Functional Renormalization and relativistic Luttinger fermions

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13th International Conference on the Exact Renormalization Group, ERG 2026, 1-5 September 2026

A method based conference . . . ?

Science across field boundaries!

ERG conference tales

ERG

=

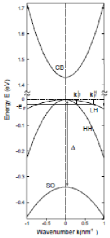
science across

Non-Fermi liquid vs (topological) Mott insulator in electronic systems with quadratic band touching in three dimensions

Igor Herbut

(Simon Fraser University, Vancouver)

Lukas Janssen (SFU)
Balazs Dora (Budapest)
Roderich Moessner (MPI PKS))



IH and Lukas Janssen,
Phys. Rev. Lett. 113, 106401 (2014)

ERG2014, Lefkada Island

An inspiration

from cond-mat

(Luttinger'56)

(Abrikosov'74)

(Moon, Xu, Kim, Balents, PRL 2014)

(Herbut, Janssen, Boettcher, Ray, Moser

...2014)

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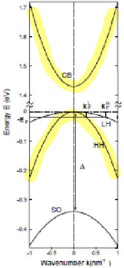

ERG2014, Lefkada Island

Luttinger fermions in nonrelativistic field theory

- Effective field theory of low energy degrees of freedom near quadratic band touching/crossing points (QBT/QBC): (Abrikosov,Beneslavskii'71) (Moon et al.'13; Janssen,Herbut'15)

$$\mathcal{L} = \psi^\dagger \left(\partial_\tau - G_{ij}\,\partial_i\partial_j\right)\psi + \mathcal{L}_{\text{int}}$$

- For $(-G_{ij}\,\partial_i\partial_j)^2 = (-\partial^2)^2$, the G_{ij} satisfy a Clifford-type algebra (Abrikosov'74)

$$\{G_{ij}, G_{kl}\} = -\frac{2}{d-1}\delta_{ij}\delta_{kl} + \frac{d}{d-1}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)$$

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- Can be spanned by Euclidean Dirac algebra, (Janssen,Herbut'15)

$$G_{ij} = a_{ij}^A\gamma_A, \quad \text{where } \{\gamma_A, \gamma_B\} = 2\delta_{AB}\mathbb{1}, \quad A, B = 1, \ldots,$$

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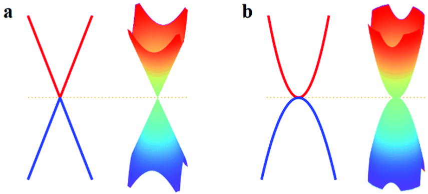

[figure: doi.org/10.1039/C8TC04813D]

cond-mat applications:

– $j = \frac{3}{2}$ Luttinger semi-metals (Luttinger'56)

– Pyrochlor-iridates, HgTe (Moon,Xu,Kim,Balents'13)

– nematic quantum criticality (Janssen,Herbut'15)

– Bernal-stacked honeycomb bilayers (Ray,Vojta,Janssen'18)

– gapless spin liquids (Dey,Maciejko'22)

Dirac vs. Luttinger fermions

Dirac algebra:

$$\{\gamma_i, \gamma_j\} = 2\delta_{ij}\mathbb{1}$$

Abrikosov algebra:

$$\{G_{ij}, G_{kl}\} = \left[-\frac{2}{d-1}\delta_{ij}\delta_{kl} + \frac{d}{d-1}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)\right]\mathbb{1}$$

▷ A simple example in (Euclidean) 2d:

Dirac operator:

$$\not{\partial} = \gamma_i \partial_i$$

representation $\gamma_1 = \sigma_1, \gamma_2 = \sigma_3$:

$$\not{\partial} = \begin{pmatrix} \partial_2 & \partial_1 \\ \partial_1 & -\partial_2 \end{pmatrix}$$

Luttinger operator:

$$G_{ij}\partial_i\partial_j$$

representation $G_{12} = G_{21} = \sigma_1$, $G_{11} = -G_{22} = \sigma_3$:

$$G_{ij}\partial_i\partial_j = \begin{pmatrix} \partial_1^2 - \partial_2^2 & \partial_1\partial_2 \\ \partial_1\partial_2 & -\partial_1^2 + \partial_2^2 \end{pmatrix}$$

Relativistic Luttinger fermions?

- ▷ Abrikosov algebra

$$\{G_{\mu\nu}, G_{\kappa\lambda}\} = -\frac{2}{d-1} g_{\mu\nu} g_{\kappa\lambda} + \frac{d}{d-1}\left(g_{\mu\kappa} g_{\nu\lambda} + g_{\mu\lambda} g_{\nu\kappa}\right), \quad g = \mathrm{diag}(+,-,-,-,\dots)$$

- ▷ kinetic action:

$$S = -\int d^d x\, \bar{\psi}\, G_{\mu\nu} \partial^\mu \partial^\nu\, \psi$$

Relativistic Luttinger fermions?

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▷ kinetic action:

$$S = -\int d^d x\, \bar{\psi}\, G_{\mu\nu} \partial^\mu \partial^\nu\, \psi$$

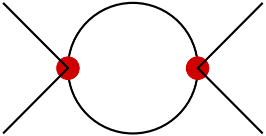
▷ A simple motivation:

canonical dimension in $d = 4$: $[\phi] = [\psi] = 1$

scalar ϕ^4

$$\beta = +\frac{3}{16\pi^2}\lambda^2$$

triviality



fermionic $(\bar{\psi}\psi)^2$

$$\beta = -\frac{\#}{16\pi^2}\lambda^2 \quad ?$$

asymptotic freedom ?

Relativistic Luttinger fermions?

▷ Abrikosov algebra

$$\{G_{\mu\nu}, G_{\kappa\lambda}\} = -\frac{2}{d-1} g_{\mu\nu} g_{\kappa\lambda} + \frac{d}{d-1}\left(g_{\mu\kappa} g_{\nu\lambda} + g_{\mu\lambda} g_{\nu\kappa}\right), \quad g = \mathrm{diag}(+,-,-,-,\dots)$$

▷ kinetic action:

$$S = -\int d^d x \, \bar{\psi} \, G_{\mu\nu} \partial^\mu \partial^\nu \, \psi$$

▷ BUT what is $\bar{\psi}$?

$$\left(\text{Dirac:} \quad \bar{\psi} = \psi^\dagger \gamma_0\right)$$

Construction of the conjugate spinor

$\Bigg($ ERG tale: fermions in gravity: type I vs. type II puzzle $\Longrightarrow$ spin base invariance $\Bigg)$

(Doná, Percacci'12) (Lippoldt, HG'13)

▷ Unitary time evolution requires a real action, suggesting:

$$\bar{\psi} = \psi^\dagger\, h, \qquad \text{spin metric: } h \qquad (\text{Dirac: } h = \gamma_0)$$

▷ action $\in \mathbb{R}$ (unitarity) requires $d_\gamma = 32$ dimensional representation, e.g.

