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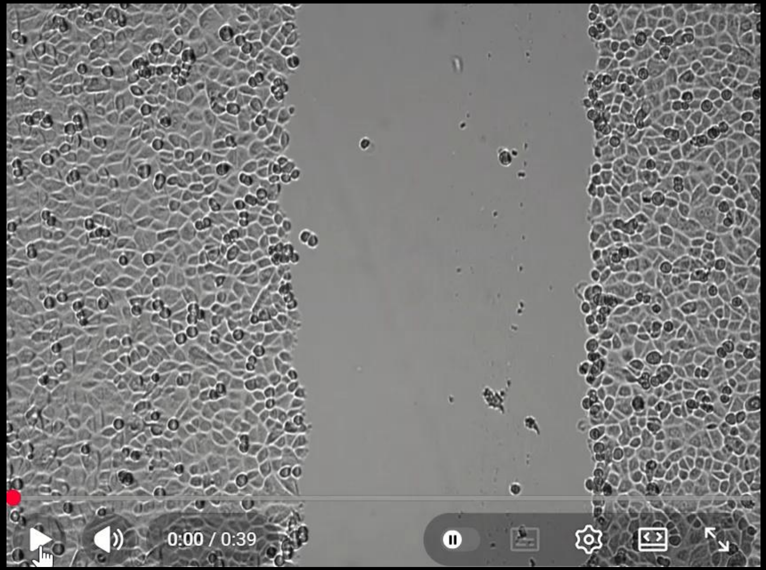
Universality Far from Equilibrium:
New Fixed Points and New Challenges for RG

Chiu Fan Lee
Department of Bioengineering, Imperial College London, UK

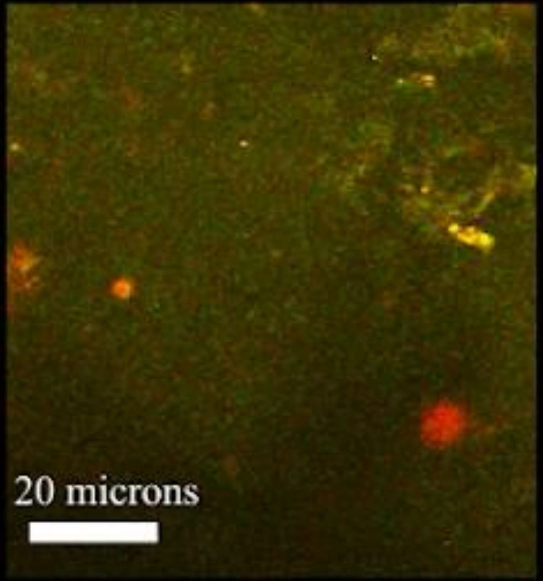
IMPERIAL

Wound healing assay



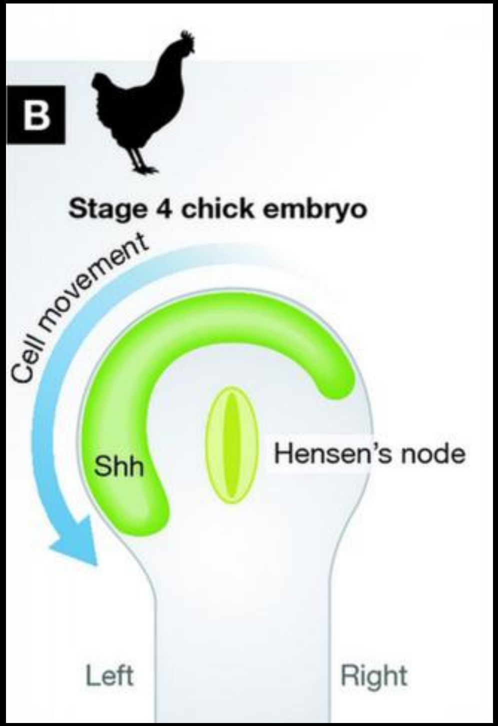
https://www.youtube.com/watch?v=v9xq_GiRXeE

Haematopoietic stem cell in bone marrow



Upadhaya et al, Cell Stem Cell (2020)

