From Partons to Strings: Scattering on the Coulomb branch of $\mathcal{N} = 4$ SYM

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Based on 2510.19909 and 2605.16034 with
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(+ work in progress)

Bootstrap 2026

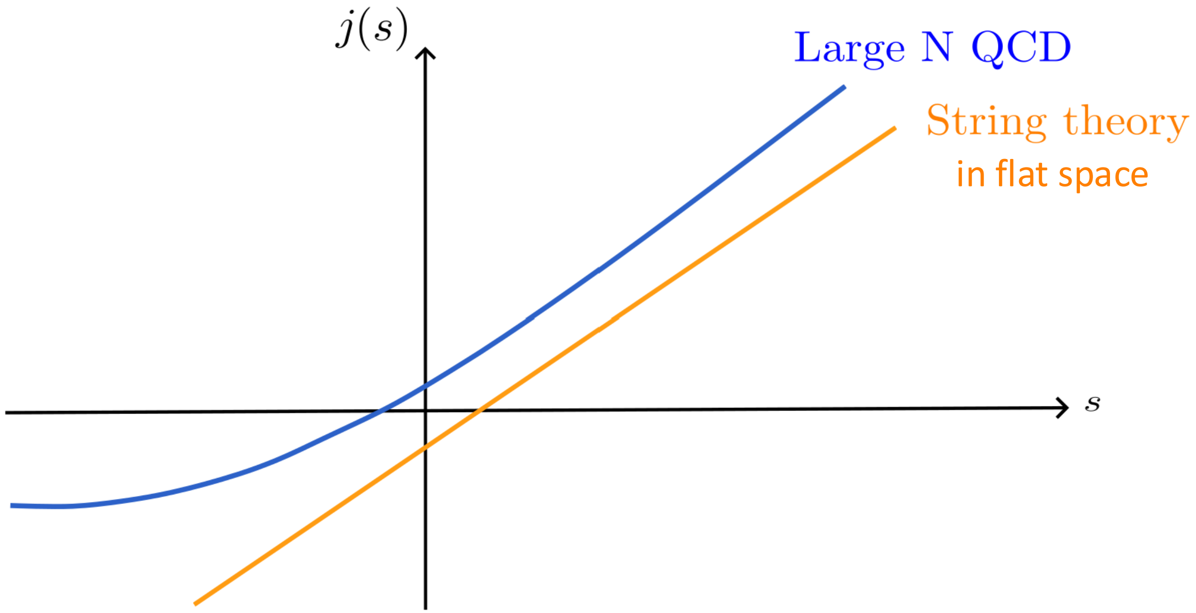
Motivation

$$T_{\mathrm{Ven}}(s,t) = \frac{\Gamma(-s)\Gamma(-t)}{\Gamma(1-s-t)}$$

- linear Regge trajectories
- degenerate spectrum

Stringy S-matrices that «step away» from these familiar cases?

Scattering on the Coulomb branch of $\mathcal{N}=4$ SYM in the planar limit



Setup [Alday, Henn, Plefka, Schuster '10]

$$U(N+4) \to U(N) \times U(1)^4$$

"light" fields of mass $|m_i - m_j|$

$$\begin{pmatrix} \Phi_{a,b} & \Phi_{a,N+j} \\ \Phi_{N+i,b} & \Phi_{N+i,N+j} \end{pmatrix} \qquad \begin{matrix} a,b = 1,\ldots,N \\ i,j = 1,\ldots,4 \end{matrix}$$

$$\mathcal{L} = \mathcal{L}_{\mathcal{N}=4} - \frac{1}{2}(m_i - m_j)^2 (\Phi_I)_{ij}(\Phi^I)_{ji} - m_i^2 (\Phi_I)_{ia}(\Phi^I)_{ai} + \ldots$$

$$M^{(I=6)}_{N+i,N+j} = \begin{pmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & m_3 & 0 \\ 0 & 0 & 0 & m_4 \end{pmatrix}$$

"heavy" W-bosons of mass m_i

Simplest case $m_i \equiv m$

$$(\chi_I)_{i,j} \equiv (\Phi_I)_{N+i,N+j}$$

$$A = \langle (\chi_I)_{1,2}(p_1)(\chi_J)_{2,3}(p_2)(\chi_I)_{3,4}(p_3)(\chi_J)_{4,1}(p_4) \rangle, \quad I, J \neq 6$$

$\longrightarrow$ 2-to-2 amplitude of massless scalars in planar, Higgsed $\mathcal{N}=4$ SYM

Setup [Alday, Henn, Plefka, Schuster '10]

$$U(N+4) \to U(N) \times U(1)^4$$

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"light" fields of mass $|m_i - m_j|$