(HG,Heinzel,Laufkötter,Picciau'23)

$$h = \gamma_1\gamma_2\gamma_3\gamma_{10}$$

$\Longrightarrow$ action is Lorentz invariant, $\mathcal{S}_{\text{Lor}} \in \text{SO}(1,3) \subset \text{SL}(32,\mathbb{C})$

$$\mathcal{S}_{\text{Lor}}^{-1}\, G_{\mu\nu}\, \mathcal{S}_{\text{Lor}} \quad = \quad G_{\kappa\lambda}\Lambda^{\kappa}{}_{\mu}\Lambda^{\lambda}{}_{\nu}, \qquad \psi \to \mathcal{S}_{\text{Lor}}\psi$$

Free theory of relativistic Luttinger fermions

▷ action

$$S = \int d^4x \left(\bar{\psi}\, G_{\mu\nu}\, (i\partial^\mu)(i\partial^\nu)\, \psi \right)$$

$\Longrightarrow$ Lorentz invariant and spin-base $\mathrm{SL}(32, \mathbb{C})$ invariant

▷ canonical mass dimension

$$[\psi] = 1$$

$\sim$ like a scalar field in $d = 3 + 1$

$\Longrightarrow$ $\sim (\bar{\psi}\psi)^2$-type self-interactions can be marginal

Free theory of relativistic Luttinger fermions

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▷ NOTE: there are $\binom{10}{0} + \binom{10}{1} + \binom{10}{2} + \binom{10}{3} + \cdots = 1024$ independent bilinears

only 16 for Dirac fermions

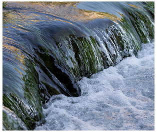
Examples: 4d self-interacting Luttinger fermionic models

▷ e.g., scalar channels *(Eucl.)*: (HG,Heinzel,Laufkötter,Picciau'24;HG,Picciau'24)

$$\mathcal{L}_{\text{int}} = \left\{ \frac{\bar\lambda}{2}(\bar\psi\psi)^2,\ -\frac{\bar\lambda}{2}(\bar\psi\gamma_{10}\psi)^2,\ -\frac{\bar\lambda}{2}(\bar\psi\gamma_{11}\psi)^2,\ \frac{\bar\lambda}{2}(\bar\psi\gamma_{01}\psi)^2 \right\}$$

▷ renormalization flow: (Wetterich'93)

$$\partial_t\Gamma_k = \frac{1}{2}\,\mathrm{Tr}\,\frac{1}{\Gamma_k^{(2)}+R_k}\,\partial_t R_k$$



▷ renormalized coupling flow (at large-N_f)

$$\partial_t\lambda = -\frac{4N_\mathrm{f}}{\pi^2}\lambda^2,$$

expansion parameter: $\dfrac{1}{d_\gamma N_\mathrm{f}}$

$\Longrightarrow$ asymptotically free purely fermionic matter models

scalars: (Berges,Gurau,Keppler,Preis'24)

Strongly coupled low-energy regime

▷ e.g., "γ_{11} model" *(Mink.)*: (HG,Picciau'24)

$$S = \int d^4x \left[-\bar{\psi} G_{\mu\nu} \partial^\mu \partial^\nu \psi + \frac{\bar{\lambda}}{2} (\bar{\psi}\gamma_{11}\psi)^2 \right].$$

▷ Hubbard-Stratonovich transformation, $\phi \sim \bar{\psi}\gamma_{11}\psi$

$$S = \int d^4x \left[-\bar{\psi} G_{\mu\nu} \partial^\mu \partial^\nu \psi + \bar{h}\phi\bar{\psi}\gamma_{11}\psi - \frac{1}{2}\bar{m}^2\phi^2 \right],$$

▷ matching condition

$$\bar{\lambda} = \frac{\bar{h}^2}{\bar{m}^2}$$

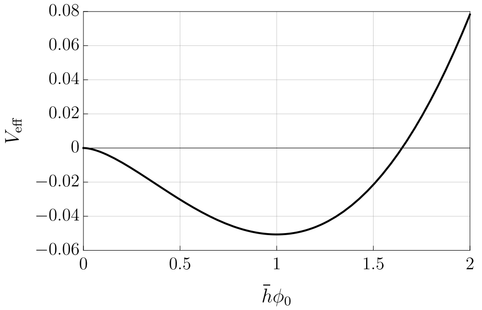
Strongly coupled low-energy regime

Mean-field theory:

▷ effective potential (large-N_f)

(HG,Picciau'24)

$$V_\mathrm{eff} = \frac{N_\mathrm{f} d_\gamma}{2^6\pi^2} (\bar{h}\phi)^2 \ln \frac{(\bar{h}\phi)^2}{e(\bar{h}v)^2},$$



- symmetry breaking with vev v, stable minimum
- "dimensional transmutation"
- low-energy spectrum: light scalar sigma-type excitation
- high-energy complete, asymptotic freedom
- no tachyons, no "ghostly" instabilities

similar for γ_{01} model

Relativistic Luttinger Yukawa model

▷ Functional RG: flowing action (Eucl.): (HG,Picciau'25)

$$\Gamma_k = \int \left[-Z_\psi \bar{\psi} G_{\mu\nu} \partial^\mu \partial^\nu \psi - \bar{h} \phi \bar{\psi} \gamma_{11} \psi + \frac{Z_\phi}{2} (\partial_\mu \phi)^2 + U(\phi) \right], \quad U(\phi) = \frac{1}{2} \bar{m}^2 \phi^2 + \frac{1}{2} \bar{\lambda}_\phi \phi^4 + \dots$$

Naive expectactions:

- 3 physical parameters
- $\bar{m}^2$, $\bar{h}$: RG relevant

  may require fine tuning

- $\bar{\lambda}_\phi$: RG marginal

  log running expected

If so:

$\Longrightarrow$ large hierarchy

phys. masses $\ll \Lambda$

expected to require fine tuning

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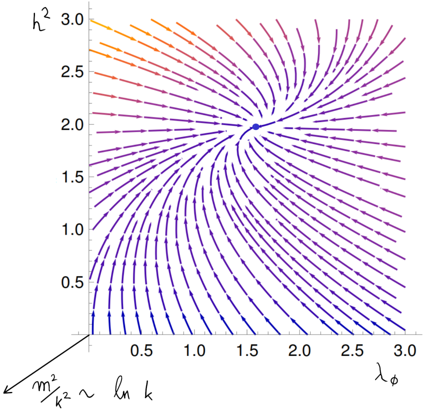
Instead we observe:

- dim'less renormalized couplings h, λ_ϕ, …approach (partial) fixed point
- scalar mass is marginal at fixed point

$$\partial_t \frac{m^2}{k^2} \simeq \text{const.}, \qquad \eta_\phi \simeq 2$$

$\Longrightarrow$ log-running towards broken regime



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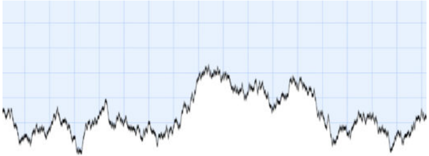
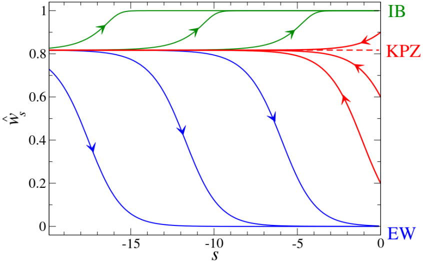
An inspiration from stat-phys:



stochastic growth of a surface

KPZ equation (Kardar,Parisi,Zhang'86)



Self-organized criticality

(Canet,Delamotte,Chaté,Wschebor'10'11)

(Canet'25) [ERG: L. Canet, L. Gosteva]

Relativistic Luttinger Yukawa model

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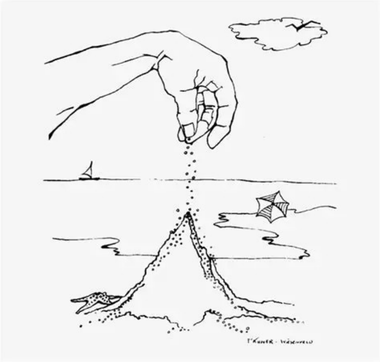
$$\Gamma_k = \int \left[-Z_\psi \bar{\psi} G_{\mu\nu} \partial^\mu \partial^\nu \psi - \bar{h} \phi \bar{\psi} \gamma_{11} \psi + \frac{Z_\phi}{2} (\partial_\mu \phi)^2 + U(\phi) \right], \qquad U(\phi) = \frac{1}{2} \bar{m}^2 \phi^2 + \frac{1}{2} \bar{\lambda}_\phi \phi^4 + \dots$$

Similar to self-organized criticality:

- approach to criticality governed by (partial) fixed point
- dynamical time $\hat{=}$ RG time
- slow driving force $\hat{=}$ scalar mass parameter

$$\partial_t \frac{m^2}{k^2} \simeq \text{const.},$$

$\Longrightarrow$ large hierarchies are natural (Wetterich,Bornholdt'92)



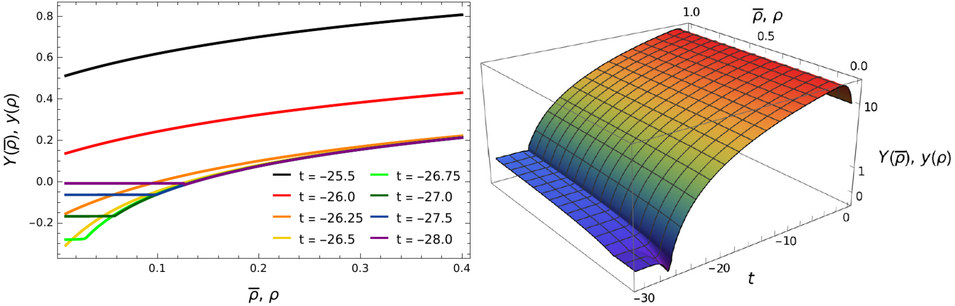
(Bak,Tang,Wiesenfeld'87)

Relativistic Luttinger Yukawa model

▷ Functional Flow (LPA'):

(HG,Eckert'wip) [ERG: Eckert]

$$\Gamma_k = \int \left[-Z_\psi \bar{\psi} G_{\mu\nu} \partial^\mu \partial^\nu \psi - \bar{h}\phi\bar{\psi}\gamma_{11}\psi + \frac{Z_\phi}{2}(\partial_\mu \phi)^2 + U(\phi) \right], \quad U(\phi) = \frac{1}{2}\bar{m}^2\phi^2 + \frac{1}{2}\bar{\lambda}_\phi \phi^4 + \ldots$$



$\Longrightarrow$ scale dependent potential: global partial fixed function $u(\rho)$

numerical methods, a la (Ihssen,Sattler,Wink'23)

Dirac vs. Luttinger fermions

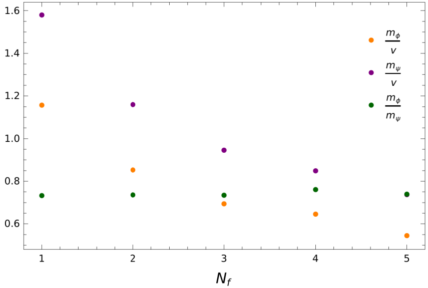
▷ generic Yukawa interaction with "chiral" symmetry (HG,Picciau'25; HG,Eckert'wip) [ERG: Eckert]

$$\mathcal{L}_{\text{int}} = \bar{h}\phi\bar{\psi}\psi + \frac{1}{2}\bar{m}^2\phi^2 + \frac{1}{8}\lambda_\phi\phi^4$$

Dirac-Yukawa:

3 parameters $\rightarrow$ 3 observables

e.g. v, m_{Higgs}, m_{top}



Luttinger-Yukawa:

3 parameters $-$ 2 pFP $\rightarrow$ 1 observable (scale)

$$\frac{\text{scalar mass}}{\text{fermion gap}} \simeq 0.732 \quad \text{(deeply bound state)}$$

Luttinger-Dirac-Yukawa model

▷ How robust is the self-organized criticality mechanism? (HG,Ye'wip) [ERG: Y. Ye]

$$\Gamma_k = \int \left[-Z_\chi \bar{\chi} G_{\mu\nu} \partial^\mu \partial^\nu \chi - \bar{h}_\chi \phi \bar{\chi} \gamma_{11} \chi + Z_\psi \bar{\psi} \gamma_\mu \partial^\mu \psi + \bar{h}_\psi \phi \bar{\psi} \psi + \frac{Z_\phi}{2} (\partial_\mu \phi)^2 + U(\phi) \right]$$