Left-right symmetry breaking



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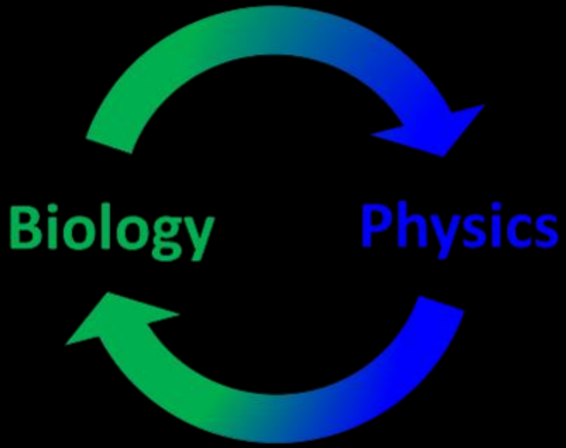

Q: Given how complicated these biological systems are, how is meaningful modelling ever possible?

A: Even if the microscopic dynamics is complicated, the underlying symmetry & conservation law may be simple

→ *universal hydrodynamic equations of motion*

+

Renormalisation group analysis

Universal hydrodynamics (more can be simpler) -Navier-Stokes

- Incompressible limit for simplicity
- Hydrodynamic variable: velocity $\vec{v}$
- Equation of motion (EOM):

$$\partial_t \vec{v} = \rho^{-1} \vec{F}\ \text{, with constraint } \nabla \cdot \vec{v} = 0$$

- What is the force field $\vec{F}$?

Symmetries + conservation

$$\boxed{\text{EOM: } \partial_t \vec{v} = \rho^{-1} \vec{F}}$$

- Symmetries
  - Temporal invariance: **F** does not depend on time
  - Translational invariance: **F** does not depend on position **r**
  - Rotational invariance: **F** does not depend on a particular direction
  - Chiral (parity) invariance: **F** is not right-handed or left-handed
  - Galilean invariance: EOM invariant under a boost
- Conservation: momentum conservation

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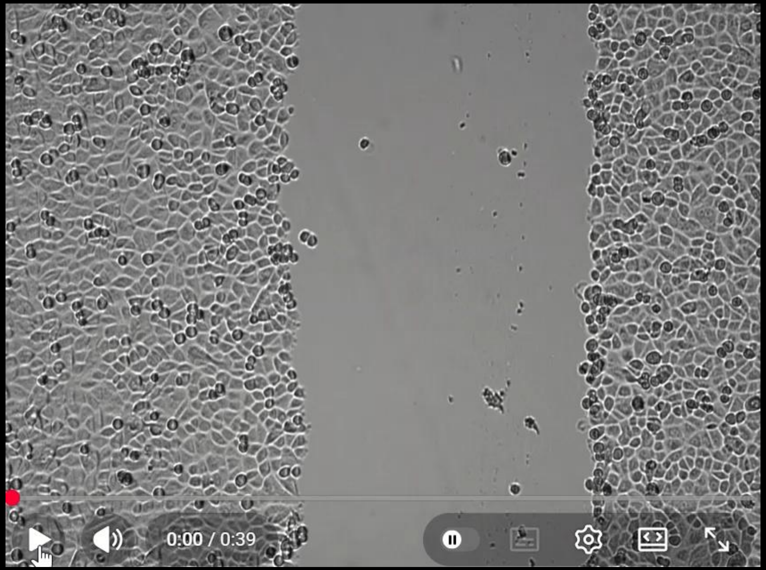
- To order $O(v^2, \partial^2)$ [hydrodynamics], incompressible Navier-Stokes EOM:

$$\partial_t \vec{v} = -\vec{\nabla} P - \left(\vec{v} \cdot \vec{\nabla}\right)\vec{v} + \mu \nabla^2 \vec{v} + \cdots + \vec{f}$$