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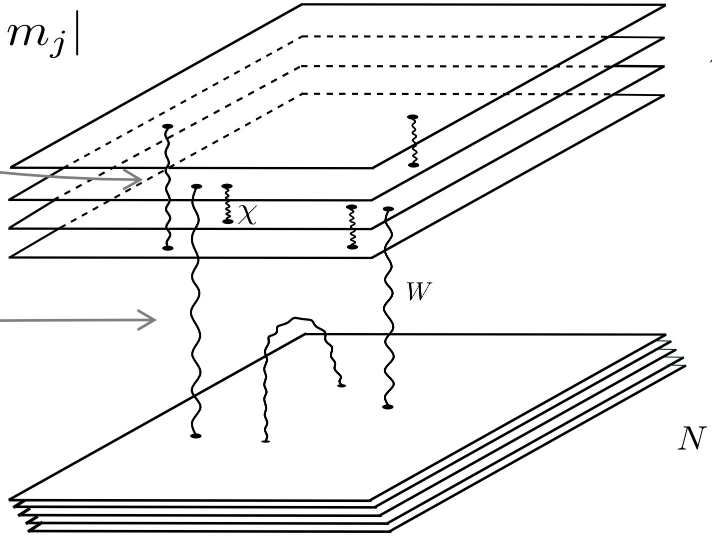

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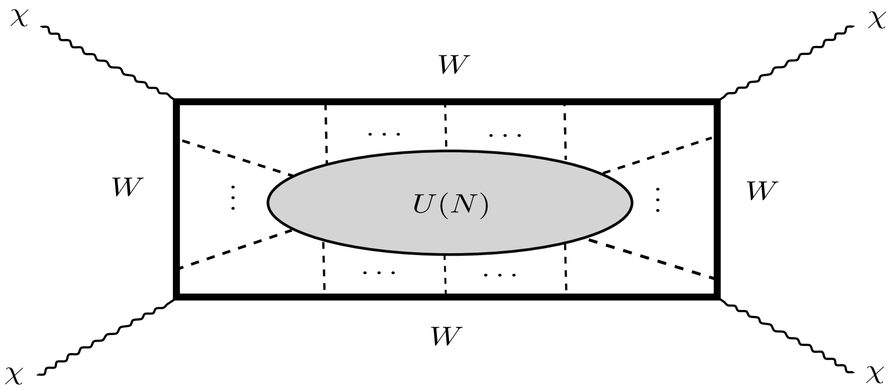
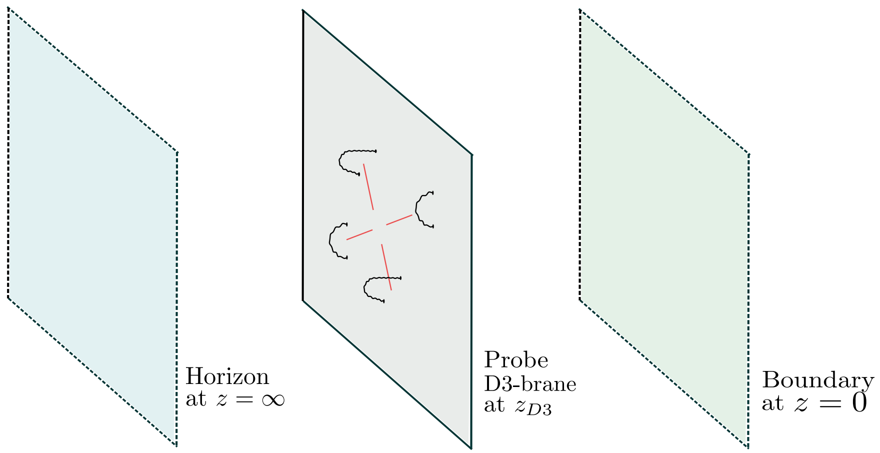
$\longrightarrow$ $\boxed{\text{2-to-2 amplitude of massless scalars in planar, Higgsed } \mathcal{N}=4 \text{ SYM}}$

Weak and strong coupling

$$f(s,t) \equiv \frac{A(-4m^2/s, -4m^2/t)}{g_{\rm YM}^2 st} \quad , \quad g^2 = \frac{g_{\rm YM}^2 N}{(4\pi)^2}$$





- IR finite
- $f(s,t) = -\frac{1}{st} + g^2(\dots)$
- One-loop: branch cut at $s = 4m^2$
- Set of bound states as g increases

$\longrightarrow$ Physics: Coulomb potential
[Klebanov, Maldacena, Thorne '06]
[Caron-Huot, Henn '14]

