Asymptotic behaviour of the derivative expansion in the ERG

ERG2026
05/09/26

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TRM, 2607.18397

- The derivative expansion of the FRG is a divergent series in any dimension, both for an exponential cutoff and more general smooth cutoffs.

- <= Within massless $\lambda\phi^4$ perturbation theory, derivative expansion diverges first at two loops.

- => the approximation technique of expanding the FRG in ever larger sets of local operators does not converge.

- The derivative expansion is an asymptotic series that initially converges towards the exact result before divergent behaviour takes over.

- Asymptotic => can nevertheless get accurate results from summing first few terms, most accurate result gained by summing to just short of the smallest term.
- <= several lines of theoretical argument and analysing infinite classes of 2-loop and 3-loop contributions.
- Talk will focus on some key points.

$$\Delta_{IR}(q,\Lambda) = \frac{C_{IR}(q^2/\Lambda^2)}{q^2} = \frac{1}{q^2 + R_\Lambda(q)}$$

$$\partial_\Lambda \Gamma = \frac{1}{2}\mathrm{tr}\left[\frac{K_\Lambda}{\Delta_{IR}}\left(1+\Delta_{IR}\Gamma^{(2)}\right)^{-1}\right]$$

$K_\Lambda = -\partial_\Lambda \Delta_{IR}$

$$\Gamma = \sum_{\ell=0}^{\infty} \Gamma_\ell$$

$$\Gamma_0 = \int d^dx \left\{\frac{1}{2}(\partial_\mu \varphi)^2 + \frac{\lambda}{4!}\varphi^4\right\}$$

(irrelevant parts)

$$\partial_\Lambda \Gamma_1 = \frac{1}{2}\mathrm{tr}\left[\frac{K_\Lambda}{\Delta_{IR}}\left(1+\Delta_{IR}\Gamma_0^{(2)}\right)^{-1}\right] \quad \Longrightarrow \quad \Gamma_1 = \frac{1}{2}\,\mathrm{tr}\ln\left(1+\Delta_{IR}\Gamma_0^{(2)}\right)$$

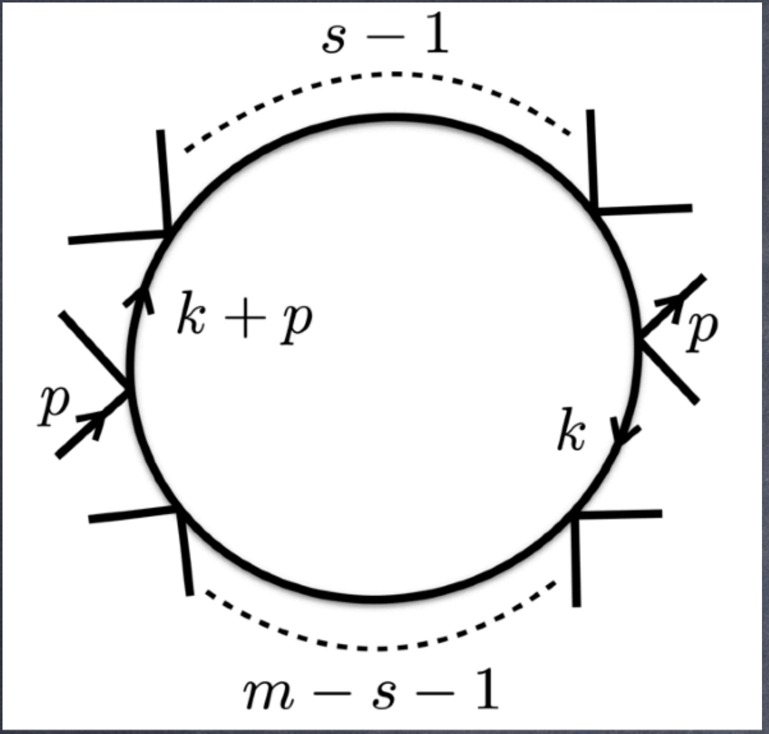
$$\partial_\Lambda \Gamma_2 = -\frac{1}{2}\mathrm{tr}\left[\frac{K_\Lambda}{\Delta_{IR}}\left(1+\Delta_{IR}\Gamma_0^{(2)}\right)^{-1}\Delta_{IR}\Gamma_1^{(2)}\left(1+\Delta_{IR}\Gamma_0^{(2)}\right)^{-1}\right]$$

Exponential cutoff

$$R_\Lambda(q) = \frac{q^2}{\mathrm{e}^{\,q^2/\Lambda^2} - 1}$$

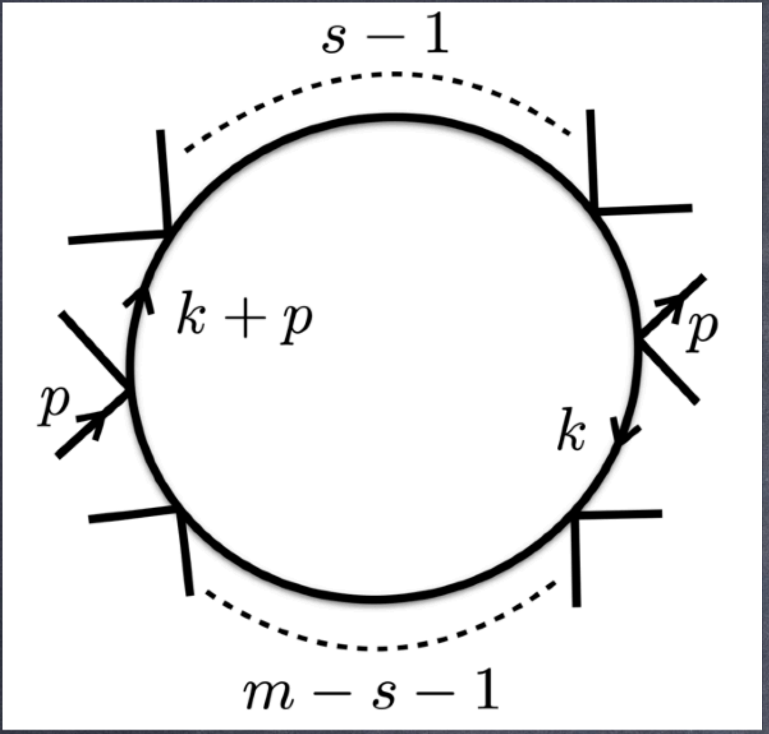
$$\Delta_{IR}(q,\Lambda) = \frac{1-\mathrm{e}^{-q^2/\Lambda^2}}{q^2} \qquad\qquad K_\Lambda(q) = \frac{2}{\Lambda^3}\,\mathrm{e}^{-q^2/\Lambda^2}$$

