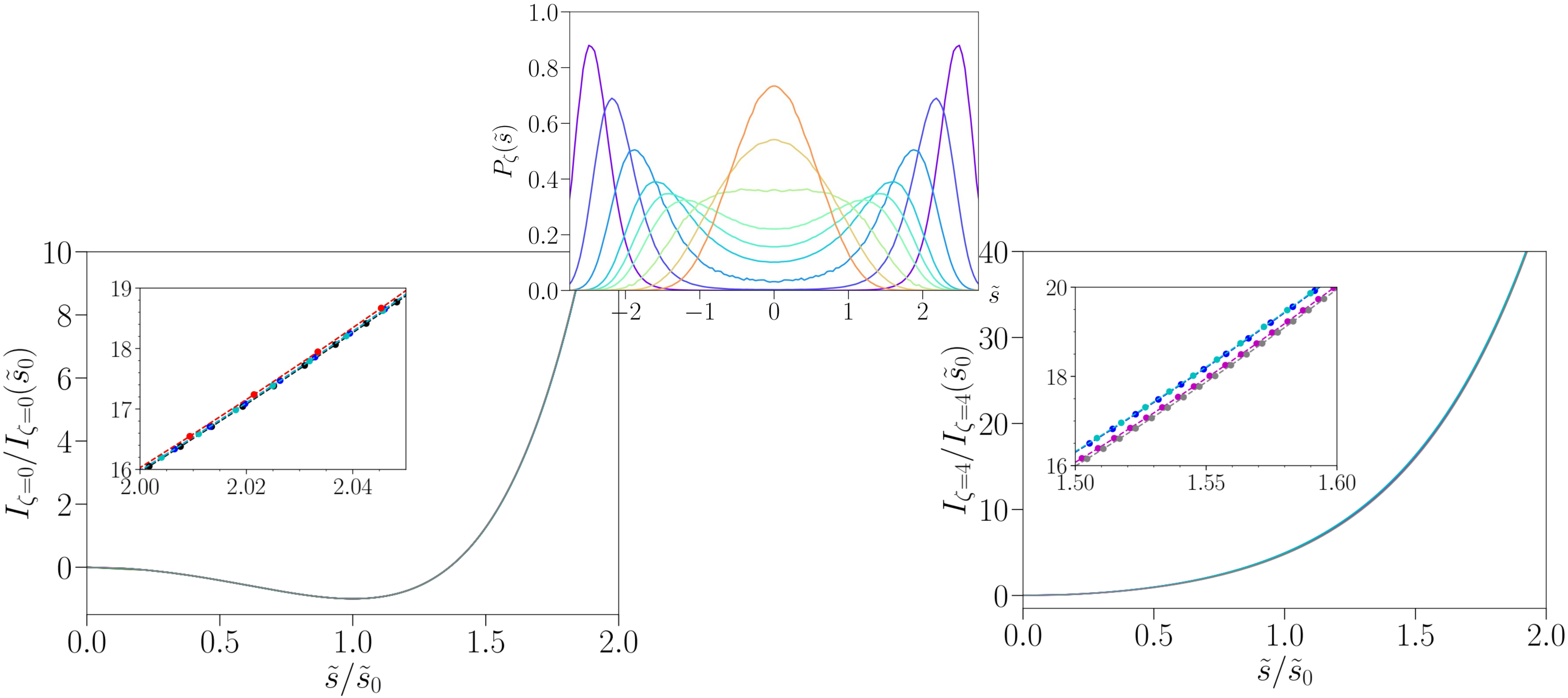


Universality of critical PDF: rate function

Choose $T(L) \to T_c$, such that $\zeta = L/\xi(T(L))$ is constant.

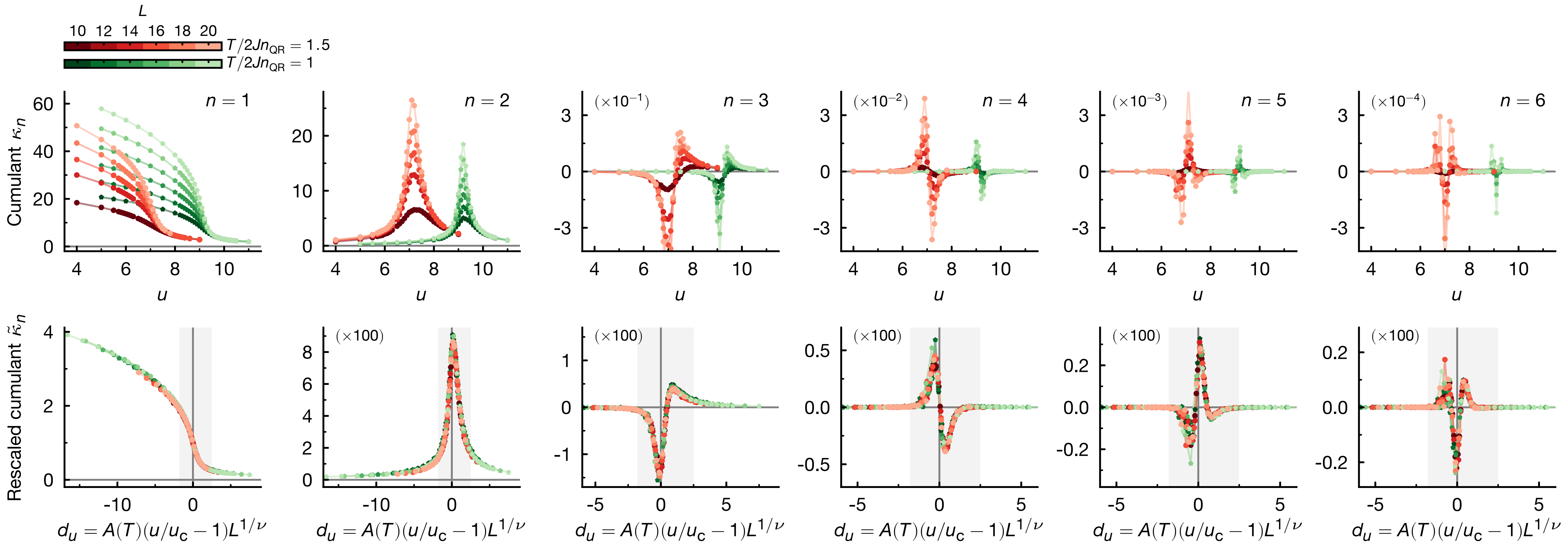
$$\tilde{m} = mL^{\beta/\nu}$$

$$P(m) \propto e^{-L^d I(m,\xi,L)} = e^{-I^*_\zeta(\tilde{m})}$$



Universality of the cumulants?

QMC simulations (by T. Roscilde) of quantum rotators (XY universality class) with periodic BC



Same cumulants than that obtain from classical MC of XY model and classical phi^4 theory

Probability distribution and RG fixed points

Direct connection between the probability distribution and RG fixed point quantities we compute?

$$H(\varphi) = \int d^dx\, \left((\nabla\varphi)^2 + r\varphi^2 + g\varphi^4\right)$$

Wilson's RG

$$H^*(\varphi) = \int d^dx\, \left((\nabla\varphi)^2 + r^*\varphi^2 + g^*\varphi^4 + \cdots\right)$$

Fixed point coupling constants

Link between Wilson's RG and probability distribution:
Bruce '81, Domb and Chen '96, Rudnik et al. '98,...

$$P(s) \propto e^{-H^*(s)}$$???

Based on perturbative RG, inclusion of finite size ad hoc...
and fixed point couplings are scheme dependent!?!

Comparison fixed point vs rate function