$\Longrightarrow$ partial fixed point & self-organized criticality persist

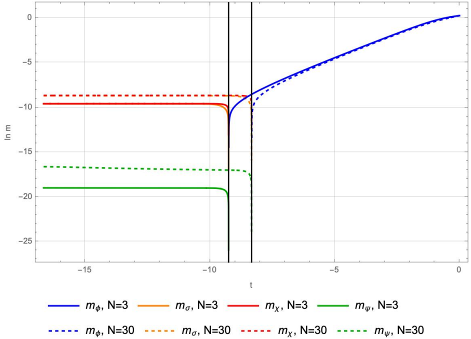
▷ anomalous dimension $\eta_\phi \simeq 2$

$\Longrightarrow$ Dirac Yukawa h_ψ RG irrelevant

$\Longrightarrow$ Strong mass hierarchy

$$m_\psi \ll m_\chi, m_\sigma, v \ll \Lambda$$

$\Longrightarrow$ novel seesaw mechanism for mass hierarchies

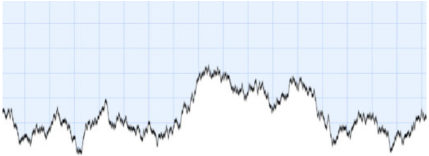


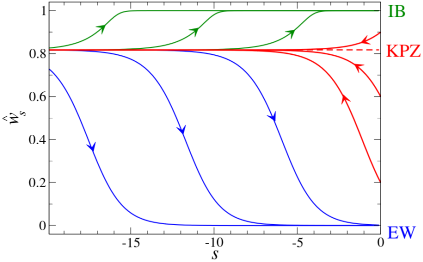
Robust fixed points / universality classes

Example: KPZ universality class

- ▷ stochastic noise + nonlinearity

(Kardar,Parisi,Zhang'86, Canet'25)

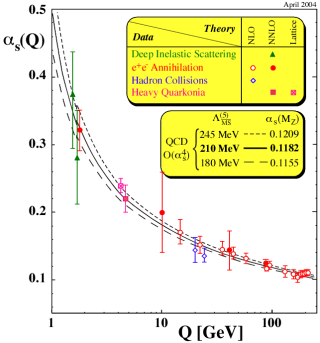




- ▷ destabilized by: long-range interactions, correlated noise, disorder, ...

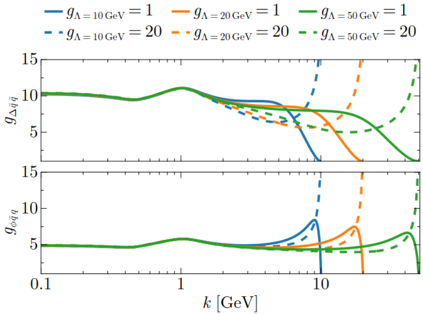
Example: QCD universality class

▷ nonabelian gauge sector + fermions + …



(Bethke'04)

(Gross,Wilczek'73;Politzer'73)



(HG,Wetterich'02'04,Pawlowski'05; Flörchinger,Wetterich'09)

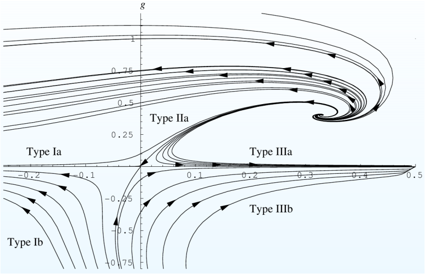
(Gholami,Mire,Rennecke,Schaefer,Yin'26)

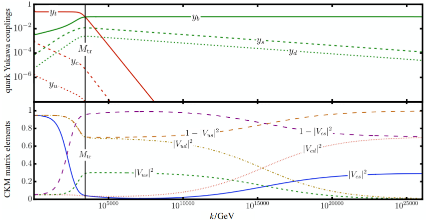
▷ destabilized by: large N_f, …

[ERG: A. Pastor Gutierrez, R. Zwicky]

Example: Quantum gravity @ Reuter fixed point

▷ Reuter fixed point in QG asymptotic safety (Reuter'96)





(Eichhorn,Gryftopoulos,Held'25)

(Reuter,Saueressig'01)

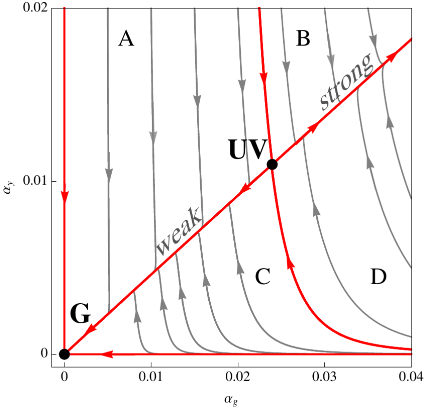
▷ destabilized by: large N_f ? ("matter matters / gravity rules")

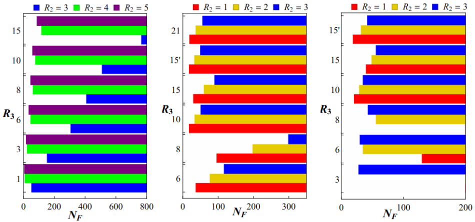
(Dona,Eichhorn,Percacci'13, Christiansen,Litim,Pawlowski,Reichert'18)

Example: Asymptotical safety guaranteed

▷ Gauge Yukawa theories, Litim-Sannino model

(Litim,Sannino'14)





(Bond,Hiller,Kowalska,Litim'17)

▷ destabilized by: small N_f, no scalar, …

Robust fixed points / universality classes

▷ many qualitative examples

▷ quantitative statements require suitable metrics in theory space

$\Longrightarrow$ theory space geometry?

(Wallace,Zia'74, Jack,Osborn'90)

(Ellwanger'18)

(Flörchinger'23)

(Gattus,Millington'26)

Conclusion

- relativistic QFTs in 3+1d spacetime with Luttinger fermions

  Lorentz + spin-base symmetry

- new building blocks for power-counting renormalizable QFTs

  new routes to UV completeness

  natural large scale / mass hierarchies from self-organized criticality

- plethora of interactions, strongly coupled long-range physics

  some models exhibit symmetry-broken massive phases with light scalar modes

- Questions: robust (partial) fixed points?