- => One loop vertices $\Gamma_1(p_1,\cdots,p_{2m})$ have convergent derivative/momentum expansions with an infinite radius of convergence



$\Gamma_1(p,-p,0^{2m-2})$ at large $O(\partial^{2n})$

$$\int \frac{d^d k}{(2\pi)^d}\,\Delta_{IR}^{m-s}(k)\Delta_{IR}^{s}(k+p)$$



$\Gamma_1(p,-p,0^{2m-2})$ at large $O(\partial^{2n})$

$$\int \frac{d^dk}{(2\pi)^d}\, \Delta_{IR}^{m-s}(k)\Delta_{IR}^{s}(k+p)$$

$$\Delta_{IR}(q,\Lambda) = \frac{1-\mathrm{e}^{-q^2/\Lambda^2}}{q^2} = \frac{1}{\Lambda^2}\int_0^1 da\ \mathrm{e}^{-aq^2/\Lambda^2}$$

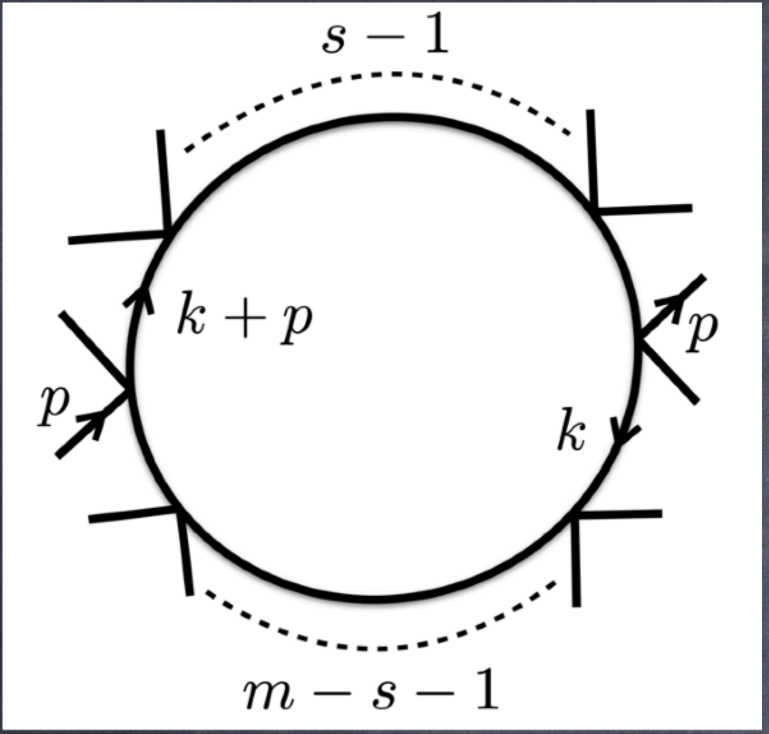
$$= \frac{\Lambda^{d-2m}}{(4\pi)^{d/2}}\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{p^2}{\Lambda^2}\right)^n \int_0^1 \frac{d^{\,m-s}a\, d^sb}{A^{d/2}B^{d/2}}\left(\frac{1}{A}+\frac{1}{B}\right)^{-n-d/2}$$

$$A = \sum_{i=1}^{m-s} a_i\,, \qquad B = \sum_{j=1}^{s} b_j$$

$$a_i = 1-\alpha_i\,, \qquad b_j = 1-\beta_j$$

$$\frac{s^n(m-s)^n}{m^{n+d/2}}\int_0^1 d^{\,m-s}\alpha\, d^s\beta \left(1+\frac{s}{m(m-s)}\sum_{i=1}^{m-s}\alpha_i + \frac{m-s}{ms}\sum_{j=1}^{s}\beta_j\right)^{-n-d/2} + \cdots$$

$$= m^{m-d/2}\left(\frac{m}{s}-1\right)^{m-2s}\frac{1}{n^m}\left(\frac{s(m-s)}{m}\right)^n + \cdots$$