'pressure' (Lagrange multiplier) to enforce incompressibility

Noise term satisfying **symmetries & conservation law**

Wound healing assay



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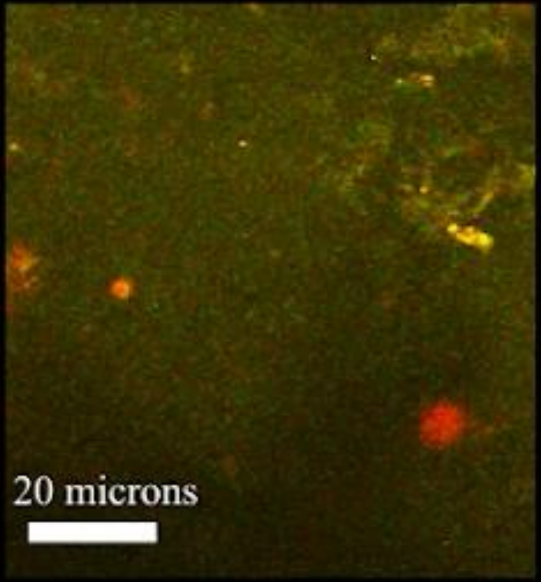

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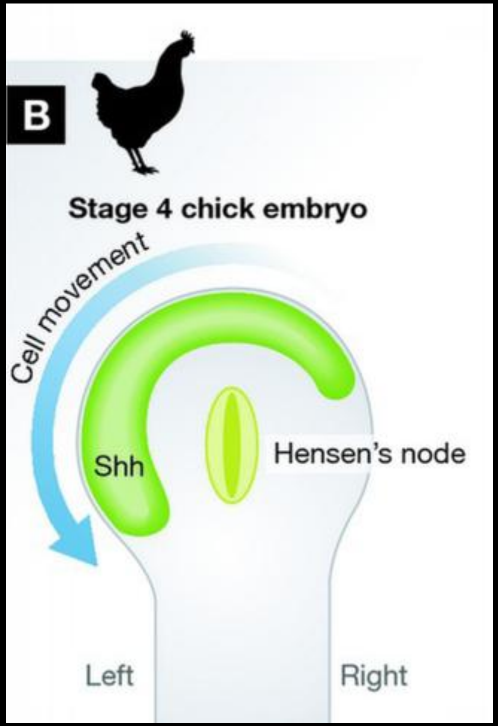

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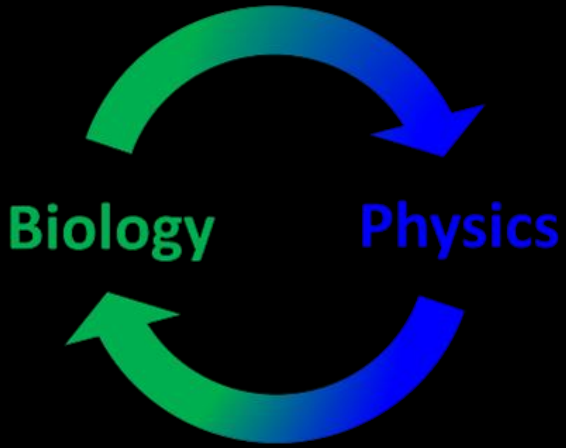

Active fluids ≠ Navier-Stokes

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- ~~Conservation: momentum conservation~~

Hydrodynamic EOM of incompressible active fluids

$$\partial_t \vec{v} + \lambda_0 (\vec{v} \cdot \vec{\nabla}) \vec{v} = \mu_0 \nabla^2 \vec{v} + (a_0 + b_0 v^2) \vec{v} - \vec{\nabla} P + \overrightarrow{f_0} + \ldots$$

Gaussian noise

'pressure' (Lagrange multiplier) to enforce incompressibility

- A.k.a Toner-Tu EOM in the incompressible limit
- = Advective O(d) model due to spin-space coupling $\lambda_0 (\vec{v} \cdot \vec{\nabla}) \vec{v}$

Ordered phase in 3D – flocking phase

Mean flocking direction is $\hat{\mathbf{x}}$ and let $\mathbf{v} = (v_0 + u_x)\hat{\mathbf{x}} + \mathbf{u}_\perp$

