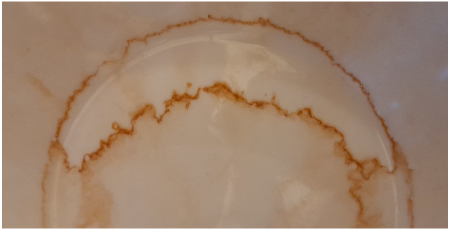
Universality classes in coupled Kardar-Parisi-Zhang equations



Léonie Canet
Université Grenoble Alpes

ERG 2026 Brighton 04/09/2026

Presentation outline

1. The Kardar-Parisi-Zhang equation
2. Its non-perturbative sides

   LC, J. Stat. Mech. **2025** 124003

3. Coupled Kardar-Parisi-Zhang equations
4. FRG analysis of Coupled Kardar-Parisi-Zhang equations

   C. Sánchez, N. Wschebor, **LC**, in preparation (2026)

In collaboration with ...

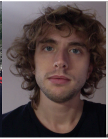




B.Delamotte	N.Wschebor	H.Chaté	M.Brachet	C.Sánchez
LPTMC Paris	Montevideo	CEA Paris	ENS Paris	Montevideo

PhD students and post-docs




T.Kloss M.Tarpin C.Fontaine F.Vercesi L.Gosteva

References

LC, B. Delamotte, H. Chaté, N. Wschebor, PRL **104** (2010), PRE **84** (2011)
C. Fontaine, F. Vercesi, M. Brachet, **LC**, PRL **131** (2023)
L. Gosteva, M. Tarpin, N. Wschebor, **LC**, PRE **110** (2024)
L. Gosteva, N. Wschebor, **LC**, J. Stat. Mech. **2025** 114002
LC, J. Stat. Mech. **2025** 124003

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   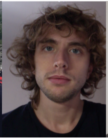 

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L. Gosteva, N. Wschebor, **LC**, J. Stat. Mech. **2025** 114002
LC, J. Stat. Mech. **2025** 124003

The Kardar-Parisi-Zhang equation

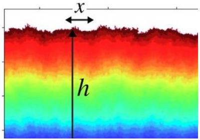
- Langevin dynamics of stochastically growing interfaces

Kardar, Parisi, Zhang, PRL **56** (1986)

$$\partial_t h = \nu \nabla^2 h + \frac{\lambda}{2} \left(\nabla h\right)^2 + \sqrt{D}\eta$$

η: Gaussian random noise of variance

$$\left\langle \eta(t, \mathbf{x}) \eta(t', \mathbf{x}') \right\rangle = 2\delta(t - t')\delta^d(\mathbf{x} - \mathbf{x}')$$



Takeuchi *et al*, Sci. Rep. **1** (2011)

The Kardar-Parisi-Zhang equation

- **Langevin dynamics** of stochastically growing interfaces

Kardar, Parisi, Zhang, PRL **56** (1986)

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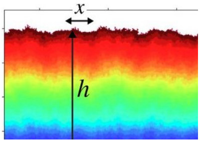


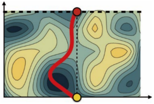
Takeuchi *et al*, Sci. Rep. **1** (2011)

- **leads to kinetic roughening:** a non-equilibrium critical phenomenon
  - generic scale-invariance
    self-organised criticality
  - universality
  - anomalous scaling

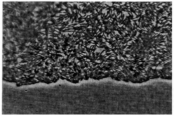
Halpin-Healy, Zhang, Phys. Rep. **254** (1995); Krug, Adv. Phys. **46** (1997); Takeuchi, Physica A **504** (2018)

The Kardar-Parisi-Zhang universality class

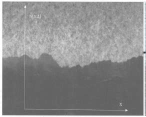
- exact mapping to:
  - Burgers equation for randomly stirred fluids
    velocity: $\vec{u} = -\lambda \vec{\nabla} h$
  - Directed Polymer in Random Media
    free energy: $\ln Z = \frac{\lambda}{2\nu} h$



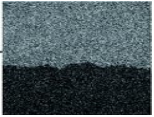
- extremely large universality class Takeuchi, Phys. A **504** (2018)



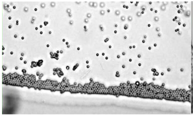
bacteries
PRL (1997)



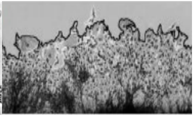
combustion
JPSJ (1997)



liquid crystals
PRL (2010)



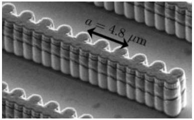
deposition
Nature (2011)



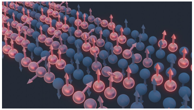
cancer cells
PRE (2012)

The Kardar-Parisi-Zhang universality class: even for quantum systems!