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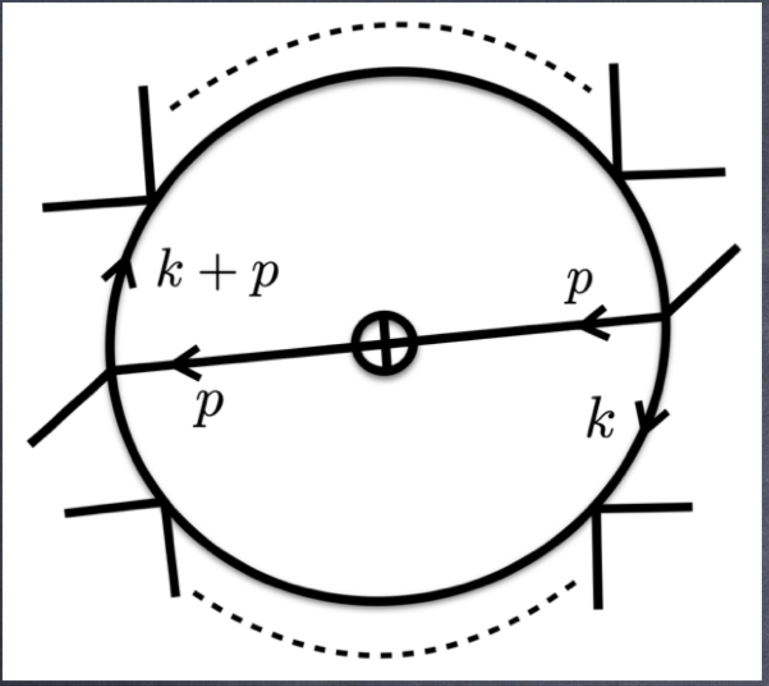
$$= m^{m-d/2}\left(\frac{m}{s}-1\right)^{m-2s}\frac{1}{n^m}\left(\frac{s(m-s)}{m}\right)^n+\cdots$$

$$\Gamma_1(p,-p,0^{2m-2})|_{\partial^{2n}} = -2(2m-2)!\frac{\Lambda^{d-2m}}{(4\pi m)^{d/2}}\left(-\frac{m}{2}\lambda\right)^m\frac{1}{n!\,n^m}\left(-\frac{m}{4}\frac{p^2}{\Lambda^2}\right)^n+\cdots \qquad (m\ \text{even})\,,$$

$$= -4\frac{m+1}{m-1}(2m-2)!\frac{\Lambda^{d-2m}}{(4\pi m)^{d/2}}\left(-\frac{m}{2}\lambda\right)^m\frac{1}{n!\,n^m}\left(-\frac{m^2-1}{4m}\frac{p^2}{\Lambda^2}\right)^n+\cdots \qquad (m\ \text{odd})\,. \quad (5.9)$$



Two loop contribution to ϕ^{2m}

$$\partial_\Lambda \Gamma_2(0^{2m})|_{\partial^{2n}} = -\frac{1}{2}\int \frac{d^dp}{(2\pi)^d} K_\Lambda(p)\,\Gamma_1(p,-p,0^{2m})|_{\partial^{2n}} \; + \cdots$$

$$K_\Lambda(p) = \frac{2}{\Lambda^3}\,\mathrm{e}^{-p^2/\Lambda^2}$$

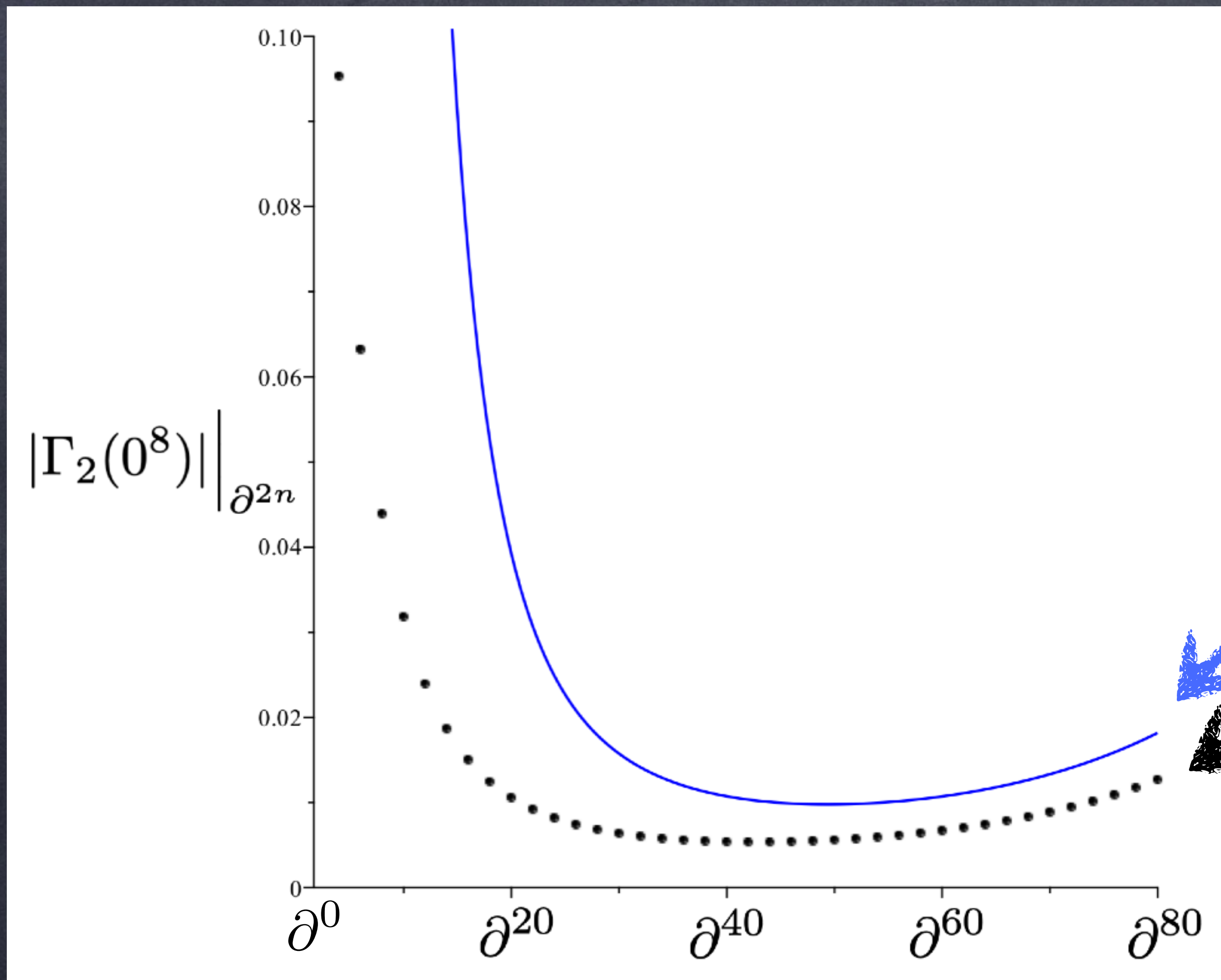
$$\frac{1}{2}\int \frac{d^dp}{(2\pi)^d} K_\Lambda(p)\,p^{2n} = \frac{\Lambda^{d-3+2n}}{(4\pi)^{d/2}}\,\frac{\Gamma(n+d/2)}{\Gamma(d/2)}$$