  ...Luttinger fermions as asymptotic states?

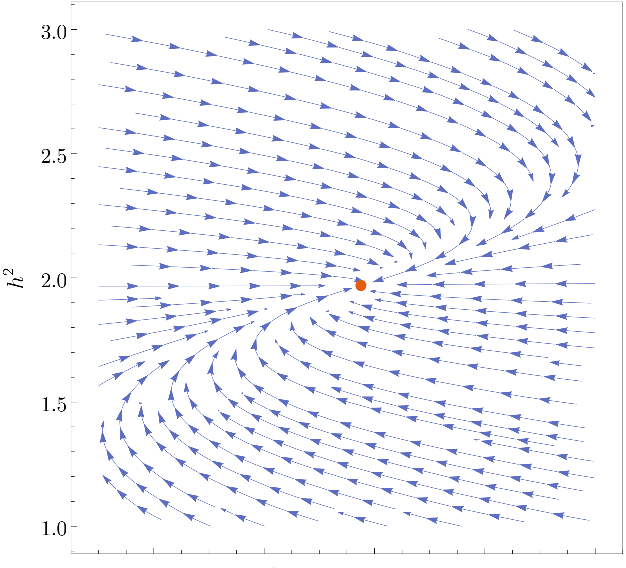
  ...explicit breaking to Dirac fermions?

  ...new ingredient for useful models?

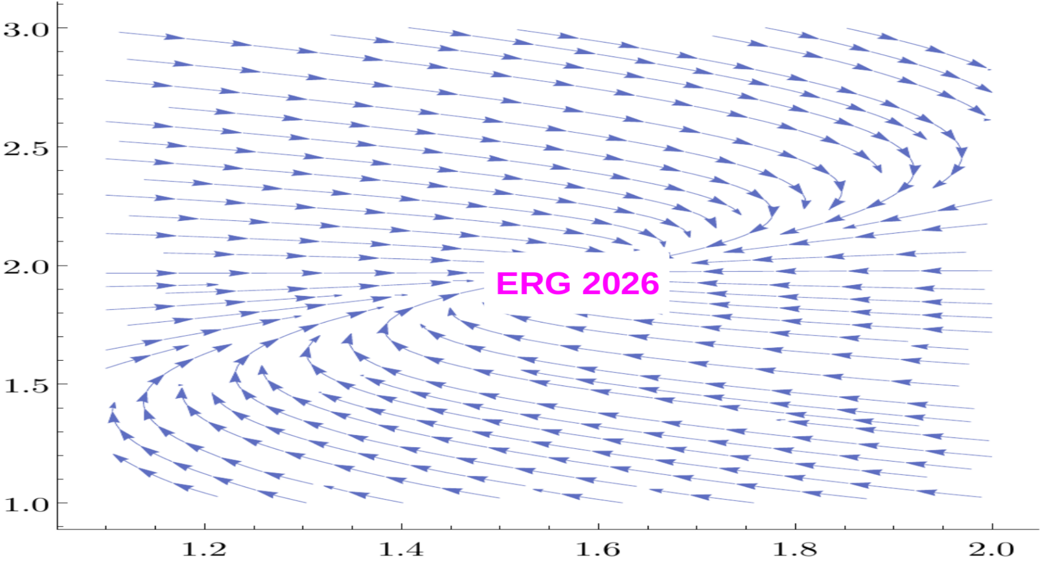
Epilogue

▷ robust fixed point

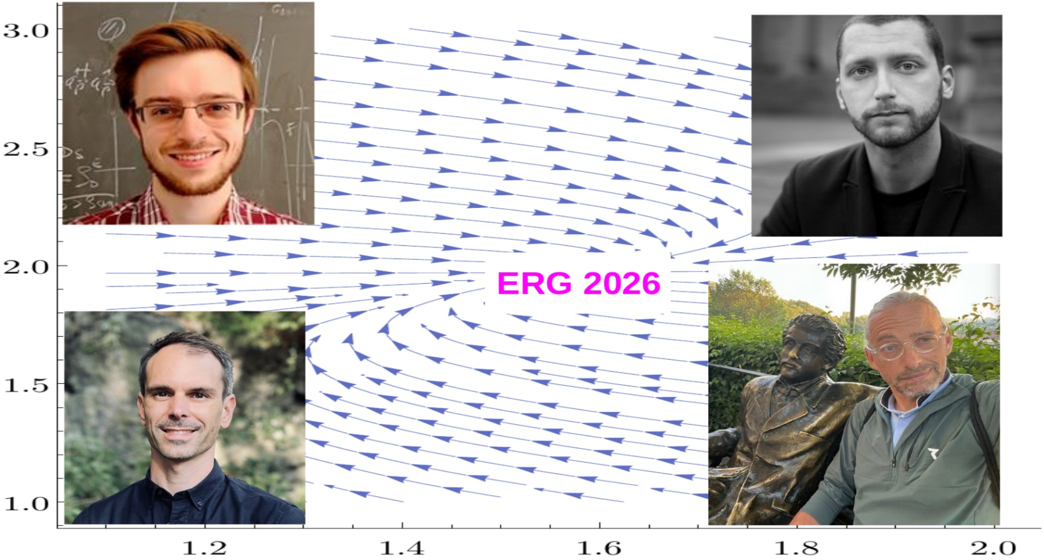
self-organized criticality



Self-organized criticality ?



Well-organized criticality !



Appendix

Abrikosov algebra

Properties of the Abrikosov algebra

- $G_{\mu\nu}$: representation in terms of $d_\gamma \times d_\gamma$ matrices for each μ, ν
- $G_{\mu\nu} = G_{\nu\mu}$ Lorentz symmetric and traceless: $\mathrm{tr_L}\, G_{\mu\nu} \equiv G^{\mu}{}_{\mu} = 0$

  …excludes the trivial Klein-Gordon part $\sim \partial^2$
- e.g., in $d = 4$

  Abrikosov algebra can be spanned by (Euclidean) Dirac algebra:

$$G_{\mu\nu} = a^{A}_{\mu\nu}\,\gamma_A, \quad \text{where } \{\gamma_A, \gamma_B\} = 2\delta_{AB}\mathbb{1}, \quad A, B = 1, \dots, 9$$

cf. (Janssen,Herbut'15)