- phase dynamics in driven-dissipative Bose-Einstein condensates
  - observed in experiments:
    - 1D exciton-polaritons Nature **608** (2022)
    - 2D exciton-polaritons arXiv:2608.07242



- finite-T dynamics in integrable Heisenberg spin chains
  - observed in experiments:
    - solid-state magnets Nature Phys. **17** (2021)
    - ultra-cold gases Science **316** (2022)

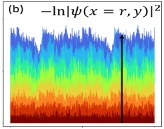


- in entanglement growth

Nahum, Ruhman, Vijay, Haah, PRX **7** (2017)

- in Anderson localisation

Mu, Gong, Lemarié, PRL **132** (2024)



The Kardar-Parisi-Zhang universality class

- KPZ equation

$$\partial_t h = \nu \nabla^2 h + \frac{\lambda}{2} \left(\nabla h\right)^2 + \sqrt{D}\eta$$

- correlation function takes Family-Vicsek scaling form

$$C(t, \mathbf{x}) = \left\langle \left(h(t, \mathbf{x}) - h(0,0)\right)^2 \right\rangle \sim \begin{cases} |\mathbf{x}|^{2\chi} & t = 0 \\ t^{2\beta} & \mathbf{x} = 0 \end{cases}$$

$\longrightarrow$ collapse onto a universal scaling function $\mathcal{F}$

$$C(t, \mathbf{x}) = |\mathbf{x}|^{2\chi} \mathcal{F}\left(t/|\mathbf{x}|^z\right), \quad z = \chi/\beta$$

$$\text{universal} \begin{cases} \text{roughness exponent} & \chi \\ \text{growth exponent} & \beta \\ \text{dynamical exponent} & z \end{cases}$$

The Kardar-Parisi-Zhang equation in $d = 1$
Scaling ...

- space-time correlation function $C(t, x) = |x|^{2\chi} \mathcal{F}(t/|x|^z)$

▷ linear growth (EW): $\lambda = 0$

Edwards, Wilkinson, Proc.R.Soc.Lond. A (1982)

$$\chi = 1/2\,, \quad z = 2$$

▷ nonlinear growth (KPZ)

Kardar, Parisi, Zhang, PRL **56** (1986)

$$\chi = 1/2\,, \quad z = 3/2$$

The Kardar-Parisi-Zhang equation in $d = 1$
. . . and beyond

▶ space-time correlation function $C(t,x) = |x|^{2\chi}\mathcal{F}\left(t/|x|^{z}\right)$

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$\chi = 1/2\,, \quad z = 2$

▷ nonlinear growth (KPZ)

Kardar, Parisi, Zhang, PRL **56** (1986)

$\chi = 1/2\,, \quad z = 3/2$

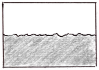
⋆ universal distribution of height fluctuations

- curved geometry – droplet (GUE-TW)

  Sasamoto, Spohn, PRL **104** (2010)

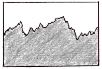
  Amir, Corwin and Quastel, Commun. Pure Appl. Math. **64** (2011)

  Calabrese, Le Doussal, Rosso, EPL **90** (2010)



- flat geometry (GOE-TW)

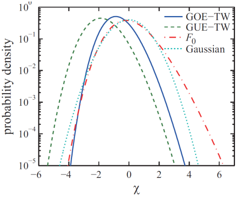
  Calabrese, Le Doussal, PRL **106** (2011)



- Brownian geometry (Baik-Rains F_0)

  Imamura, Sasamoto, PRL (2012)

  Borodin, Corwin, Ferrari, Vetö, Math. Phys. Ann. Geom. **18** (2015)

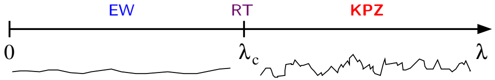


⋆ universal scaling functions $\mathcal{F}$: in terms of Airy processes

Prahöfer, Spohn, J. Stat. Phys. (2004), Sasamoto, J. Phys. A (2005), Imamura, Sasamoto, PRL (2012)

The Kardar-Parisi-Zhang equation in $d \neq 1$

- $d > 2$: Roughening Transition and nonpertubative fixed point



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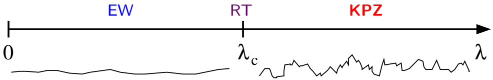

- field theory for the KPZ equation

Martin, Siggia, Rose, PRA **8** (1973), Janssen, Z. Phys. B **23** (1976), de Dominicis, J. Phys. Paris **37** (1976)

$$\mathcal{S}_{\mathrm{KPZ}}[h,\bar{h}] = \int_{t,\mathbf{x}} \left\{ \bar{h} \left[\partial_t h - \nu \nabla^2 h - \frac{\lambda}{2} \left(\nabla h \right)^2 \right] - D\bar{h}^2 \right\} \qquad g = \frac{\lambda^2 D}{\nu^3}$$