EOM: $\boxed{\partial_t \mathbf{u}_\perp + \lambda(\mathbf{u}_\perp \cdot \nabla_\perp)\mathbf{u}_\perp = -\nabla_\perp P + \mu_\perp \nabla_\perp^2 \mathbf{u}_\perp + \mu_x \partial_x^2 \mathbf{u}_\perp + \mathbf{f}_\perp}$

$$u_n^\perp \partial_n^\perp u_m^\perp = \partial_n^\perp (u_n^\perp u_m^\perp) + (\cancel{\partial_x u_x}) u_m^\perp .$$

- Neither μ_x nor noise strength get renormalised
- Also, λ is not renormalised because of an emergent pseudo-Galilean invariance of the ordered-phase EOM

3 exact non-renormalisation conditions
→ 3 scaling relations:

$$z = 2\zeta, \quad z + \chi = 1, \quad z - 2\chi - \zeta = d - 1$$

$$\mathbf{r}_\perp \to b\,\mathbf{r}_\perp, \quad x \to b^\zeta x, \quad t \to b^z t, \quad \mathbf{u}_\perp \to b^\chi \mathbf{u}_\perp$$

'Exact' result: $\zeta = \frac{4}{5}, \quad z = \frac{8}{5}, \quad \chi = -\frac{3}{5}, \quad (d = 3)$

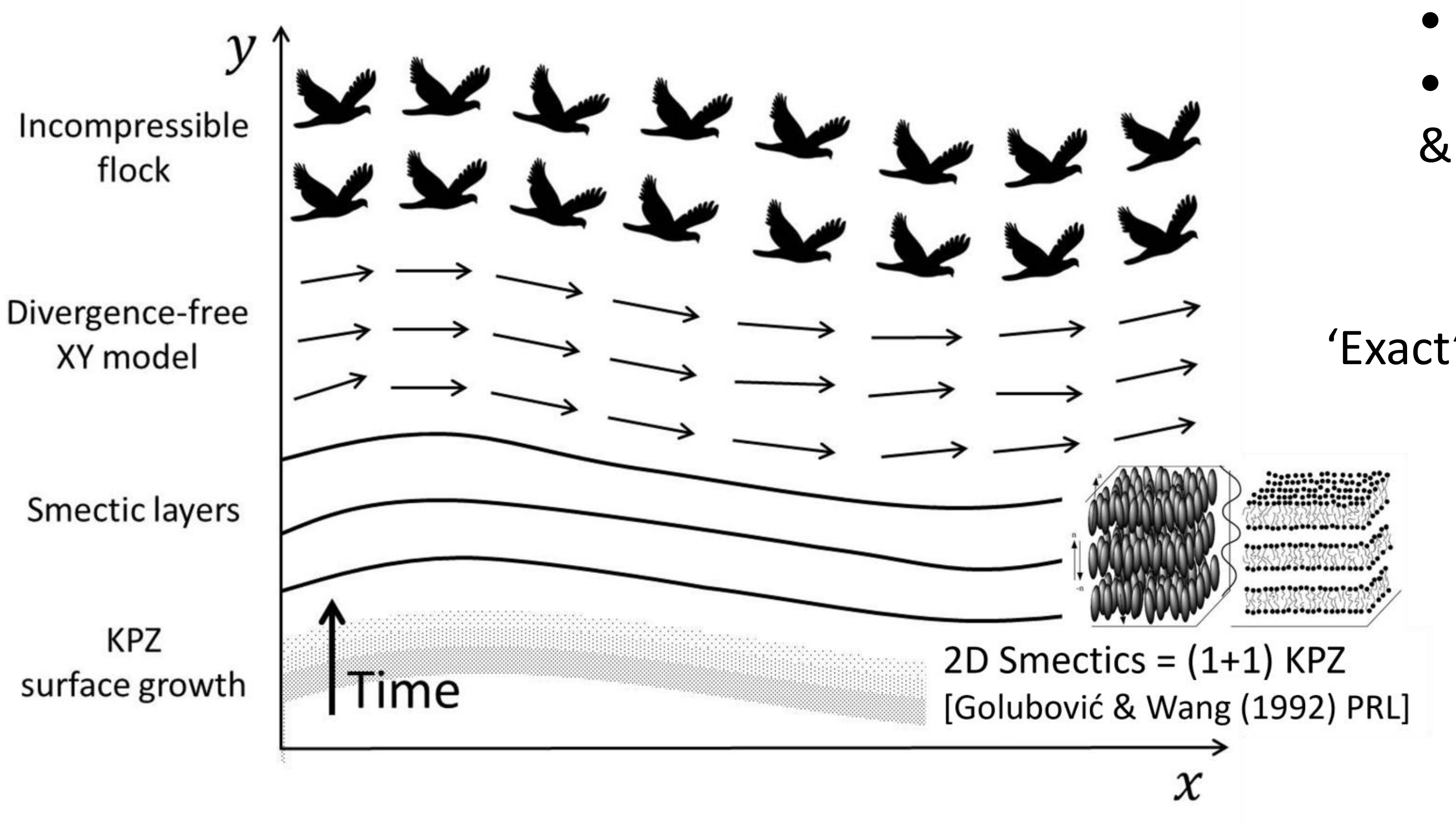
Further support from functional RG analysis:
P. Jentsch and C.F. Lee. Can exact scaling exponents be obtained using the renormalization group? Affirmative evidence from incompressible polar active fluids. E-print: arXiv:2307.06725