- Flat space tree-level scattering

$$f(s,t) = -\frac{g^2}{\pi^2 m^4} \frac{\Gamma\left(-\frac{gs}{m^2\pi}\right)\Gamma\left(-\frac{gt}{m^2\pi}\right)}{\Gamma\left(1-\frac{gs}{m^2\pi}-\frac{gt}{m^2\pi}\right)} + \dots$$

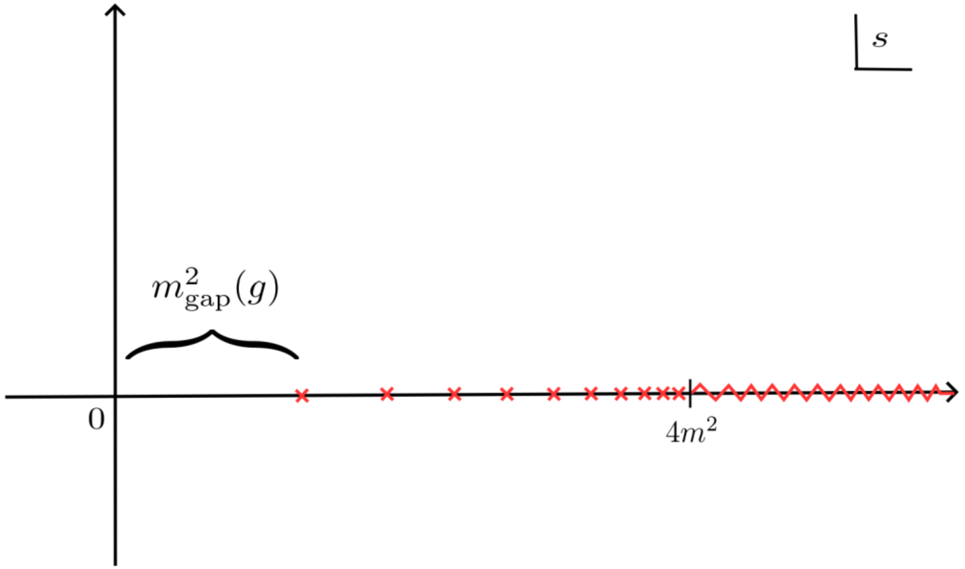
[Maldacena, Remmen '22]

- Worldsheet bootstrap for the $1/g$ correction

$$\hat{f}^{(1)}(S,T) = \int_0^1 \frac{x^{-S}(1-x)^{-T}}{x(1-x)} h(S,T;x)dx$$

[Alday, Hansen, Silva, Chester, Nocchi, …]

Finite coupling



$$m_{\text{gap}}(g) = 2m + \mathcal{O}(g^4)\,, \quad g \ll 1$$

$$m_{\text{gap}}(g) = \frac{m\sqrt{\pi}}{\sqrt{g}} + \mathcal{O}(g^{-3/2})\,, \quad g \gg 1$$

At finite coupling:

- Dual conformal invariance
- Integrability
- S-matrix bootstrap

$\longrightarrow$ conformal symmetry in T-dual space

$$u = \frac{4m_1 m_3}{-s + (m_1 - m_3)^2}$$

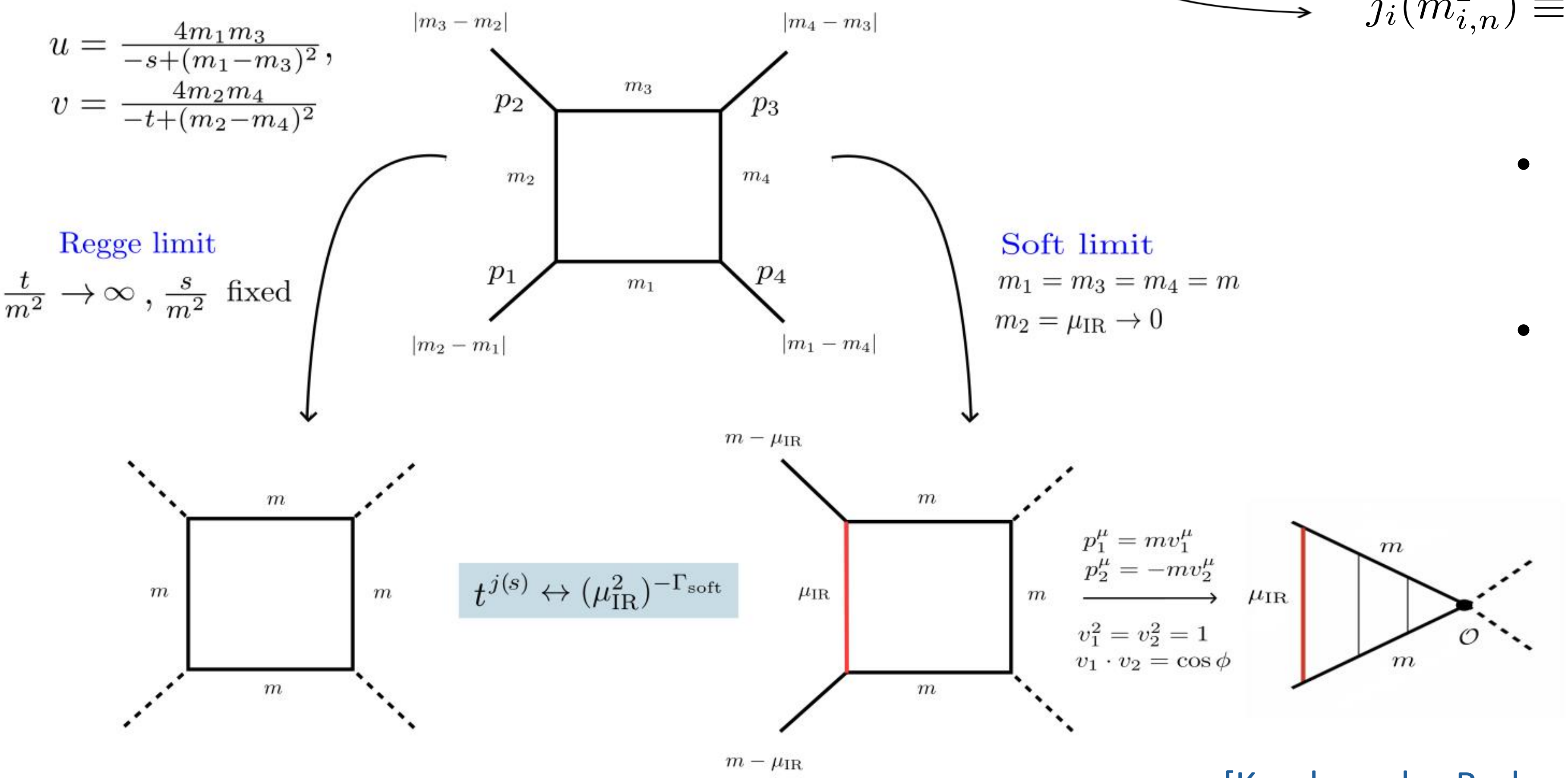
$$v = \frac{4m_2 m_4}{-t + (m_2 - m_4)^2}$$

Regge/cusp correspondence

Regge expansion: $f(s,t) \sim \sum_i r_i(s) Y^{-j_i(s)-1}\,, \quad t \to \infty\,,\ \ s\,, m^2 \text{ fixed}$

$j_i(m_{i,n}^2) \equiv n \in \mathbb{Z}_{\geq 0}$

[Brüser, Caron-Huot, Henn ‘18]



- Quark lines: Wilson lines + corrections
- Integrate out the W-bosons: insertions of operators

[Korchemsky, Radyushkin ‘92]

Our proposal: $j_n(s) + 1 = -\Gamma_{\mathrm{cusp},\mathcal{O}_n}(\phi)\ \ ,\ \ s = 4m^2 \sin^2(\phi/2)$

Integrability and QSC

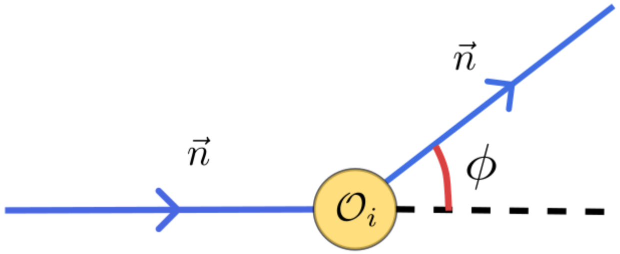
The leading and first subleading trajectories:

[Correa, Henn, Maldacena, Sever '12], [Brüser, Caron-Huot, Henn '18]

$$\boxed{j_0(s)+1=-\Gamma_{\text{cusp}}(g,\phi)\,,\quad j_1(s)+1=-\Gamma_{\text{cusp},\Phi_{||}}(g,\phi)}$$

