Renormalisation Group approach to General Relativity

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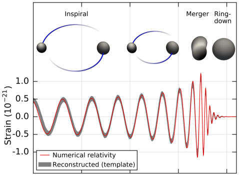
Introduction

- In 1915 Einstein wrote his field equations for gravity which predicted waves in the fabric of space and time.
- One hundred years later these oscillations were measured for the first time by the LIGO and VIRGO interferometer experiments.
- This has opened up a new era for General Relativity (GR) where we have access to the strongly coupled regime of the theory.



Introduction

- The two body model in GR describes the inspiralling, merger and ring-down of black hole pairs or other astrophysical relevant situations which produce gravitational waves.

- Obtaining an exact solution to the two body model is hard.
  1. either one settles for a perturbative expansion, which will fail when the system becomes strongly coupled.
  2. or one uses numerical general relativity which is computationally expensive.

Introduction

- In this talk I will discuss a third option: an exact renormalisation group (RG) equation for classical physics.
- **Talk by Facundo Gutiérrez tomorrow: Renormalization Group Approach to the Gravitational Two-Body Problem**
- Based on work with F. Gutiérrez and A. Codello: **arXiv:2605.22037, arXiv:2510.27676**

Introduction

- The idea is to start with the two body model, coupled to gravitational through the Einstein equations, and "integrate out" the gravitational field to arrive at an effective two body description.

- This is achieved using Wilson's idea of *coarse graining* the fields applied only to the gravitational field i.e. the metric tensor $g_{\mu\nu}(x)$.

- Work flow:

$$S[g, x_1, x_2] \to S_k[g, x_1, x_2] \to S_{k=0}[g, x_1, x_2] \to S_{k=0}[\eta, x_1, x_2] = S_{\text{eff}}[x_1, x_2]$$

Introduction

Numerical Relativity $\longleftrightarrow$ Perturbative Methods

(Non-perturbative, but slow) (Approximate, but fast)

$\updownarrow$

RG for GR

(Non-perturbative and fast?)

The Polchinski Equation

- RG: Integrate out high-momentum degrees of freedom while keeping Physics invariant
- The UV regulated cutoff is given by

$$P_k = \frac{1}{-\nabla^2} - \frac{1}{-\nabla^2 + R_k(-\nabla^2)} \implies \frac{1}{-\nabla^2} = P_k + \frac{1}{-\nabla^2 + R_k(-\nabla^2)}$$

  The total action reads:

$$\mathcal{S}_{\text{tot},k} = \int_x \int_y \frac{1}{2}\phi(x) P_k^{-1}(x,y)\phi(y) + \mathcal{S}_k$$

- The modified kinetic term ensures that the high-momentum modes are suppressed.
- At $k = 0$ we have $P_{k=0} = 0 \implies$ all modes are suppressed.
- Trivial solution to the equation of motion: $\phi_0 = 0$

The Polchinski Equation

- RG: Integrate out high-momentum degrees of freedom while keeping Physics invariant

- Defining the IR regulated cutoff by

$$\mathcal{G}_k = \frac{1}{-\nabla^2 + R_k(-\nabla^2)}\,, \qquad \dot{\mathcal{G}}_k = k\partial_k \frac{1}{-\nabla^2 + R_k(-\nabla^2)}$$

  The Polchinski equation reads:

$$k\frac{\partial}{\partial k}\mathcal{S}_k = -\int_x \int_y \frac{1}{2}\frac{\delta \mathcal{S}_k}{\delta\phi(x)}\dot{\mathcal{G}}_k(x,y)\frac{\delta \mathcal{S}_k}{\delta\phi(y)} + \frac{\hbar}{2}\int_x \int_y \dot{\mathcal{G}}_k(x,y)\frac{\delta^2 \mathcal{S}_k}{\delta\phi(y)\delta\phi(x)}$$

- The flow contains:
  - a classical term,
  - a quantum fluctuation term.

- This equation provides a practical realisation of Wilsonian RG.