Toner & Tu (1995) PRL
Chen, Lee, Toner (2018) NJP

Ordered (flocking) phase in 2D

In $d = 2$, incompressibility removes u_x as an independent degree of freedom → a single scalar field

$$\boxed{u_x = -v_0 \partial_y h, \qquad u_y = v_0 \partial_x h.}$$



2 constraints:
- stationary/FDT relation
- KPZ tilt (Galilean) invariance

& 2 scaling exponents

'Exact' result: $\zeta = \frac{2}{3}, \quad \chi = -\frac{1}{3}, \qquad (d = 2)$

Chen, Lee, Toner (2016) Nat Comm

Critical compressible flocks with an easy axis

A.k.a. Active Ising model (ρ: number density; ϕ: coarse-grained spin/polarisation field):

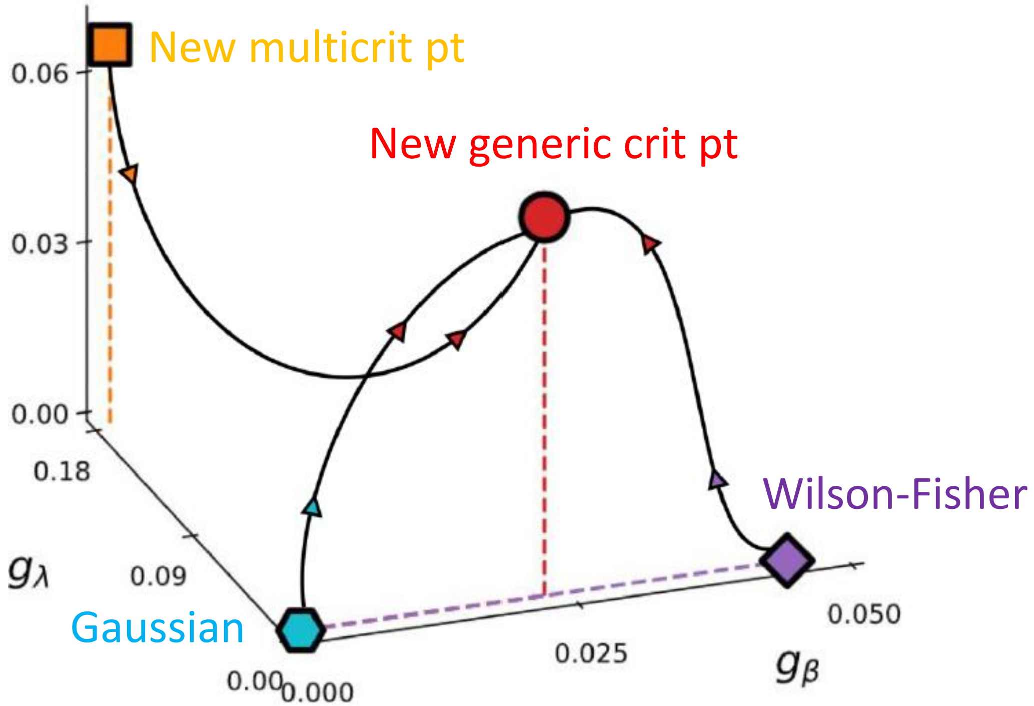
Continuity eq.: $\partial_t \rho = -\gamma\, \partial_x \phi + K_x \partial_x^2 \rho + K_\perp \nabla_\perp^2 \rho,$