for a Maldacena-Wilson line with a cusp

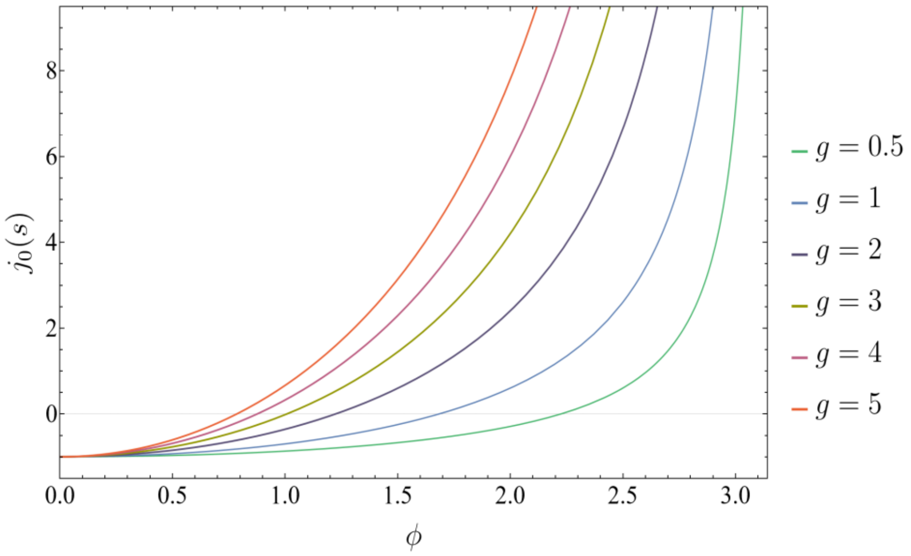
$$W = \mathrm{Tr}\mathcal{P}\exp\left(ig_{\text{YM}}\int_C \left(A_\mu(x)dx_\mu - i\Phi_{||}(x)|dx|\right)\right)$$



$$\langle W\rangle = \left(\frac{\Lambda_{\text{UV}}}{\Lambda_{\text{IR}}}\right)^{-\Gamma_{\text{cusp}}(g,\phi)}$$

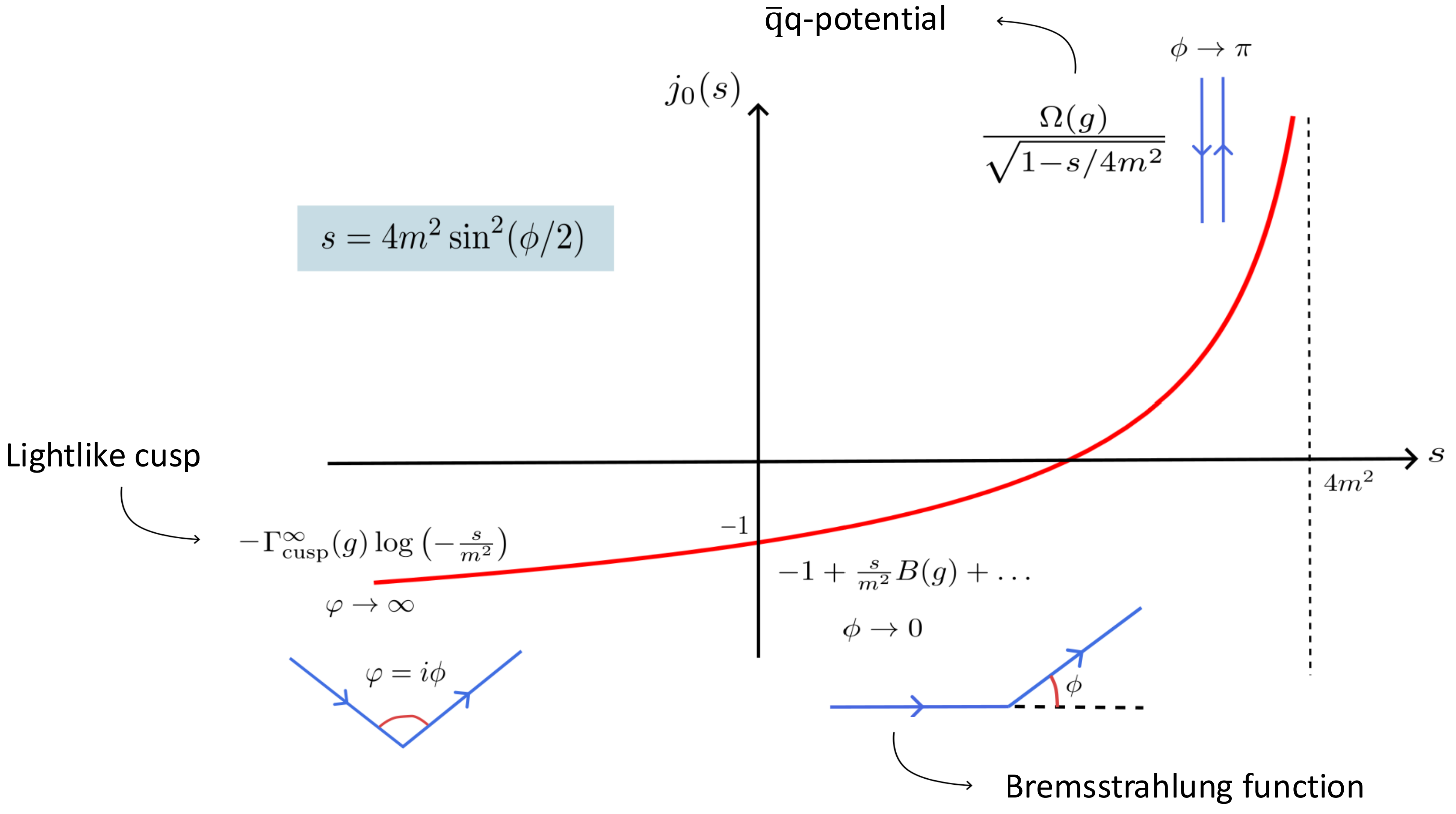
- Can be computed using integrability of $\mathcal{N}=4$ SYM
- Know the spectrum exactly $\quad j_i(s)=n\,,\quad n\in\mathbb{Z}_{\geq 0}$