The Kardar-Parisi-Zhang equation in $d \neq 1$

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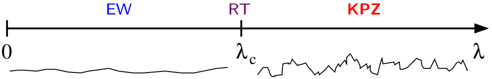

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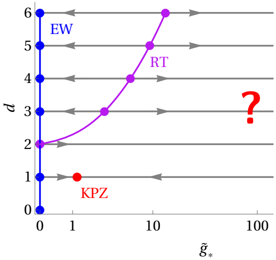
- Perturbative Renormalisation Group



▷ resummed to all orders in $d = 2 + \varepsilon$

EW: $z = 2$ and $\chi = \frac{2-d}{2}$

RT: $z = 2$ and $\chi = 0$ for all ε

Wiese, PRE **56** (1997), J. Stat. Phys. **93** (1998)

but fails to find the KPZ strong-coupling fixed-point !

The strong-coupling KPZ fixed point from Functional Renormalisation Group

- field theory for the KPZ equation

$$\mathcal{S}_{\mathrm{KPZ}}[h,\bar{h}] = \int_{t,\mathbf{x}} \left\{ \bar{h} \left[\partial_t h - \nu \nabla^2 h - \frac{\lambda}{2} (\nabla h)^2 \right] - D \bar{h}^2 \right\} \qquad g = \frac{\lambda^2 D}{\nu^3}$$

- FRG with simplest approximation: $\nu \longrightarrow \nu_\kappa$ and $D \longrightarrow D_\kappa$
- regulator matrix: $\Delta\mathcal{S}_\kappa[h,\bar{h}] = \int_{\omega,\mathbf{q}} \left\{ \nu_\kappa \mathbf{q}^2 \bar{h}\, R_\kappa(\mathbf{q})\, h - D_\kappa \bar{h}\, R_\kappa(\mathbf{q})\, \bar{h} \right\}$

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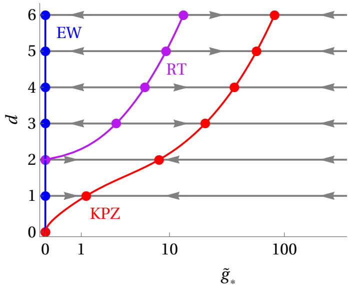
phase diagram



KPZ fixed point in all d!

LC, Chaté, Delamotte, Wschebor, PRL **104** (2010)

T. Kloss, **LC**, N. Wschebor, PRE **86** (2012)

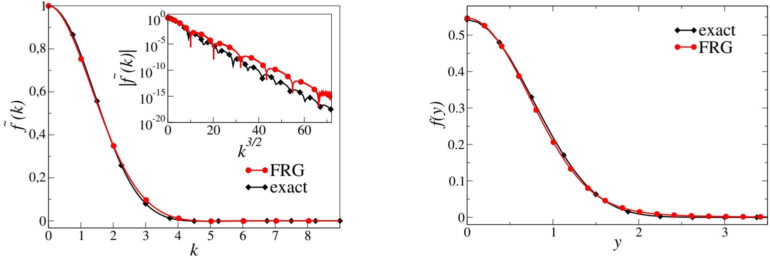
The strong-coupling KPZ fixed point from Functional Renormalisation Group

- FRG with functional approximation: $\nu \longrightarrow \nu_\kappa(\omega, \mathbf{p})$, $D \longrightarrow D_\kappa(\omega, \mathbf{p})$

▷ correlation function : shows generic scaling

$$C_\kappa(\omega, \mathbf{p}) = \frac{2D_\kappa(\omega, \mathbf{p})}{\omega^2 + \mathbf{p}^4 \nu_\kappa^2(\omega, \mathbf{p})} \quad \xrightarrow{\kappa \to 0} \quad \frac{1}{\mathbf{p}^{d+2+\chi}} \tilde{f}\left(\omega/\mathbf{p}^z\right)$$

universal scaling function $\tilde{f}$ in $d = 1$



LC, Chaté, Delamotte, Wschebor PRE **84**, (2011), Prähofer and Spohn, J. Stat. Phys. **115** (2004)

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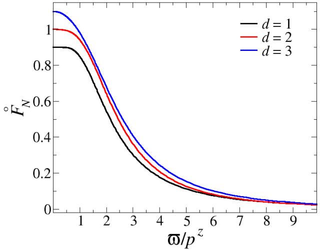
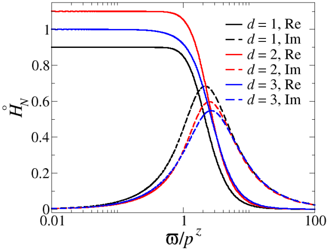
universal scaling functions for correlation $\mathring{F}$ and response $\mathring{H}$ in $d > 1$





T. Kloss, LC, N. Wschebor, PRE **86** (2012)

In this all ?