Spinbase invariant Dirac fermions

Canonical scaling: e.g., fermions a la Dirac

▷ free Dirac action

$$S = \int d^4x \, \bar{\psi} i \gamma_\mu \partial^\mu \psi$$

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$$S = \int d^4x \; \psi^\dagger h \, i\gamma_\mu \partial^\mu \, \psi$$

▷ Dirac-Clifford algebra

$$\{\gamma_\mu, \gamma_\nu\} = 2 g_{\mu\nu} \mathbb{1}_{d_\gamma}$$

▷ e.g., $g = \mathrm{diag}(+, -, -, -)$

$$\gamma_0^2 = 1, \quad \gamma_{1,2,3}^2 = -1$$

$\Longrightarrow$ for $S \in \mathbb{R}$, the spin metric h has to satisfy

$$[h, \gamma_0] = 0, \quad \{h, \gamma_i\} = 0 \quad \Longleftrightarrow \quad h = \gamma_0$$

Relativistic and spin-base invariance

▷ Dirac-Clifford algebra

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}\mathbb{1}_{d_\gamma}$$

invariant under Lorentz transformations $\Lambda \in \mathrm{SO}(1,3)$ $\quad$ $(g_{\mu\nu} = g_{\kappa\lambda}\Lambda^\kappa{}_\mu\Lambda^\lambda{}_\nu)$

$$\gamma_\mu \to \gamma'_\mu = \gamma_\nu \Lambda^\nu{}_\mu$$

$\implies$ Dirac operator $i\gamma_\mu\partial^\mu$ Lorentz invariant:

$$i\gamma_\mu\partial^\mu = i\gamma'_\mu\partial'^\mu$$

BUT: Dirac matrices change from Lorentz frame to Lorentz frame

Relativistic and spin-base invariance

▷ Dirac-Clifford algebra

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}\mathbb{1}_{d_\gamma}$$

also invariant under spin-base transformations (Schrödinger'32;Bargman'32)

$$\gamma_\mu \to \mathcal{S}\gamma_\mu\mathcal{S}^{-1}, \quad \mathcal{S} \in \mathrm{SL}(d_\gamma, \mathbb{C}),$$

$\Longrightarrow$ can be used to transform γ'_μ back to original form:

$$\gamma_\mu \quad \xrightarrow{Lor.} \quad \gamma'_\mu = \gamma_\nu \Lambda^\nu{}_\mu \quad \xrightarrow{SB} \quad \mathcal{S}\gamma_\nu\Lambda^\nu{}_\mu\mathcal{S}^{-1} \overset{!}{=} \gamma_\mu$$

$\Longrightarrow$ induces a mapping of SO(1,3) $\to$ SL($d_\gamma, \mathbb{C}$)

$$\mathcal{S}_{\mathrm{Lor}}^{-1}\gamma_\mu\mathcal{S}_{\mathrm{Lor}} \quad = \quad \gamma_\nu\Lambda^\nu{}_\mu$$

Relativistic and spin-base invariance

& "Lorentz transformation of spinors"

$$\psi \to \mathcal{S}_{\mathsf{Lor}} \psi$$

▷ The Dirac action then is Lorentz invariant

$$S = \int d^4x \; \psi^\dagger h \, i\gamma_\mu \partial^\mu \, \psi$$

provided that spin metric transforms as (Weldon'00; HG,Lippoldt'13)

$$h \;\to\; (\mathcal{S}_{\mathsf{Lor}}^\dagger)^{-1} \, h \, \mathcal{S}_{\mathsf{Lor}}^{-1} \qquad \text{solution} : h = \gamma_0$$

▷ spin metric h acts as a "hermitizer" (Bargman'32)

Luttinger-Gross-Neveu model beyond large N_f

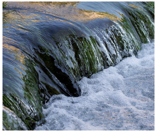
Example: 4d self-interacting Luttinger fermionic model

▷ $\sim$ “Gross-Neveu-Thirring”-type model:

$$S[\psi,\bar\psi] = \int d^4x \left[\bar\psi G_{\mu\nu}(i\partial^\mu)(i\partial^\nu)\psi + \frac{\bar\lambda_0}{2}(\bar\psi\psi)^2 + \frac{\bar\lambda_t}{2}(\bar\psi G_{\mu\nu}\psi)^2\right]$$

▷ renormalization flow: (Wetterich'93)

$$\partial_t \Gamma_k = \frac{1}{2}\,\mathrm{Tr}\,\frac{1}{\Gamma_k^{(2)} + R_k}\,\partial_t R_k$$



▷ renormalized coupling flow (point-like truncation), e.g., for $N_\mathrm{f} = 1$

$$\partial_t\lambda_0 = 2\eta_\psi\lambda_0 - \frac{1}{16\pi^2} l_2^{4,\mathrm{B}}(0;\eta_\psi)\left[30\,\lambda_0^2 - \frac{88}{3}\,\lambda_0\,\lambda_t\right]$$

$$\partial_t\lambda_t = 2\eta_\psi\lambda_t - \frac{1}{16\pi^2} l_2^{4,\mathrm{B}}(0;\eta_\psi)\left[-\frac{486}{9}\lambda_t^2 + \frac{16}{9}\,\lambda_0\,\lambda_t - \frac{1}{6}\lambda_0^2\right]$$

Example: 4d self-interacting Luttinger fermionic model

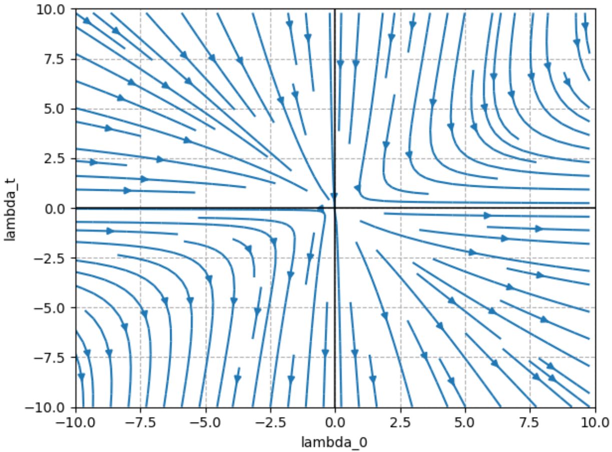
▷ $N_\mathrm{f} = 1$ phase diagram of the model:
$(\eta_\psi \sim \lambda_i^2$, LPA: $\to 0)$