[Gromov, Levkovich-Maslyuk '15], [Grabner, Gromov, Julius '20]



Leading Regge trajectory

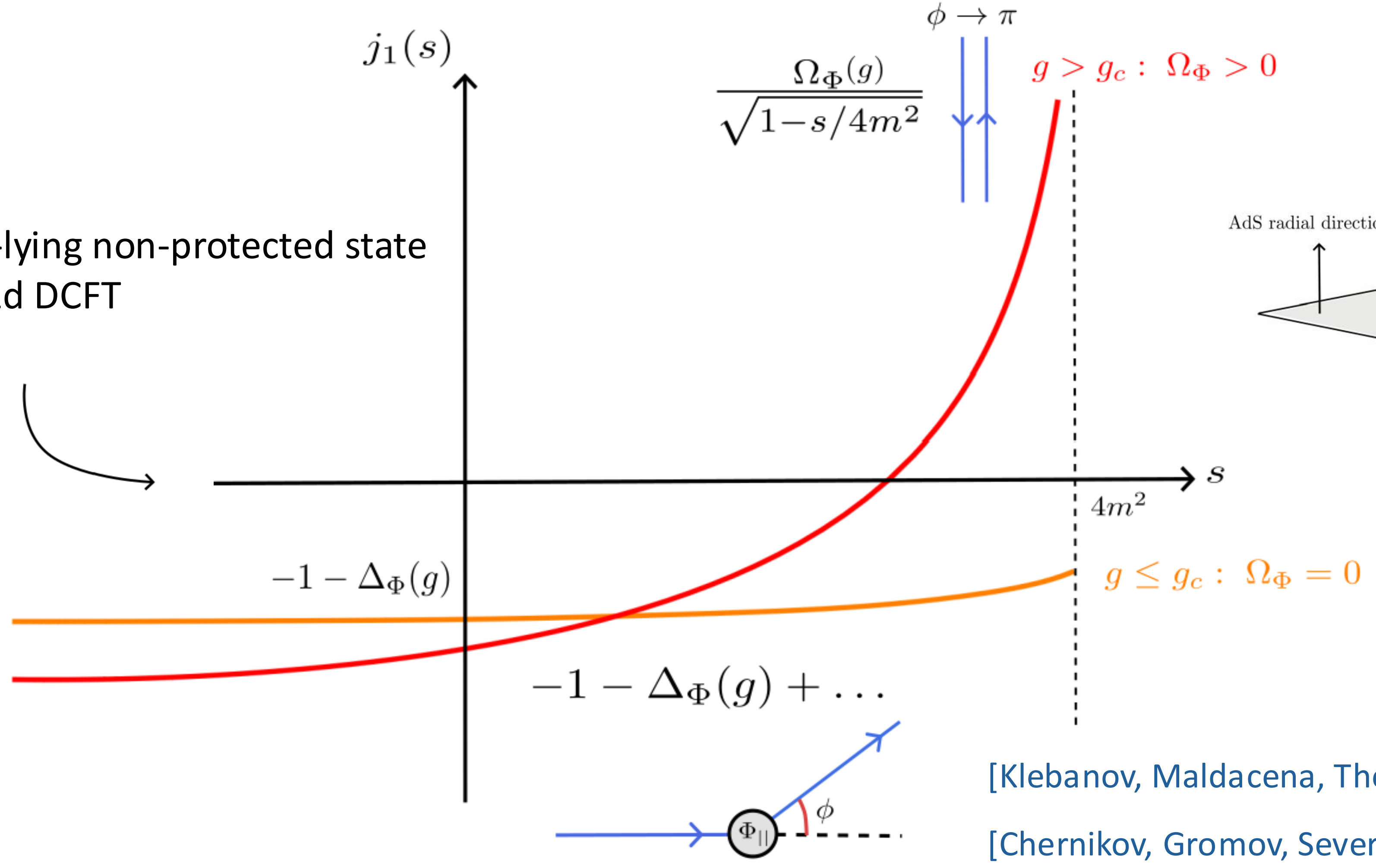
$$j_0(s) + 1 = -\Gamma_{\mathrm{cusp}}(g, \phi)$$