$$\partial_t \phi + \frac{\lambda}{2}\partial_x \phi^2 = -\kappa_1 \partial_x \rho - \kappa_2 \partial_x \rho^2 + \mu_x \partial_x^2 \phi + \mu_\perp \nabla_\perp^2 \phi - \alpha_0 \phi - \beta \phi^3 + f_x - \alpha_1 \rho\phi - \alpha_2 \rho^2 \phi.$$

Terms in standard Ising Model A in blue

Critical order-disorder transition occurs when $\alpha_0 = \alpha_1 = 0$
[Nesbitt, Pruessner & Lee (2021) NJP]

Perturbative 1-loop calculation

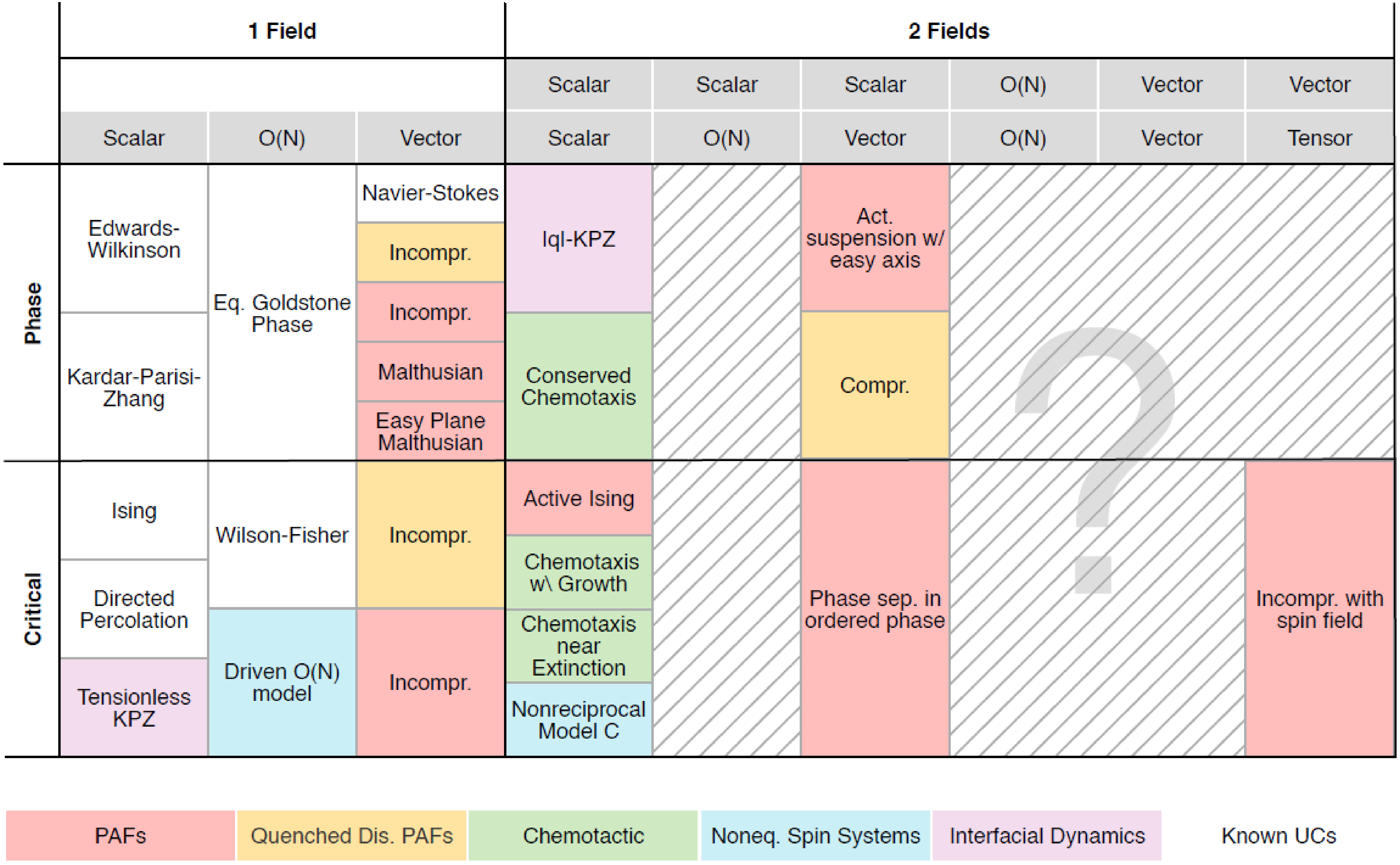


Wong & Lee (2026) Six universality classes govern the critical and multicritical behavior of an active Ising model. Phys. Rev. E **114**, L022103

9 new phases + 10 new critical points uncovered using RG **since 2015** in classical (non-quantum) nonequilibrium systems!

Jentsch & Lee (2026) Hydrodynamics, Renormalization Group, and Universality Classes Far from Equilibrium. E-print: arXiv:2607.02318

Sect.	System	Relevant terms	d_c	Exponents	Method
		Phases			
4.1.2	Interfacial dynamics with quasi-static bulk phases[18]	$\lvert\nabla\rvert(\nabla h)^2$	2	$z=3-\frac{\varepsilon}{3}$, $\chi=\frac{\varepsilon}{3}$	1-loop
4.2.1	PAF with quenched disorder[24,25]	$(\mathbf{v}_T\cdot\nabla_\perp)\mathbf{v}_T$	5	$z=\zeta=\frac{4}{3}$, $\chi=-\frac{1}{3}$	Exact (3D)
4.3.1	Malthusian PAF[32,33]	$(\mathbf{v}_\perp\cdot\nabla_\perp)\mathbf{v}_\perp$	4	$z=2-\frac{6\varepsilon}{11}$, $\zeta=1-\frac{3\varepsilon}{11}$, $\chi=-1+\frac{6\varepsilon}{11}$	1-loop
4.3.2	Malthusian PAF with an easy plane (xy-plane)[34]	$\partial_y v_y^2$	4	$z=\frac{3}{2}$, $\zeta=\frac{3}{4}$, $\chi=-\frac{1}{2}$	Exact (3D)
4.4.1	Incomp. PAF in 3D[35]	$(\mathbf{v}_T\cdot\nabla_\perp)\mathbf{v}_T$	4	$z=\frac{8}{5}$, $\zeta=\frac{4}{5}$, $\chi=-\frac{3}{5}$	Exact (3D)
4.4.2	Incomp. PAF in 2D[36,37]	v_y^3	$\frac{5}{2}$	$z=1+\frac{\varepsilon}{11}$, $\zeta=\frac{2}{3}$, $\chi=-\frac{1}{3}$	1-loop
4.4.3	Incomp. PAF in 2D with quenched disorder[39,40]	v_y^3	$\frac{7}{3}$	$z=\frac{2}{3}-\frac{3\varepsilon}{9}$, $\zeta=\frac{2+\varepsilon}{3}$, $\chi=-\frac{1-\varepsilon}{3}$	1-loop