The modern era

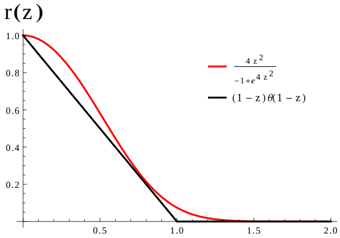
- In the modern era we use the Wetterich-Morris equation rather than the Polchinski equation.
- Mathematically the two equations are related by a Legendre transformation (Morris 1994).
- UV cut-off for the microscopic action $\to$ IR cutoff in the macroscopic action: $\phi \to \Phi = \langle \varphi \rangle$.
- While the Polchinski equation has a classical term, there is no classical term in the Wetterich-Morris equation.

Heuristic derivation of the Wetterich-Morris equation

- Let's start from the one-loop effective action supplemented by an IR cutoff

$$S[\Phi] + \frac{1}{2}\mathrm{Tr}\log\left(S^{(2)}[\Phi]\right) \to \Gamma_k[\Phi] = S[\Phi] + \frac{1}{2}\mathrm{Tr}\log\left(S^{(2)}[\Phi] + R_k\right)$$

- The function $R_k(p^2) = k^2 r(p^2/k^2)$ diverges for $k \to \infty$ and goes to zero when $k = 0$



- One gets

$$\partial_k \Gamma_k[\Phi] = \frac{1}{2}\mathrm{Tr}\frac{\partial_k R_k}{S^{(2)}[\Phi] + R_k}$$

Heuristic derivation of the Wetterich-Morris equation

- Performing an RG improvement one set's $S \to \Gamma_k$ in the rhs

$$\partial_k \Gamma_k[\Phi] = \frac{1}{2} \mathrm{Tr} \frac{\partial_k R_k}{\Gamma_k^{(2)}[\Phi] + R_k}$$

  this equations is exact!

- In the limit $k \to \infty$ we have $\Gamma_\infty[\Phi] = S[\Phi]$ is the bare action.

- In the limit $k \to 0$ we have $\Gamma_0[\Phi] = \Gamma[\Phi]$ is the full effective action.

Setting up the two body problem

- For the two body problem we have the action

$$S[g,x_1,x_2] = \frac{1}{\kappa} S_{\mathrm{g}}[g] + S_{\mathrm{pp}}[g,x_1,x_2]\,,$$

- Here $\frac{1}{\kappa} S_{\mathrm{g}}[g] = \frac{2c^4}{\kappa} \int dt\, d^3\mathbf{x} \sqrt{-g}\, R + S_{\mathrm{gf}}$ is the Einstein-Hilbert action with a gauge fixing term added, with $\kappa = 32\pi G_N$

- and $S_{\mathrm{pp}}[g,x_1,x_2] = -\sum_{n=1,2} m_n c^2 \int d\tau_n$ is the sum of the point-particle actions.

The effective action

- The central object of interest is the *effective action*, defined as the total action S evaluated on the metric that solves the equations of motion:

  $$S_{\mathrm{eff}}[x_n] = S[g_*[x_n], x_n] \,,$$

  where g_* satisfies the Einstein equations

  $$\frac{\delta S}{\delta g_{\mu\nu}(x)}[g_*[x_n], x_n] = 0 \,.$$

- Our aim is to construct a scale-dependent action $S_k[g, x_n]$ such that

  $$\lim_{k\to 0} S_k[\eta, x_n] = S_{\mathrm{eff}}[x_n] \,,$$

  where η denotes the Minkowski background.