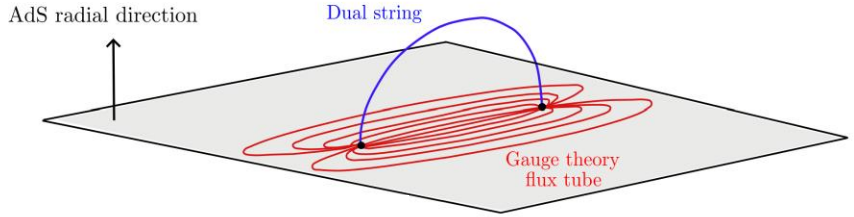


Subleading Regge trajectory

$$j_1(s) + 1 = -\Gamma_{\text{cusp},\Phi_{||}}(g, \phi)$$

Lowest-lying non-protected state of the 1d DCFT





[Klebanov, Maldacena, Thorne ‘06]

[Chernikov, Gromov, Sever, ‘26]

Presence of excited bound states

stifness of the dual string

Enhanced unitarity

$SO(3)$ unitarity:

$$\mathrm{Im}(f(s,t)) = s^{-2} \sum_{J=0}^{\infty} c_J(s)\, P_J^{\mathrm{SO}(3)}(\cos\theta)\,, \quad c_J(s) \geq 0\,, \quad s \geq 0\,, \quad \cos\theta = 1 + \frac{2t}{s}$$

Dual conformal symmetry $SO(2,4)$ extra degeneracy [Caron-Huot, Henn '14]

- Hydrogen atom in QM: $SO(3)$ ⊕ Runge-Lenz $\longrightarrow$
  - $SO(4)$ on the bound states
  - $SO(1,3)$ on the scattering states [Fock, 1935]

$SO(3)$ + dual conformal symmetry

$SO(4)$ unitarity: $\tilde{c}_J(s) \geq 0$

$$\mathrm{Im}(f(s,t)) = s^{-2} \sum_{J=0}^{\infty} \tilde{c}_J(s)\, \tilde{P}_J^{\mathrm{SO}(4)}(\cos\tilde{\theta})\,,$$

$$0 \leq s < 4m^2\,, \quad \cos\tilde{\theta} = 1 + \frac{2t}{s} - \frac{t}{2m^2}$$

[Brüser, Caron-Huot, Henn '18]

$SO(1,3)$ unitarity: $\tilde{c}_\nu(s) \geq 0$

$$\mathrm{Im}(f(s,t)) = s^{-2} \int_0^{\infty} d\nu\; \tilde{c}_\nu(s)\, \tilde{\Omega}_\nu^{\mathrm{SO}(1,3)}(\cosh\tilde{\theta})\,,$$