In this all ?

The Galerkin-truncated Burgers equation: Crossover from inviscid-thermalised to Kardar-Parisi-Zhang scaling

C. Cartes[1], E. Tirapegui[2], R. Pandit[3] and M. Brachet[4]

Phil. Trans. A **380** (2022)

Family-Vicsek Scaling of Roughness Growth in a Strongly Interacting Bose Gas

Kazuya Fujimoto[1,2] Ryusuke Hamazaki[3,4] and Yuki Kawaguchi[2]

Phys. Rev. Lett. **124** (2020)

(c)

Model	α	β	z
KPZ	$1/2$	$1/3$	$3/2$
EW	$1/2$	$1/4$	2
BHM ($\nu \gg 1$)	0.517 ± 0.030	0.255 ± 0.012	2.07 ± 0.20
BHM ($\nu \simeq 1/2$)	0.500 ± 0.003	0.489 ± 0.004	1.00 ± 0.01

this Letter

Anomalous ballistic scaling in the tensionless or inviscid Kardar-Parisi-Zhang equation

Enrique Rodríguez-Fernández[1,*] Silvia N. Santalla[2,†] Mario Castro[3,‡] and Rodolfo Cuerno[1,§]

Phys. Rev. E **106** (2022)

observation of an unpredicted scaling $z = 1$
in the inviscid limit $\nu \to 0$ of the KPZ equation

An even stronger-coupling fixed point: The Inviscid Burgers fixed point

- FRG with simplest approximation: $\nu \longrightarrow \nu_\kappa$ and $D \longrightarrow D_\kappa$

▷ one effective coupling $\hat{g}_\kappa = \kappa^{d-2}\lambda^2 D_\kappa/\nu_\kappa^3$
▷ define $\hat{w}_\kappa = \hat{g}_\kappa/(1+\hat{g}_\kappa) \in [0,1]$

- FRG flow equation for $\hat{w}_\kappa$ in $d=1$

$$\kappa\partial_\kappa \hat{w}_\kappa = \hat{w}_\kappa \left(3 - 2z_\kappa(\hat{w}_\kappa)\right)\left(1-\hat{w}_\kappa\right)$$

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- 3 fixed point solutions

▷ EW: $\hat{w}_* = 0$	▷ KPZ: $0 < \hat{w}_* < 1$	▷ IB: $\hat{w}_* = 1$
$\chi = 1/2$, $z_{\mathrm{EW}} = 2$	$\chi = 1/2$, $z_{\mathrm{KPZ}} = 3/2$	$\chi = 1/2$, $z_{\mathrm{IB}} = 1$
UV stable	IR stable	UV stable

An even stronger-coupling fixed point: The Inviscid Burgers fixed point

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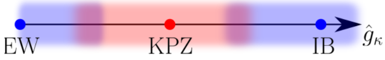
- 3 fixed point solutions

EW: $\hat{w}_* = 0$	KPZ: $0 < \hat{w}_* < 1$	IB: $\hat{w}_* = 1$
$\chi = 1/2$, $z_{\rm EW} = 2$	$\chi = 1/2$, $z_{\rm KPZ} = 3/2$	$\chi = 1/2$, $z_{\rm IB} = 1$
UV stable	IR stable	UV stable

evidences the existence of a new fixed point of the KPZ equation in $d = 1$!

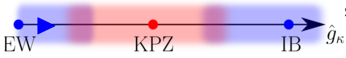


Fontaine, Vercesi, Brachet, LC, PRL **131** (2023)

An even stronger-coupling fixed point: The Inviscid Burgers fixed point

- FRG with functional approximation: $\nu \longrightarrow \nu_\kappa(\omega, \mathbf{p})$, $D \longrightarrow D_\kappa(\omega, \mathbf{p})$

results for large viscosity in $d = 1$

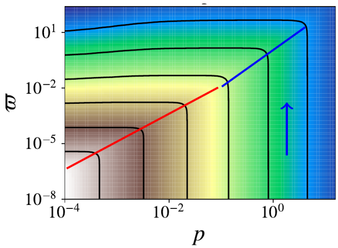


small momenta (IR) controlled by KPZ fixed point

large momenta (UV) controlled by EW fixed point

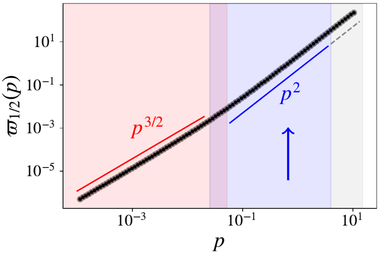
correlation function

$$C_\kappa(\omega, p) = \frac{2D_\kappa(\omega,\mathbf{p})}{\omega^2 + \left(\mathbf{p}^2 \nu_\kappa(\omega,\mathbf{p})\right)^2}$$



half decay frequency $\omega_{1/2}(p) \sim p^z$

$$C(\omega_{1/2}(p), \mathbf{p}) = C(0, \mathbf{p})/2$$

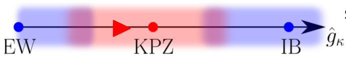


L. Gosteva, N. Wschebor, LC, J. Stat. Mech. **2025**

An even stronger-coupling fixed point: The Inviscid Burgers fixed point

- FRG with functional approximation: $\nu \longrightarrow \nu_\kappa(\omega, \mathbf{p})$, $D \longrightarrow D_\kappa(\omega, \mathbf{p})$