The effective action

- The first correction to the effective action is given by

$$S_{\text{eff}} = S_{\text{pp}} - \frac{\kappa}{2} S_{\text{pp}}^{(1)} \cdot \Delta^{-1} \cdot S_{\text{pp}}^{(1)} + \mathcal{O}(\kappa^2)\,,$$

evaluated at $g_{\mu\nu}(x) = \eta_{\mu\nu}$. Here $\cdot$ denotes a sum over pairs of indices and an integral over spacetime e.g. $g \cdot J = \int_x g_{\mu\nu}(x) J^{\mu\nu}(x)$

$$\Delta^{\mu\nu,\rho\lambda}(x,y) = \left.\frac{\delta^2 S_{\text{g}}}{\delta g_{\mu\nu}(x)\delta g_{\rho\lambda}(y)}\right|_{g=\eta}$$

Heuristic route to the classical flow equation

- Again, within the RG framework, the propagator is modified by introducing a k-dependent regulator function R_k with $\Delta \to \Delta_k = \Delta + R_k$

- This leads to a scale-dependent effective action of the form

$$S_k = S_{\rm pp} - \frac{\kappa}{2} S_{\rm pp}^{(1)} \cdot \Delta_k^{-1} \cdot S_{\rm pp}^{(1)} + \mathcal{O}(\kappa^2)\,,$$

- Differentiating this expression with respect to k, one obtains the approximate flow equation

$$\partial_k S_k = -\frac{\kappa}{2} S_{\rm pp}^{(1)} \cdot \partial_k \Delta_k^{-1} \cdot S_{\rm pp}^{(1)}\,.$$

The flowing effective action

- Now one can promote $S^{(1)}_{\rm pp} \to S^{(1)}_{k\ \ \mu\nu}$ to the running first derivative of the effective action (aka the energy-momentum tensor) as in the heuristic derivation of the Wetterich-Morris equation.

$$S^{(1)}_{k,\mu\nu} = -\frac{1}{2}\sqrt{-g}T_{\mu\nu}$$

- Additionally we replace the Minkowski metric with an arbitrary metric $\eta_{\mu\nu} \to g_{\mu\nu}(x)$.

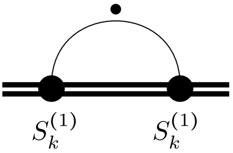
- The flow equation then takes the final form

$$\boxed{\partial_k S_k[g] = -\frac{\kappa}{2}\, S^{(1)}_k[g] \cdot \partial_k G_k[g] \cdot S^{(1)}_k[g]}$$

- Here we have the regularised propagator with the full gravity propagator

$$G_k[g] = \left(S^{(2)}_{\rm g}[g] + R_k\right)^{-1} .$$

$$\partial_k S_k = -\frac{\kappa}{2}$$



Diagrammatic representation of the functional RG equation for GR. Each big dot over the double lines represent the first functional derivative of the running effective action S_k for the two-particle system. The thin line corresponds to the regularized propagator G_k, and the dot above indicates the derivative ∂_k acting on it.

Post-Minkowskian expansion

- The fundamental requirement of the flow equation is that it must exactly reproduce the standard perturbative series.

- To show this, we start by expanding the effective action at scale k in powers of κ:

$$S_k = S_{\mathrm{pp}} + \kappa S_1 + \kappa^2 S_2 + \kappa^3 S_3 + \ldots$$

- Substituting this expansion into the flow equation, we obtain flows at each order in κ.

Post-Minkowskian expansion

- At first post-Minkowskian (1PM) order, the flow reduces to:

$$\partial_k S_1 = -\frac{1}{2} S_{\mathrm{pp}}^{(1)} \cdot \partial_k G \cdot S_{\mathrm{pp}}^{(1)} .$$

- Integrating both sides from $k = 0$ to $k = \infty$ yields:

$$S_1(\infty) - S_1(0) = -\frac{1}{2} S_{\mathrm{pp}}^{(1)} \cdot [G(\infty) - G(0)] \cdot S_{\mathrm{pp}}^{(1)} .$$

- Using the limiting behavior of R_k and the initial condition $S_{k\to\infty} = S_{\mathrm{pp}}$, we obtain the 1PM effective action:

$$S_1(k = 0) = -\frac{1}{2} S_{\mathrm{pp}}^{(1)} \cdot (\Delta^{-1}) \cdot S_{\mathrm{pp}}^{(1)} ,$$

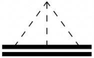
which coincides with the perturbative result.