$$\partial_\Lambda \Gamma_2(0^{2m})|_{\partial^{2n}} = \tilde{C}_m\,\Lambda^{2d-2m-5} n^{d/2-2-m}(-c_m)^n \; + \cdots$$

$$c_m = \frac{m+1}{4} \quad (m \text{ odd}), \qquad c_m = \frac{m+1}{4}\left(1-\frac{1}{(m+1)^2}\right) = \frac{m(m+2)}{4(m+1)} \quad (m \text{ even})$$

- ϕ^2 and ϕ^4 have convergent derivative expansions
- ϕ^6 converges only in d<10 dimensions
- ϕ^8 etc have divergent derivative expansions

Two loop contribution to ϕ^8 in d=3 dimensions

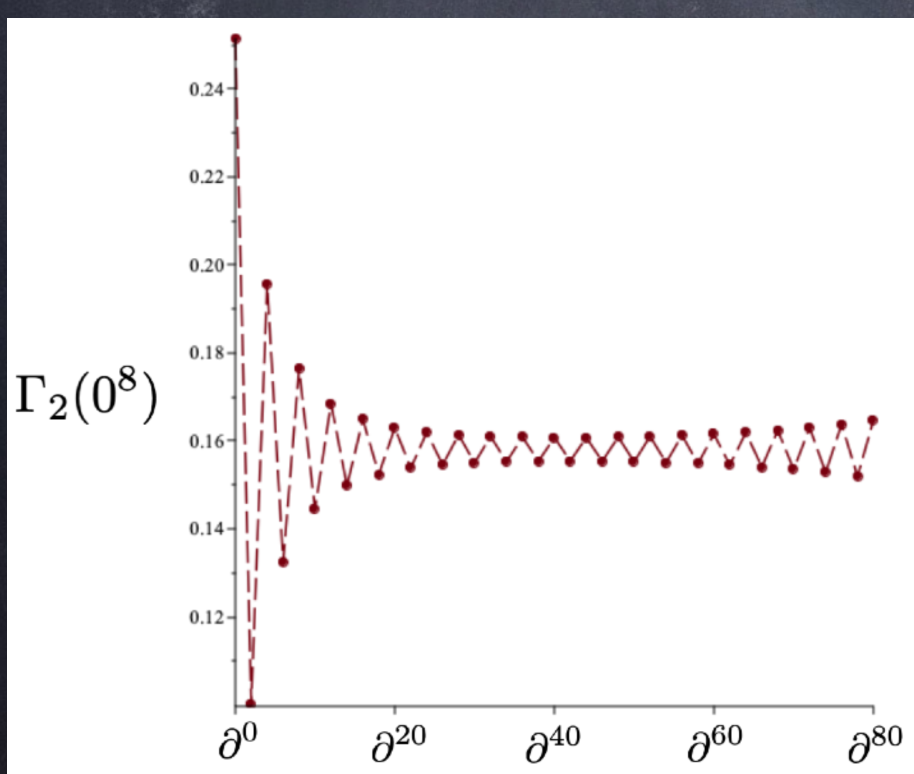


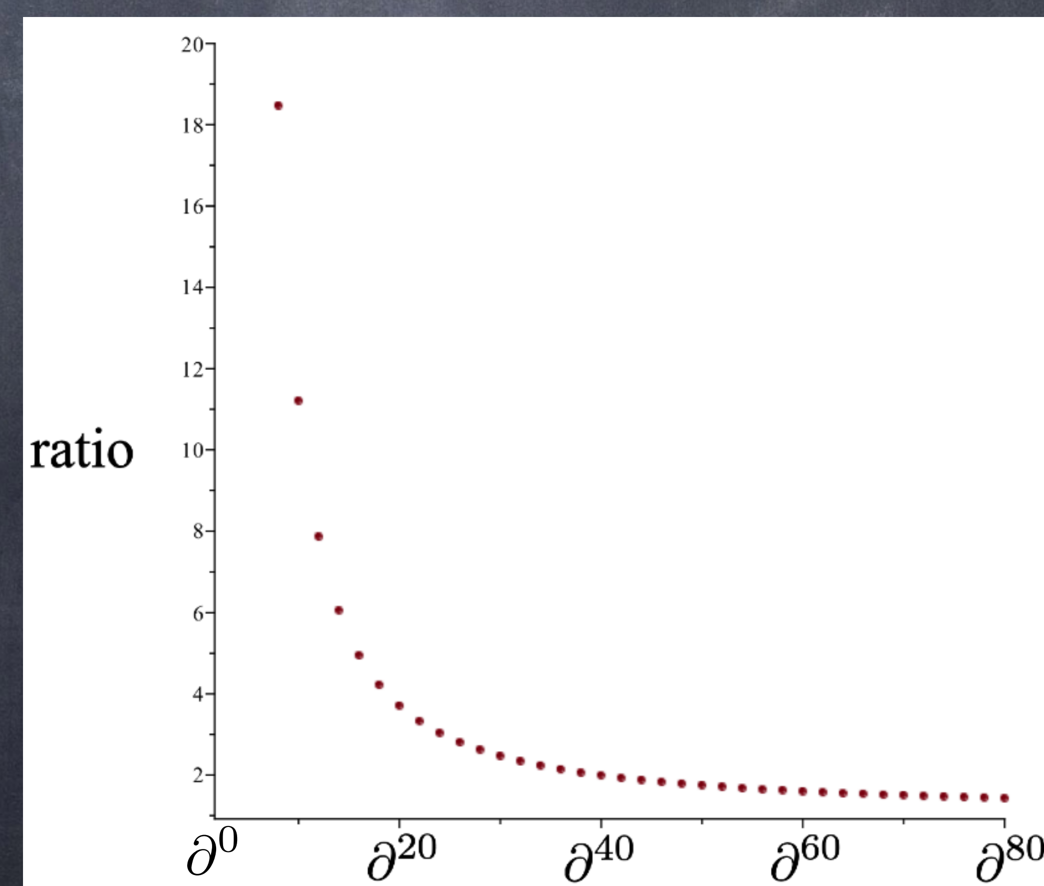
Leading n formula

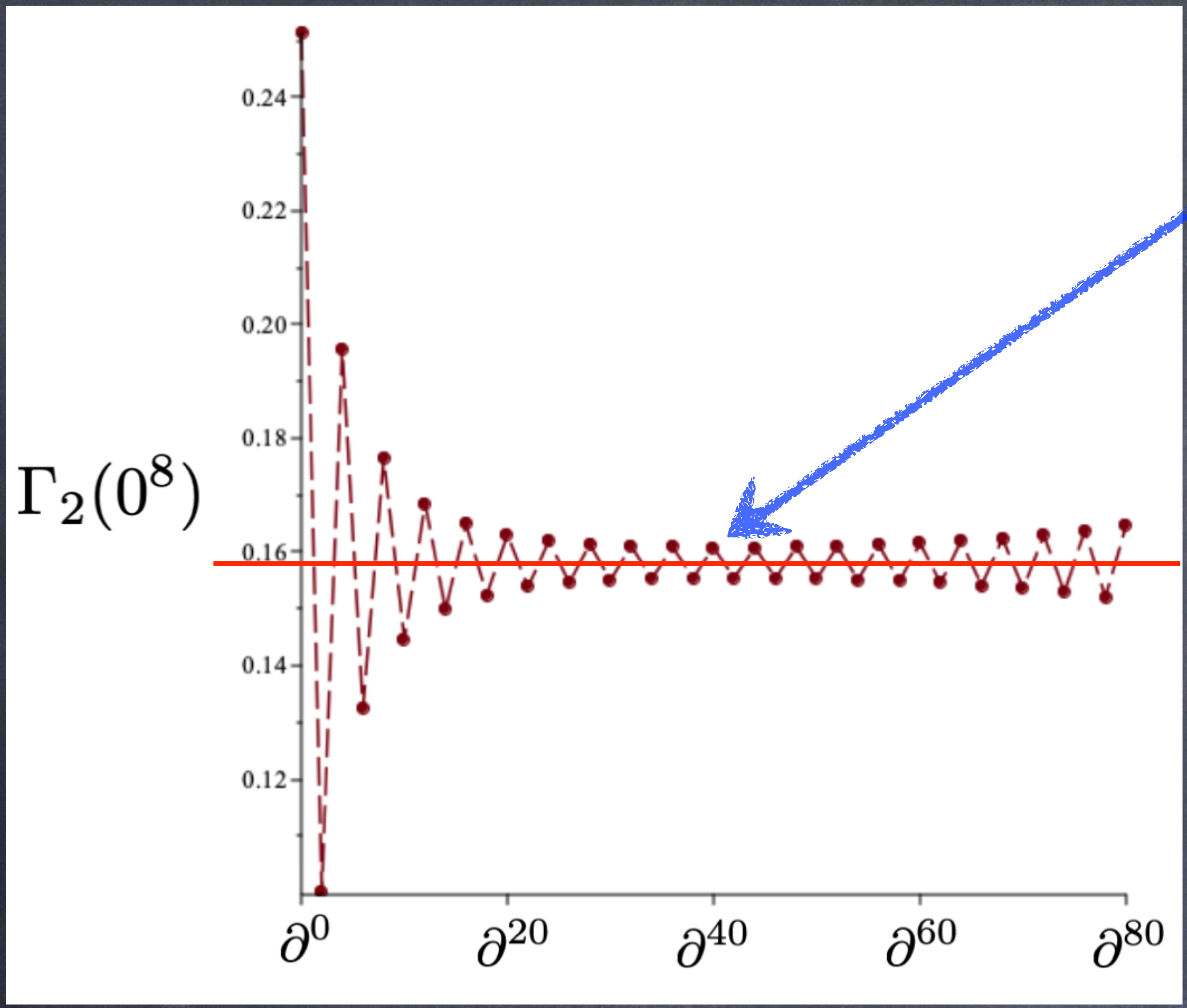
Exact result from

$$\int \frac{d^dp}{(2\pi)^d} K_\Lambda(p)\, \Gamma_1(p,-p,0^{2m})|_{\partial^{2n}}$$

in units λ^5/Λ^6







Asymptotic series rule of thumb: best result is sum to just before smallest term

Exact answer here?