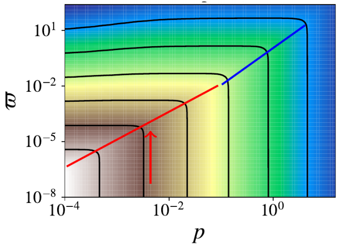
results for large viscosity in $d = 1$



small momenta (IR) controlled by KPZ fixed point
large momenta (UV) controlled by EW fixed point

correlation function

$$C_\kappa(\omega, p) = \frac{2D_\kappa(\omega, \mathbf{p})}{\omega^2 + \left(\mathbf{p}^2 \nu_\kappa(\omega, \mathbf{p})\right)^2}$$



half decay frequency $\omega_{1/2}(p) \sim p^z$

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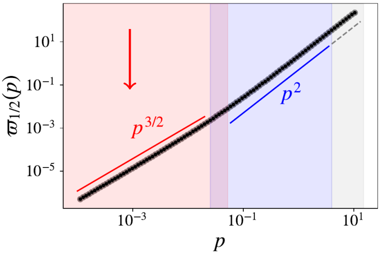

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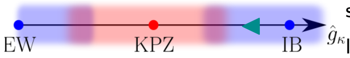

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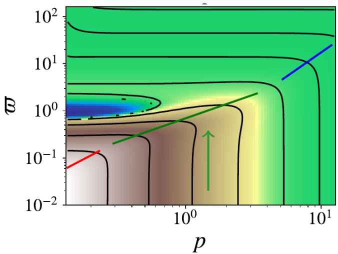

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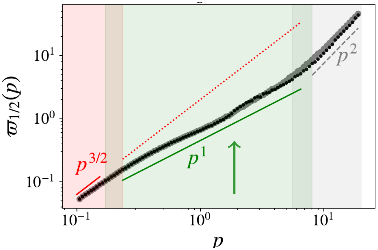

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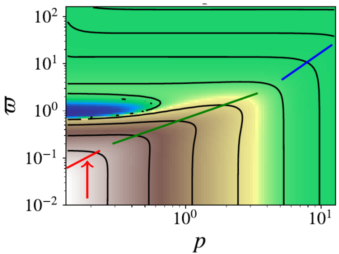

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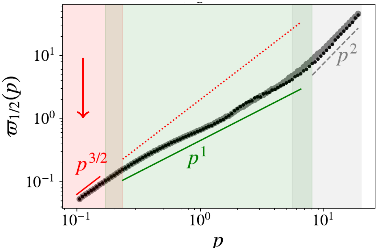


L. Gosteva, N. Wschebor, LC, J. Stat. Mech. **2025**

Realm of the Inviscid Burgers fixed point ?

Realm of the Inviscid Burgers fixed point

IB fixed point generically appears in :

- Kuramoto-Sivashinsky equation

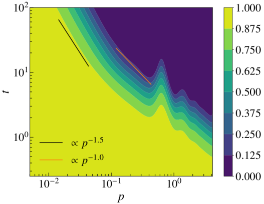
$$\partial_t h = \nu \nabla^2 h + \tau \nabla^4 h + \lambda (\nabla h)^2$$

  - microscopic scale: $\nu < 0$
  - macroscopic scale: KPZ dynamics $\nu_{\text{eff}} > 0$

$\Longrightarrow$ intermediate scales: $\nu_{\text{eff}} \simeq 0$!

Gosteva, Roy, Wschebor, LC, arXiv:2605.23364

Gosteva, Wschebor, LC, arXiv:2607.15784

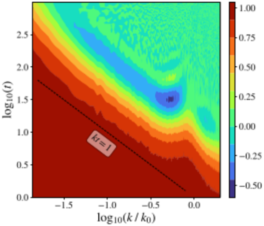


- complex Ginzburg-Landau equation

$$i\partial_t \psi = i\psi + (c_2 - i)|\psi|^2\psi - (c_1 - i)\nabla^2\psi$$

▷ dynamics of the phase θ maps in the turbulence phase regime to the KS equation

Vercesi, Poirier, Minguzzi, LC, PRE **109** (2024)



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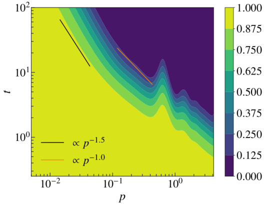


$\Longrightarrow$ **talk by L. Gosteva**

What about higher dimensions ?