$\Longrightarrow$ Gaußian fixed point:
attractive & repulsive

$\Longrightarrow$ asymptotically free for

$$\lambda_0 > 0, \quad \lambda_t < 0$$

$\Longrightarrow$ UV complete
& long-range interacting



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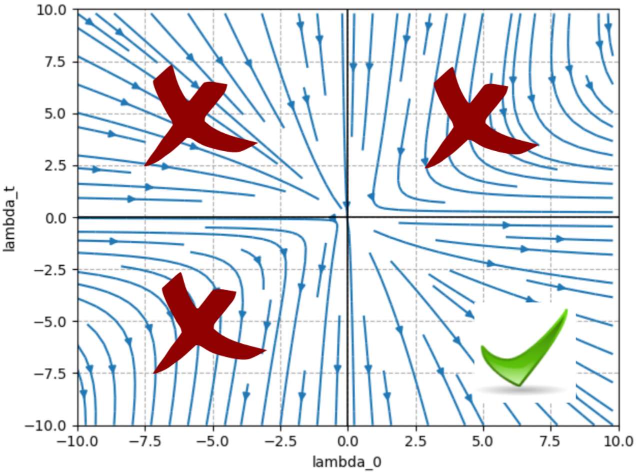


Example: 4d self-interacting Luttinger fermionic model

- ▷ initial condition

$$\tan\alpha = \frac{\lambda_0}{\lambda_t}$$

- ▷ channel competition

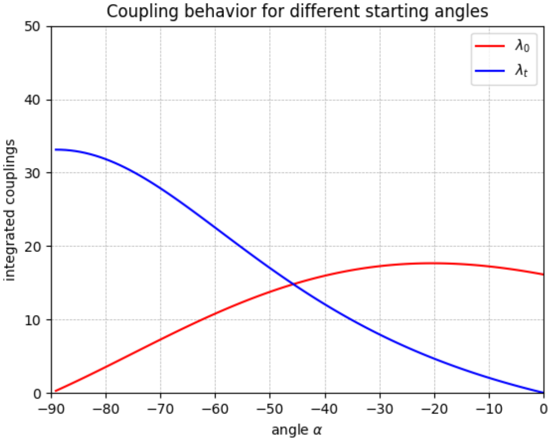
scalar vs. tensor channel

- ▷ quantum phase transition ?

$$\alpha_{\rm cr} \simeq -46^\circ$$

- ▷ long-range order?

$\Longrightarrow$ FRG



Luttinger QED and QCD

Example: gauge theories with relativistic Luttinger fermions

▷ Luttinger quantum electrodynamics (LQED) (HG,Heinzel,Laufkötter,Picciau'23)

$$S = \int d^4x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} G_{\mu\nu} (iD^\mu[A])(iD^\nu[A])\psi - m^2 \bar{\psi}\psi \right]$$

▷ First glance at Heisenberg-Euler-type effective action:

$$\Gamma^{1\ell}[A] = -i \ln\det(G_{\mu\nu} iD^\mu iD^\nu - m^2) = -\frac{i}{2} \ln\det[-(G_{\mu\nu} D^\mu D^\nu)^2 + m^4)]$$

▷ spin-field coupling

$$\sim \frac{ie}{2} [G_{\mu\nu}, G_{\kappa\lambda}] F^{\nu\lambda} \{D^\mu, D^\kappa\} + \mathcal{O}(\partial F)$$

$\left(\text{Dirac: } \sim -\frac{e}{2} \sigma_{\mu\nu} F^{\mu\nu} \right)$

$\Longrightarrow$ enhancement of "paramagnetic" effects

RG flow of LQED

▷ One-loop β function

$$\beta_{e^2} = +\frac{4 \cdot 19}{9\pi^2} e^4 = \frac{4}{9\pi^2} \left(22\Big|_{\text{para}} - 3\Big|_{\text{dia}} \right) e^4$$

$\Longrightarrow$ RG flow near Gaußian fixed point as in ordinary QED

(UV-incomplete) ✗

$\Longrightarrow$ strong para-magnetic dominance

(Nink,Reuter'15)

RG flow of LQCD

▷ One-loop β function for the nonabelian Luttinger QCD

$$\beta_{g^2} = -\frac{1}{3\pi^2}\left(\frac{11}{8}N_\mathrm{c} - \frac{2\cdot 19}{3}N_\mathrm{f}\right)g^4$$

$\Longrightarrow$ asymptotic freedom and UV completeness depends on N_c vs. N_f:

$$N_{\mathrm{c},cr} = \frac{304}{33}N_\mathrm{f} \simeq 9.21\,N_\mathrm{f}$$

$\Longrightarrow$ UV completeness for $\mathrm{SU}(N_\mathrm{c} > 10)$ for $N_\mathrm{f} = 1$.

MFT for the Luttinger-Gross-Neveu model

Low-energy regime

▷ e.g., Luttinger-Gross-Neveu model *(Mink.)*: (HG,Picciau'24)

$$S = \int d^4x \left[-\bar{\psi} G_{\mu\nu} \partial^\mu \partial^\nu \psi - \frac{\bar{\lambda}}{2} (\bar{\psi}\psi)^2 \right].$$

▷ Hubbard-Stratonovich transformation, $\phi \sim \bar{\psi}\psi$

$$S = \int d^4x \left[-\bar{\psi} G_{\mu\nu} \partial^\mu \partial^\nu \psi + \bar{h}\phi\bar{\psi}\psi - \frac{1}{2}\bar{m}^2\phi^2 \right],$$

▷ matching condition

$$\bar{\lambda} = -\frac{\bar{h}^2}{\bar{m}^2}$$

$\Longrightarrow$ only IR-free coupling branch accessible.

Low-energy regime

▷ fluctuation contribution to mean field action (exact at large N_f)

$$\Gamma_{MF} = -\frac{i}{2} \ln \det[(-\partial^2)^2 - (\bar{h}\phi_0)^2]$$

$\Longrightarrow$ tachyonic modes for $p^2 < \bar{h}\phi_0$

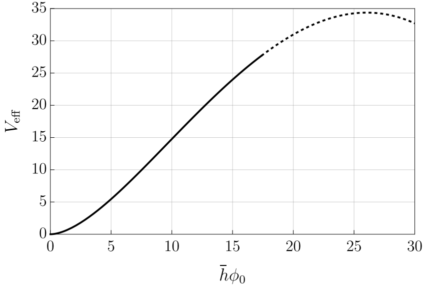
similar for γ_{10} model and Luttinger-NJL model