Leading n formula:

$$-\mathrm{Li}_\varsigma(-x) = -\sum_{n=1}^{\infty} \frac{(-x)^n}{n^\varsigma}$$

$$x = c_m > 0\,, \quad \varsigma = m + 2 - d/2$$

$$-\mathrm{Li}_\varsigma(-x) = \frac{1}{\Gamma(\varsigma)} \int_0^\infty dt\, t^{\varsigma-1} \frac{x}{\mathrm{e}^t + x}$$

$$\frac{x}{\mathrm{e}^t + x} = \frac{x\,\mathrm{e}^{-t}}{1 + x\,\mathrm{e}^{-t}} = -\sum_{n=1}^{\infty} \mathrm{e}^{-nt}(-x)^n$$

$$= -\sum_{n=1}^{N} \mathrm{e}^{-nt}(-x)^n - \frac{1}{1 + x\,\mathrm{e}^{-t}}\,\mathrm{e}^{-(N+1)t}(-x)^{N+1}$$

Exact remainder is same sign but less than the next term!

=> exact answer always lies between successive partial sums!

Watson's lemma

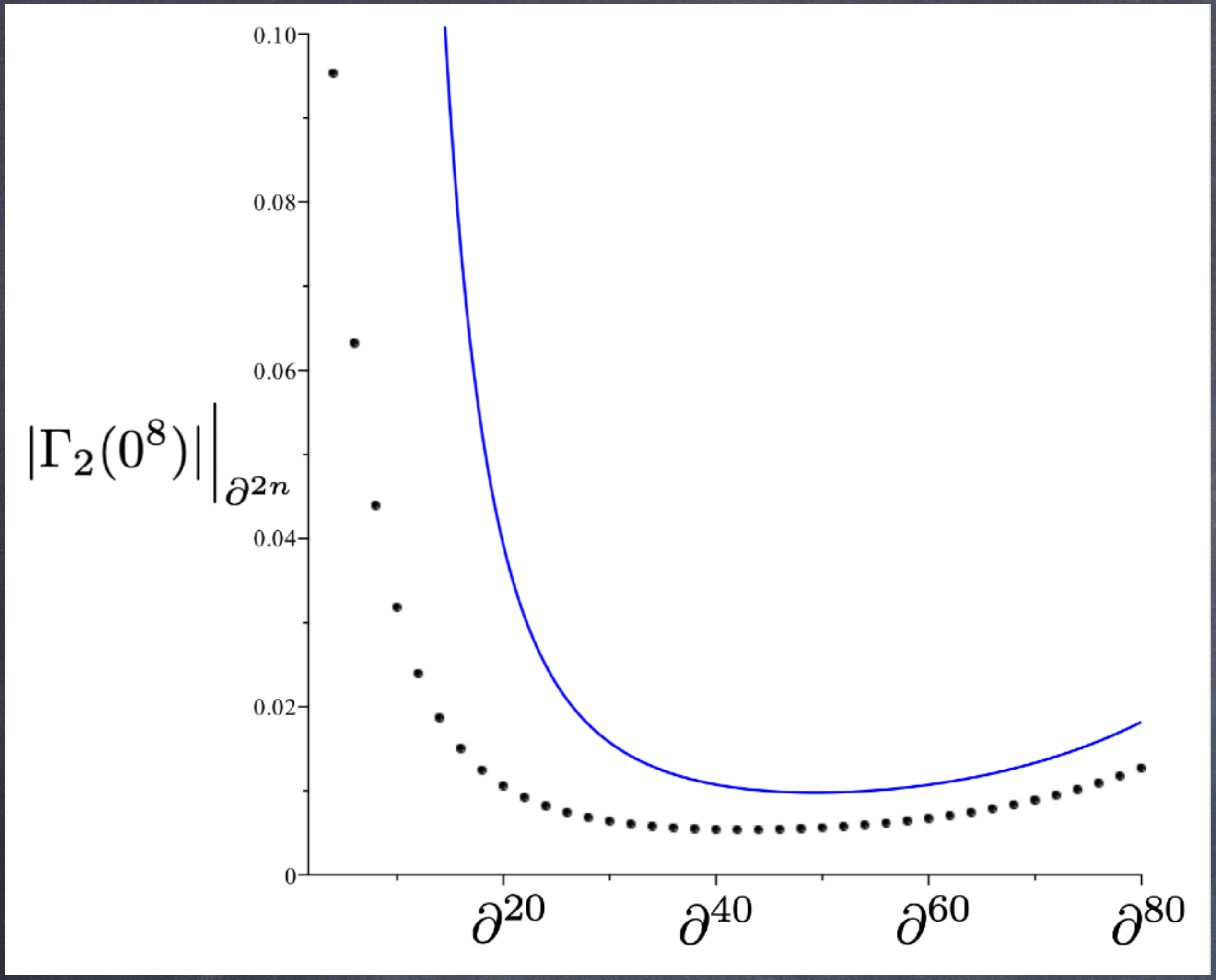
$$F(g) = \int_0^\infty du\, \mathrm{e}^{-u/g}\, u^\sigma\, f(u)$$

has an asymptotic expansion in g>0 if f(u) is bounded and Taylor expandable about the origin u=0, and σ>-1.

$$\partial_\Lambda \Gamma = \frac{1}{2}\mathrm{tr}\left[\frac{K_\Lambda}{\Delta_{IR}}\left(1+\Delta_{IR}\Gamma^{(2)}\right)^{-1}\right] \qquad K_\Lambda(q) = \frac{2}{\Lambda^3}\,\mathrm{e}^{-q^2/\Lambda^2}$$