Functional Renormalisation Group: Existence of IB fixed-point in all d

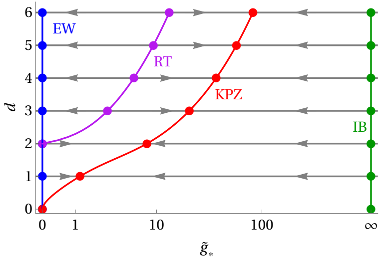
- FRG with simplest approximation: $\nu \longrightarrow \nu_\kappa$ and $D \longrightarrow D_\kappa$

▷ one effective coupling $\hat{g}_\kappa = \kappa^{d-2}\lambda^2 D_\kappa/\nu_\kappa^3$
▷ define $\hat{w}_\kappa = \hat{g}_\kappa/(1+\hat{g}_\kappa) \in [0,1]$

- FRG flow equation for $\hat{w}_\kappa$ in $d > 1$

$$\partial_s \hat{w}_\kappa = 2\hat{w}_\kappa \left(\chi_\kappa(\hat{w}_\kappa) + z_\kappa(\hat{w}_\kappa) - 2\right)\left(1 - \hat{w}_\kappa\right)$$

- The IB fixed point exists in all d



$z = 1$ in all d
super-universal

$\Longrightarrow$ exact result proved from FRG using extended symmetries

Gosteva, Tarpin, Wschebor, LC PRE **110** (2024)

Outline

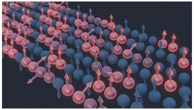
Model and symmetries

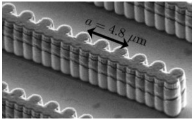
- coupled KPZ equations in $d = 1$

$$\begin{cases} \partial_t h_1 = \frac{1}{2}\nabla^2 h_1 + \left(\nabla h_1\right)^2 + Y\left(\nabla h_2\right)^2 + \eta_1 \\ \partial_t h_2 = \frac{1}{2} T \nabla^2 h_2 + 2X \nabla h_1 \nabla h_2 + \sqrt{T}\eta_2 \end{cases}$$

- arise in vastly different contexts
  - stretched polymers immersed in random media
    Ertas, Kardar, PRL **69** (1992)
  - nonlinear fluctuating hydrodynamics
    Spohn, Stoltz, J. Stat. Phys. **160** (2015)
  - sliding particles on a fluctuating landscape
    Chakraborty, Chatterjee, Barma, PRE **100** (2019)
  - isotropic Heisenberg spin chains
    de Nardis, Gopalakrishnan, Vasseur, PRL **131** (2023)
  - coupled driven-dissipative Bose-Einstein condensates
    Weinberger, Comaron, Szymanska, Nat. Comm. Phys. **8** (2025)

Valei *etal*
Newton (2026)





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- stability: $XY > 0$

- symmetries:
  - $\mathbb{Z}_2$ symmetry $h_2 \longrightarrow -h_2$
  - Time reversal symmetry (TRS) for $T = 1$ and $X = Y$

    $\Longrightarrow \chi_1 = \chi_2 = 1/2$
  - Extended Galilean symmetry for $T = 1$ and $X = 1$

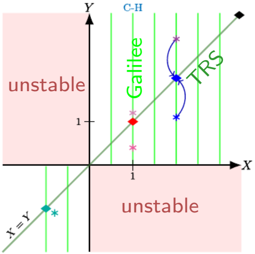
    $\Longrightarrow \chi_1 + z_1 = 2, \quad \chi_2 + z_2 = 2$
  - TRS+Galilee for $T = 1$ and $X = Y = 1$

    $\Longrightarrow z_1 = z_2 = 3/2$: two decoupled KPZ

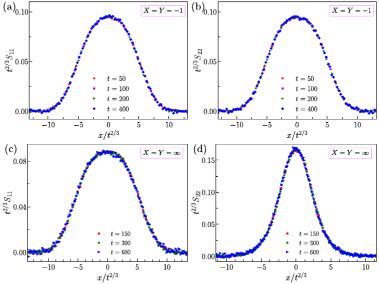
Results from analytics and numerics

- Fixed points and universality classes Roy, Dhar, Kulkarni, Spohn, J. Stat. Mech **2025**
  - attractive line of fixed points at $X = Y$, $T = 1$ with $z_1 = z_2 = 3/2$ but X dependent universal scaling functions

phase diagram for $T = 1$



scaling functions



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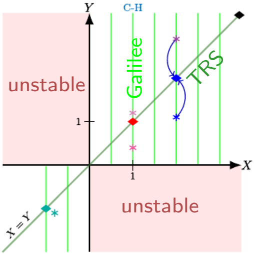