Low-energy regime

$\Longrightarrow$ renormalized effective potential:

$$V_{\text{eff}}(\phi_0) = \frac{\phi_0^2}{2}\left\{m^2(\mu) + \frac{N_f d_\gamma}{2^5\pi^2}\bar{h}^2\left[1 - \gamma - \ln\frac{(\bar{h}\phi_0)^2}{\mu^4}\right]\right\} + i\,\frac{N_f d_\gamma}{2^6\pi}\bar{h}^2\phi_0^2$$

- non asymptotically free branch
- symmetric ground state
- low-energy spectrum: massive scalar excitation
- tachyonic instability for finite ϕ_0



similar for γ_{10} model and Luttinger-NJL model

MFT for the Relativistic mass terms

Relativistic mass terms

▷ Naively: free massive action: (HG,Picciau'24)

$$S = \int d^4x \left(\bar{\psi}\, G_{\mu\nu}\, (i\partial^\mu)(i\partial^\nu)\, \psi - m^2\, \bar{\psi}\psi \right)$$

$\Longrightarrow$ Equation of motion in momentum space:

$$(G_{\mu\nu}p^\mu p^\nu - m^2)\psi = 0$$

Multiplying by $(G_{\kappa\lambda}p^\kappa p^\lambda + m^2)$ yields

$$(p^4 - m^4)\psi = (p^2 - m^2)(p^2 + m^2)\psi = 0,$$

$\Longrightarrow$ conventional massive mode, $p^2 = m^2$, and tachyonic mode $p^2 = -m^2$.

(

An immediate problem of higher derivative theories

▷ e.g. a massive scalar:

$$S = \int d^4x \left(\frac{1}{2}\phi(\partial^2)^2\phi - \frac{1}{2}m^4\phi^2 \right)$$

$\Longrightarrow$ propagator:

$$\frac{1}{p^4 - m^4} = \frac{1}{2}\left(\frac{1}{p^2 - m^2} + \frac{-1}{p^2 + m^2} \right)$$

- A negative norm state (ghost)
- A tachyonic mode with dispersion relation $p^2 = -m^2$

$\rightarrow$ instability

BUT:

Higher derivative theories need not have tachyonic instabilities

e.g., higher derivative gravity in certain coupling regions (Stelle'77)

)

Variants of mass

▷ Further possible mass terms (HG,Picciau'24)

$$-\left(m^2\,\bar\psi\psi\right),\quad -\left(im_{10}^2\,\bar\psi\gamma_{10}\psi\right),\quad -\left(m_{11}^2\,\bar\psi\gamma_{11}\psi\right),\quad -\left(im_{01}^2\,\bar\psi\gamma_{01}\psi\right),\gamma_{01}=-i\gamma_{10}\gamma_{11}$$

$\Longrightarrow$ Solution to equation of motion with γ_{11} mass terms satisfies

$$(p^4+m_{11}^4)\psi=0$$

$\Longrightarrow$ gapped spectrum, no tachyonic mode.

similar for γ_{01} mass

Analytic structure of the propagator

▷ Minkowski space propagator, e.g., with naive mass term: (HG,Picciau'24)

$$-iS(p) = \frac{1}{G_{\mu\nu}p^\mu p^\nu - m^2} = \frac{1}{p^4 - m^4}\left(G_{\mu\nu}p^\mu p^\nu + m^2\right),$$

▷ Eigenvalues:

$$\mathrm{eig}\left(-iS(p)\right) = \left\{\frac{1}{p^2 - m^2}\,[\mathrm{deg.}16], \frac{-1}{p^2+m^2}\,[\mathrm{deg.}16]\right\},$$

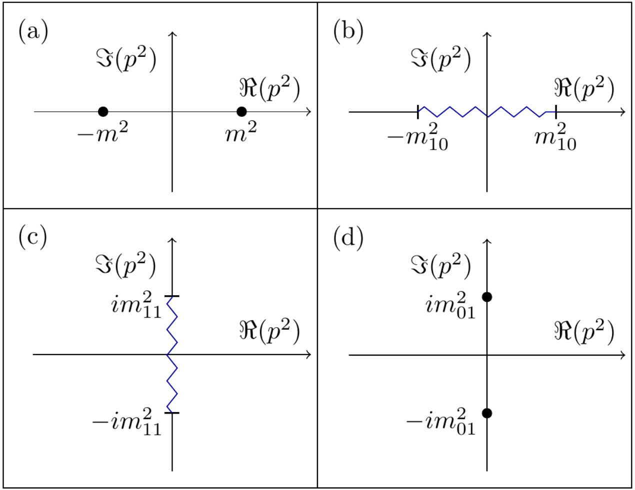
$\Longrightarrow$ as expected:

- tachyonic poles at $p^2 = -m^2$
- tachyons are also ghosts
- source of instability, cf. mean field theory

Analytic structure of the propagator

(a) naive mass

(b) γ_{10} mass



(c) γ_{11} mass

(d) γ_{01} mass

Analytic structure of the propagator

▷ specifically for the γ_{11} and γ_{01} models: (HG,Picciau'24)

- no tachyons
- complex conjugate poles or branch cut
  cf. Gribov-Stingl type propagators in low-energy QCD (Stingl'86; Alkofer,v.Smekal'00)
- no spectral representation
- (presumably) no (LSZ-type construction of an) S matrix
- no instabilities (observed so far)

Red flags

No spectral representation:

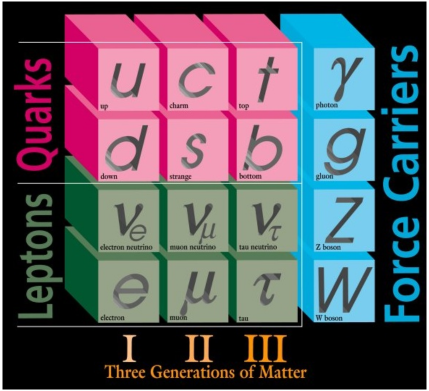
- Positivity of the state space? existence and fate of ghosts?
- applicability of spin-statistics theorem?
- applicability of CPT theorem?
- existence of relativistic Luttinger fermions as asymptotic states?

$\Longrightarrow$ some aspects may become more transparent in canonical quantization (HG,Ye wip)

A possible cure: confine Luttinger fermions by a coupling to a nonabelian gauge theory.

Luttinger numerology

▷ Observation: relativistic QFT with Luttinger fermions requires a 32-component spinor



▷ Dirac fermions in each family:

u-type :	3	(color)
d-type :	3	(color)
e-like :	1	
ν :	1	(incl. right-handed)
sum:	8	Dirac fermions

$\Longrightarrow$ 32 components