4.5	Active suspension with an easy axis[45]	$\mathbf{u}\cdot\nabla c$	4	$z=\frac{3}{2}$, $\zeta=1$, $\chi=-\frac{3}{2}$	Exact (3D)
4.6.2	Conserved chemotactic systems with quasi-static chemotactic field[48]	$\nabla\cdot[\delta\rho\nabla c]$, $\nabla^2(\nabla c)^2$	4	$z=\frac{d+2}{3}$, $\chi=-\frac{d+2}{3}$	Exact
		Critical behaviour			
4.1.1	Tensionless KPZ[12,13]	$\lvert\nabla h\rvert^2$	∞	$z=1$, $\chi=\frac{1}{2}$ (for $d=1$)	Exact
4.2.2	PAF in the ordered phase[26]	$\nabla_\perp\delta\rho^3$	5	$z=2$, $\zeta=2$, $\chi=-2+\frac{\varepsilon}{2}$, $\chi_\rho=-1+\frac{\varepsilon}{2}$	1-loop
4.2.3	PAF with an easy axis[30]	g_x^3 , $\delta\rho^2 g_x$	4	$z=2$, $\chi=-1+\frac{\varepsilon}{2}$	1-loop
4.4.4	Incomp. PAF[41]	$(\mathbf{v}\cdot\nabla)\mathbf{v}$, $\lvert\mathbf{v}\rvert^2\mathbf{v}$	4	$z=2-\frac{31\varepsilon}{113}$, $\chi=-1+\frac{41\varepsilon}{113}$	1-loop
4.4.5	Incomp. PAF with quenched disorder[42]	$(\mathbf{v}\cdot\nabla)\mathbf{v}$	6	$z=2-\frac{8\varepsilon}{31}$, $\chi=-1+\frac{15\varepsilon}{62}$	1-loop
4.4.6	PAF with an additional rank-2 spin field s[43]	9 terms, see Eqs. (46, 47)	4	$z=2-0.65(2)\varepsilon$, χ not calculated	1-loop
4.6.1	Chemotactic bacterial growth with quasi-static chemotactic field[48]	$\nabla\cdot[\delta\rho\nabla c]$, $\nabla^2(\nabla c)^2$	6	$z=\frac{d}{3}$, $\chi=-\frac{d}{3}$	Exact
4.6.3	Extinction of bacterial growth with quasi-static chemotactic field[51]	$\nabla\cdot(\rho\nabla c)$, $\rho\nabla^2 c$, ρ^2	4	CA: $z=2+\frac{\varepsilon}{18}$, CR: $z=2-\frac{\varepsilon}{2}$	1-loop
4.7.1	Driven $O(N)$ with inertia[52]	$\lvert\vec\phi\rvert^2\vec\phi$, $\lvert\vec\phi\rvert^2\partial_t\vec\phi$, $\partial_t\lvert\vec\phi\rvert^2\vec\phi$	4	$z=2+0.0072\varepsilon^2$, $\chi=-1+\frac{\varepsilon}{2}+0.1765\varepsilon^2$, (for $N=2$)	2-loop
4.7.2	Nonreciprocal Model C (Brownian Ising model)[53,54]	ψ^3 , $\psi\rho$	4	$z=2+0.0198\varepsilon^2$, $\chi=-1+\frac{9\varepsilon}{16}$	2-loop