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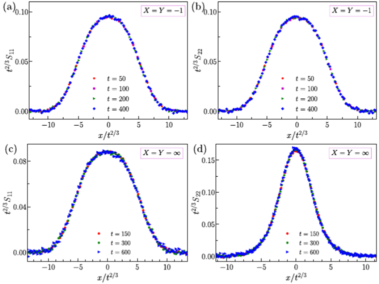


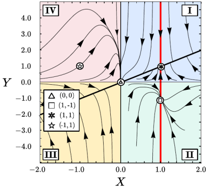
  - (X, Y, T) in the same universality class as $(X, X, 1)$

Results from one-loop RG and numerics

- ▶ confirmed by one-loop Wilson RG

Weinberger, Comaron, Szymanska

Nat. Comm. Phys. **8** (2025)



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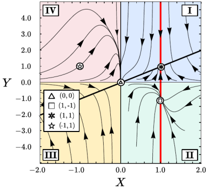
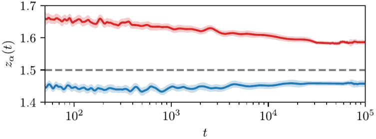


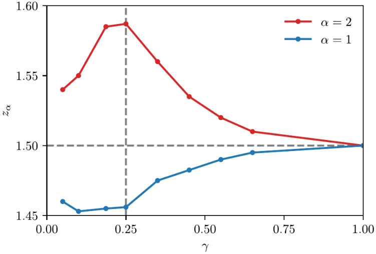
- but challenged by high-resolution numerics

critical exponents at $X = Y = 2$



$$z_1 = 1.456 \pm 0.003$$
$$z_2 = 1.587 \pm 0.005$$

critical exponents vs $\gamma = \left(1 - \frac{1}{\sqrt{X}}\right)^{-2}$



Schmidt, Krajnik, Popkov, J. Stat. Mech **2025**

Outline

Functional Renormalisation Group approach

- field theory for coupled KPZ equations

$$\mathcal{S}[h_1,h_2,\bar{h}_1,\bar{h}_2] = \int_{t,\mathbf{x}} \Big\{ \bar{h}_1 \Big[\partial_t h_1 - \nu_1 \nabla^2 h_1 - \frac{\lambda_{11}}{2} (\nabla h_1)^2 - \frac{\lambda_{22}}{2} (\nabla h_2)^2 - D_1 \bar{h}_1^2 \Big]$$
$$+ \bar{h}_2 \Big[\partial_t h_2 - \nu_2 \nabla^2 h_2 - \lambda_{12} (\nabla h_1)(\nabla h_2) - D_2 \bar{h}_2^2 \Big] \Big\}$$

- regulator matrix

$$\Delta\mathcal{S}_\kappa[h_1,h_2,\bar{h}_1,\bar{h}_2] = \int_{\omega,\mathbf{q}} \Big\{ \nu_{1_\kappa} \mathbf{q}^2 \bar{h}_1 R_\kappa(\mathbf{q}) h_1 - D_{1_\kappa} \bar{h}_1 R_\kappa(\mathbf{q}) \bar{h}_1$$
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- FRG approximation

▷ precision on exponents required to settle the debate $\Longrightarrow$ derivative expansion

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- FRG approximation

▷ precision on exponents required to settle the debate $\Longrightarrow$ **derivative expansion**

▷ key counting:

- $t \sim k^{-2}$: diffusive scaling
- $\bar{h}_i \sim k^2$: TRS in $d=1$ $\begin{cases} h_i(t) \to -h_i(-t) \\ \bar{h}_i(t) \to \bar{h}_i(-t) + \frac{\nu_i}{D_i} \nabla^2 h_i(-t) \end{cases}$

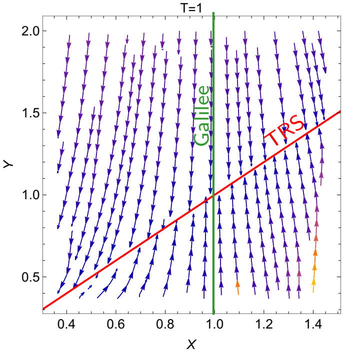
demonstrated on single KPZ at order DE8! Sánchez, Wschebor, Canet, in preparation (2026)

Functional Renormalisation Group at order DE4

- Ansatz for Γ_κ at DE4: 7 couplings

$$\Gamma_\kappa[h_1, h_2, \bar{h}_1, \bar{h}_2] = \int_{t,\mathbf{x}} \Big\{ \bar{h}_1 \Big[\partial_t h_1 - \nu_{1_\kappa} \nabla^2 h_1 - \frac{\lambda_{11_\kappa}}{2} (\nabla h_1)^2 - \frac{\lambda_{22_\kappa}}{2} (\nabla h_2)^2 - D_{1_\kappa} \bar{h}_1^2 \Big] + \bar{h}_2 \Big[\partial_t h_2 - \nu_{2_\kappa} \nabla^2 h_2 - \lambda_{12_\kappa} (\nabla h_1)(\nabla h_2) - D_{2_\kappa} \bar{h}_2^2 \Big] \Big\}$$

- flow diagram in the plane $T = \nu_1/\nu_2 = 1$



▷ similar results than 1-loop RG ...