PM expansion

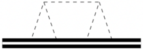
- We have checked that the PM expansion is recovered from the flow equation to order 3PM:

$$S_{\text{eff}} = S_{\text{pp}} - \frac{\kappa}{2}$$

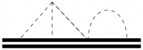

$$+ \kappa^2 \Big(\frac{1}{2}$$


$$- \frac{1}{6}$$


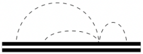
$$\Big)$$

$$+ \kappa^3 \Big(-\frac{1}{8}$$


$$+ \frac{1}{24}$$


$$+ \frac{1}{2}$$


$$- \frac{1}{2}$$


$$- \frac{1}{6}$$


$$\Big) + \mathcal{O}(\kappa^4)$$

The RG flow exactly reproduces the known 1PM, 2PM, and 3PM topologies, proving its consistency.

Beyond perturbation theory

- It is satisfying to reproduce perturbation theory, but we still would like to show that the equation is exact

- We know that the classical Polchinski equation for gravity is exact:

$$\boxed{\partial_k \mathcal{S}_k[h] = -\frac{\kappa}{2}\mathcal{S}_k^{(1)}[h] \cdot \partial_k \mathcal{G}_k \cdot \mathcal{S}_k^{(1)}[h]}$$

$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

- This equation differs from ours in two important ways
  1. The propagator $\mathcal{G}_k = (-\partial^2 + R_k)^{-1}$ is the free one.
  2. Here $\mathcal{S}_k[h]$ contains pure gravity parts where as $S_k[g]$ in our equation is the running point particle action, with no pure gravity part.

Classical average action

- We know that the Wetterich-Morris equation is more useful in practical computations
- Let's see if we can arrive at our flow equation by a Legendre transformation of the Polchinski equation
- The trick is to only make the Legendre transformation of the pure gravity part, hence we write

$$\mathcal{S}_k[h, x_1, x_2] = \frac{1}{\kappa}\mathcal{S}_{-1,k}[h] + \mathcal{S}_{\mathrm{pp},\,k}[h, x_1, x_2]\,,$$

Classical average action

- The flow of the pure gravity part is given by

$$\partial_k \mathcal{S}_{-1} = \frac{1}{2}\,\mathcal{S}_{-1}^{(1)} \cdot \frac{\partial_k R_k}{(\Delta + R_k)^2} \cdot \mathcal{S}_{-1}^{(1)}\,.$$

- We can solve this equation exactly by means of a Legendre transformation:

$$\boxed{S_{-1}[H] = \frac{1}{2} H \cdot \Delta \cdot H + \mathcal{S}_{-1}[h] - \frac{1}{2}(h - H) \cdot (\Delta + R) \cdot (h - H)}$$

- One can show that $S_{-1}[H]$ is independent of k hence

$$S_{-1}[H] = S_{\mathrm{g}}[\eta + H]$$

is just the Einstein-Hilbert action

Classical average action

- We now define the classical effective average action for the point-particles as

$$S_k[\eta + H, x_1, x_2] \equiv \mathcal{S}_{\mathrm{pp},k}[h_k[H], x_1, x_2]\,.$$

in other words we just re-express the point-particle action of the Polchinski equation in terms of the field H.

- In a few steps you can show that

$$\boxed{\partial_k S_k[g] = -\frac{\kappa}{2}\, S_k^{(1)}[g] \cdot \partial_k \frac{1}{S_g^{(2)}[g] + R_k} \cdot S_k^{(1)}[g]}$$

which is exactly our flow equation.

- By the relation to the Polchinski equation one can also prove that when $k = 0$ and $g_{\mu\nu} = \eta_{\mu\nu}$ one recovers exactly the effective action.

Conclusions

- A new middle path to solving problems in classical GR: a flow equation
- Reproduces perturbation theory
- New non-perturbative approximations can be constructed, e.g. Keep all operators with scaling dimension up to some fixed order.
- Open problem: control of diffeomorphism invariance
- First results are on the way (see talk by **Facundo Gutiérrez**)!