Average over angles q^μ and set $u = q^2/\Lambda^2$

=> g = 1 and σ = d/2-1



Higher point vertices:

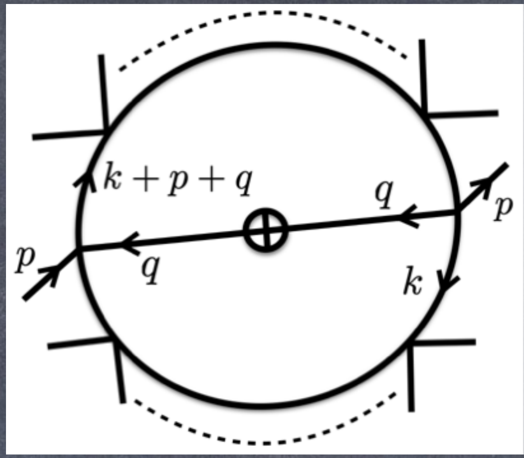
$$\partial_\Lambda \Gamma_2(0^{2m})|_{\partial^{2n}} = \tilde{C}_m \, \Lambda^{2d-2m-5} n^{d/2-2-m} (-c_m)^n \, + \cdots$$

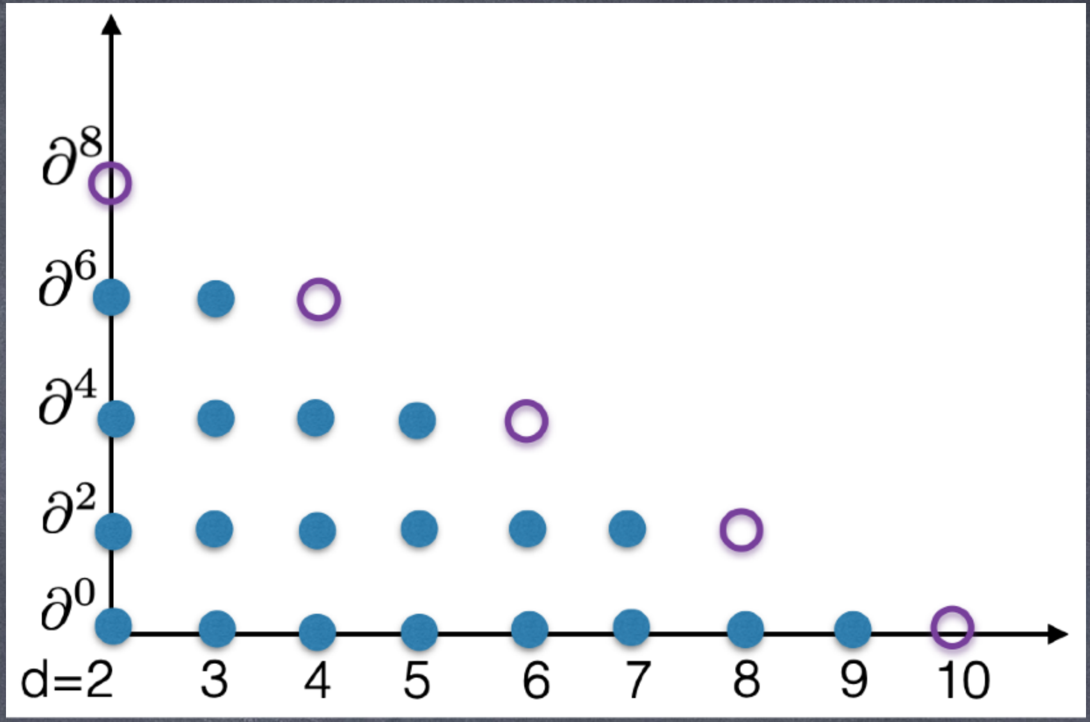
convergent up to $\sim O(\partial^{18})$

but leading contribution becomes more inaccurate around minimum with increasing m.

Paper also treats $\varphi^{2m}\partial^{2r}$

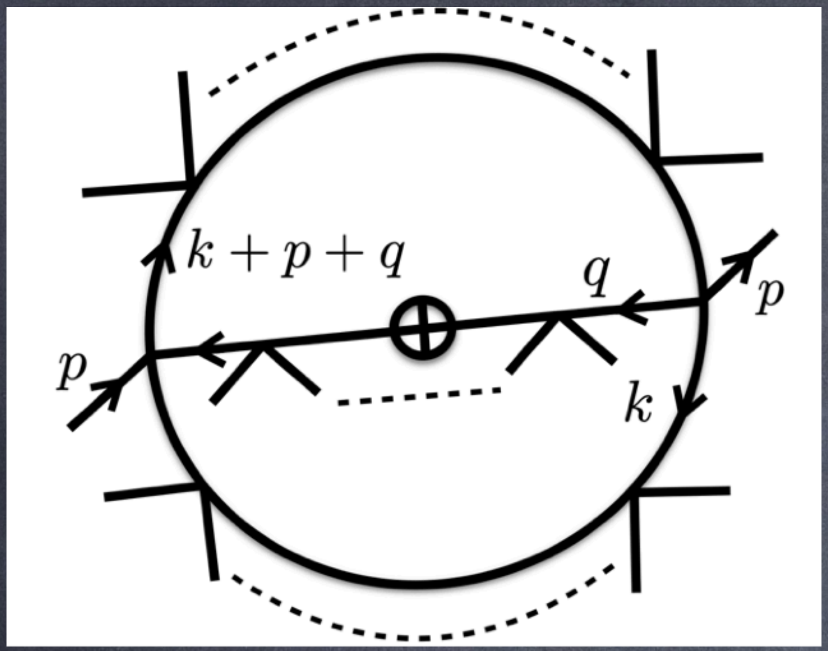
At two loops:



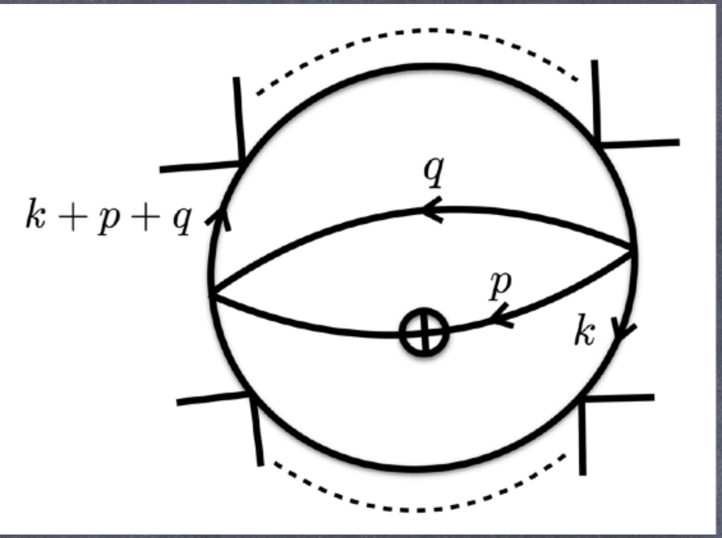


- $2-$point and $4-$point vertices have convergent derivative expansions, for any r or d
- 6-point converges in d<10-2r dimensions, oscillates in d=10-2r, and diverges for d>10-2r (and at leading order in n not in a good way)
- If d<10-2r, $8-$point and higher have divergent but, at leading order, asymptotic derivative expansions

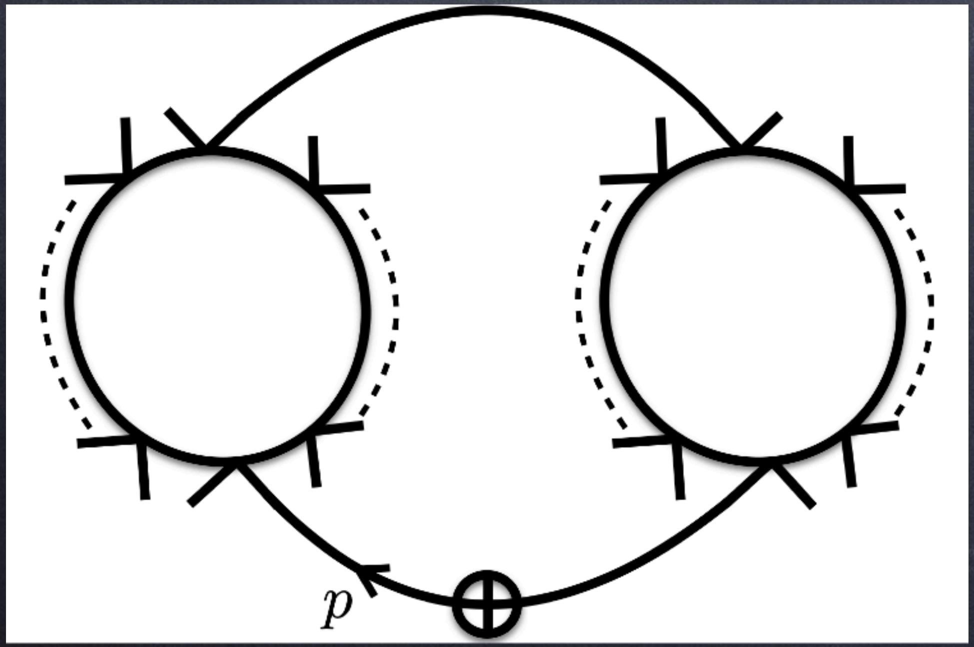
And other infinite sets at leading order:



subleading



three loops:
asymptotic for all ϕ^{2m}
convergent to $\sim O(\partial^{10})$



three loops, asymptotic
and slightly better than 2 loops.

General smooth cutoffs:

Balog et al. 1907.01829

$$W_\Lambda(q) = \frac{\alpha\, q^2}{\mathrm{e}^{\,q^2/\Lambda^2} - 1} \qquad (\alpha \neq 1) \qquad\qquad E_\Lambda(q) = \alpha\Lambda^2\, \mathrm{e}^{-q^2/\Lambda^2}$$

Resulting dependence is not analytic in q, so one-loop:

$$\Gamma_1(p) = \sum_{n=0}^{\infty} a_n\, p^{2n}/\Lambda^{2n} \qquad\qquad a_n = n^\gamma (-1)^n / r^n(\alpha) + \cdots$$

$$\int \frac{d^dp}{(2\pi)^d} K_\Lambda(p)\, p^{2n} \sim n! \qquad \Longrightarrow \qquad \Gamma_2 \sim n^{d/2+\gamma'} n! (-1/r)^n + \cdots$$

Asymptotic again. Can show also that at large n, Watson's lemma applies again.

Summary

- The derivative expansion of the FRG is a divergent series in any dimension, both for an exponential cutoff and more general smooth cutoffs.
- <= Within massless $\lambda\phi^4$ perturbation theory, derivative expansion diverges first at two loops.
- => the approximation technique of expanding the FRG in ever larger sets of local operators does not converge.
- The derivative expansion is an asymptotic series that initially converges towards the exact result before divergent behaviour takes over.