Weinberger, Comaron, Szymanska, Nat. Comm. Phys. **8** (2025)

...although one equation not correct

...and no stable fixed point for $Y < 0$

attractive fixed-point line at $X = Y$ with $z_1 = z_2 = 3/2$

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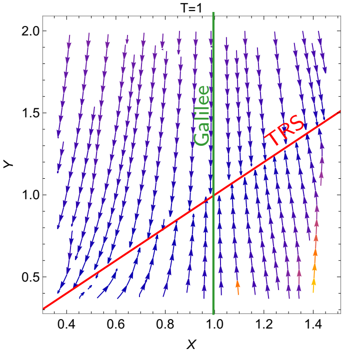

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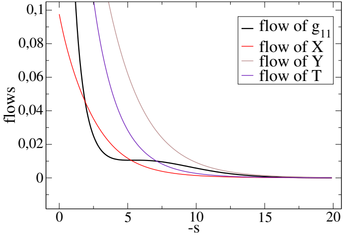

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  - line of fixed points still exist
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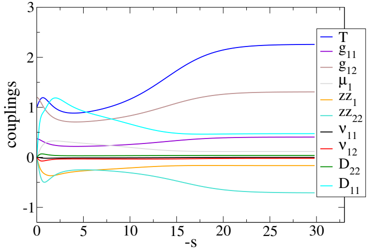

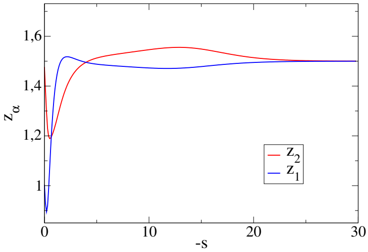

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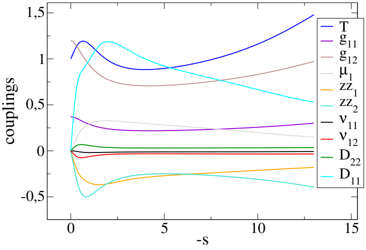
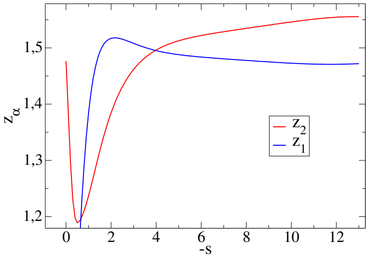
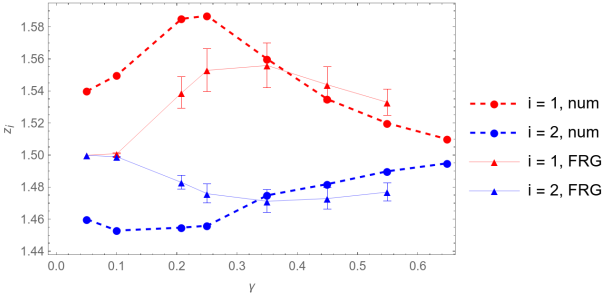




Functional Renormalisation Group at order DE6

- Exponents as a function of $\gamma = \left(1 - \frac{\lambda_{11}}{\lambda_{12}}\right)^{-2}$



Schmidt, Krajnik, Popkov
J. Stat. Mech **2025**

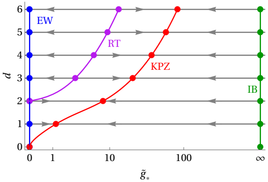
Sánchez, Wschebor, Canet
in preparation (2026)

conclusion from FRG: $z_1 = z_2 = 3/2$ for all X
but huge finite size effects, compatible with the numerics

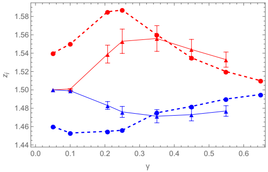
Summary and perspectives

The non-perturbative sides of the KPZ equation

- complete phase diagram of the KPZ equation



- Universality classes for coupled KPZ equations



Further exploration of the realm of stochastic interface growth

- high-precision KPZ exponents at DE8
- universality of coupled KPZ equations in $d = 2$ and $d = 3$
- open questions in many variants of the KPZ equation

Thank you for your attention !