Jentsch & Lee (2026) Hydrodynamics, Renormalization Group, and Universality Classes Far from Equilibrium. E-print: arXiv:2607.02318

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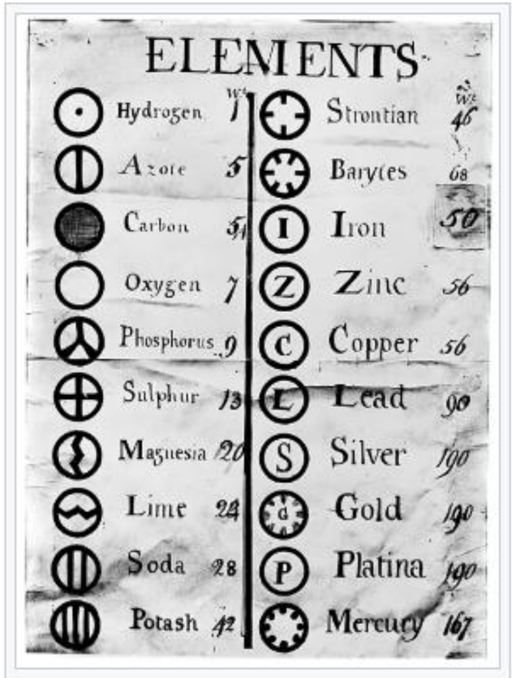
Summary: 2 messages

- Hydrodynamics → simple universal model for complex systems
- RG classifies their universality and determines their scaling behaviour

Outlook

- An embarrassment of richness: many universality classes
- The challenge now is to organise them



Dalton (1808): cataloguing the elements

We need

A periodic table of UCs