$$s \geq 4m^2\,, \quad \cosh\tilde{\theta} = 1 + \frac{2t}{s} - \frac{t}{2m^2}$$

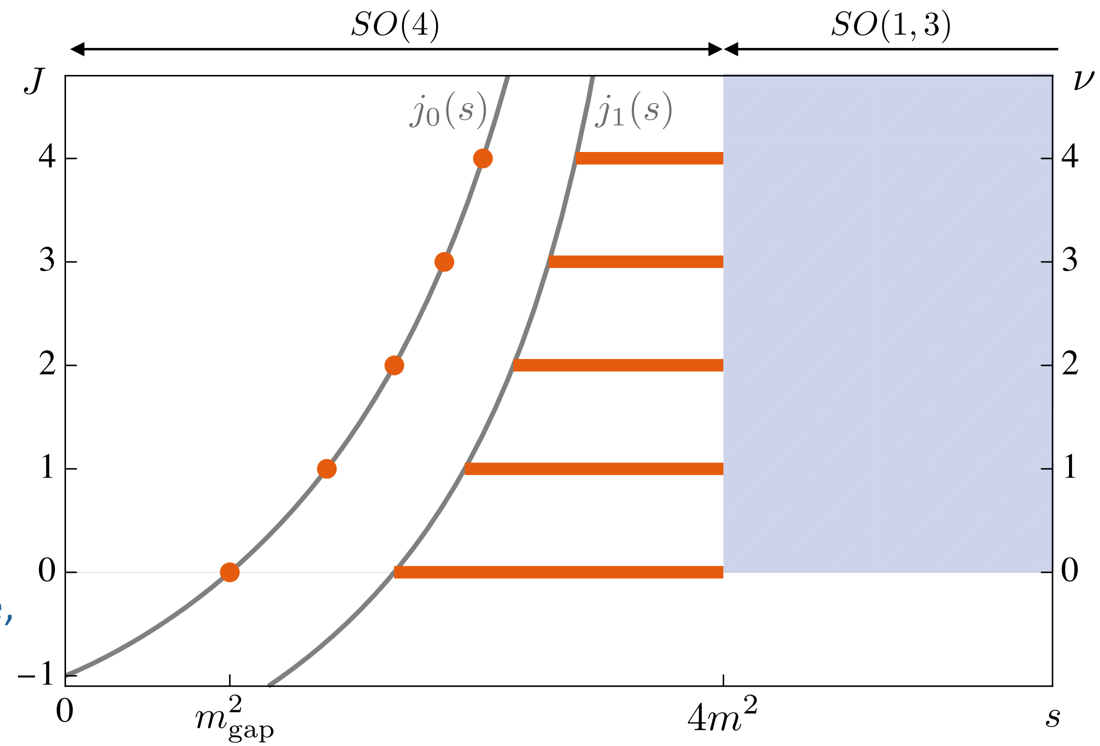
S-matrix bootstrap

- Integrability: spectrum of bound states
- Enhanced unitarity: $SO(4)$ on the bound states,

  $SO(1,3)$ on the cut
- Inputs in the S-matrix bootstrap analysis

[Adams, Arkani-Hamed, Dubovsky, Nicolis, Rattazzi '06], [Caron-Huot, Van Duong '20], [Albert, Knop, Rastelli '24], [Berman, Elvang, Herderschee '23], …

[Caron-Huot, Coronado, Zahree '24], [Dempsey, Karlsson, Pufu, Zahree, Zhiboedov '25]

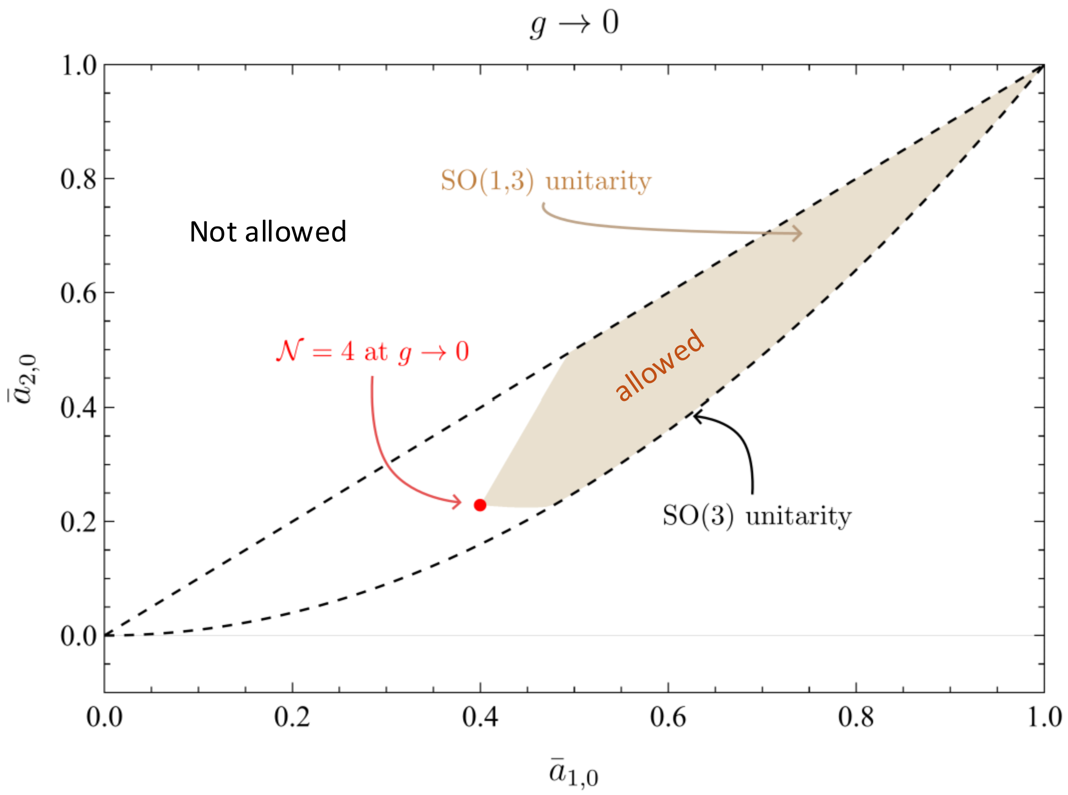


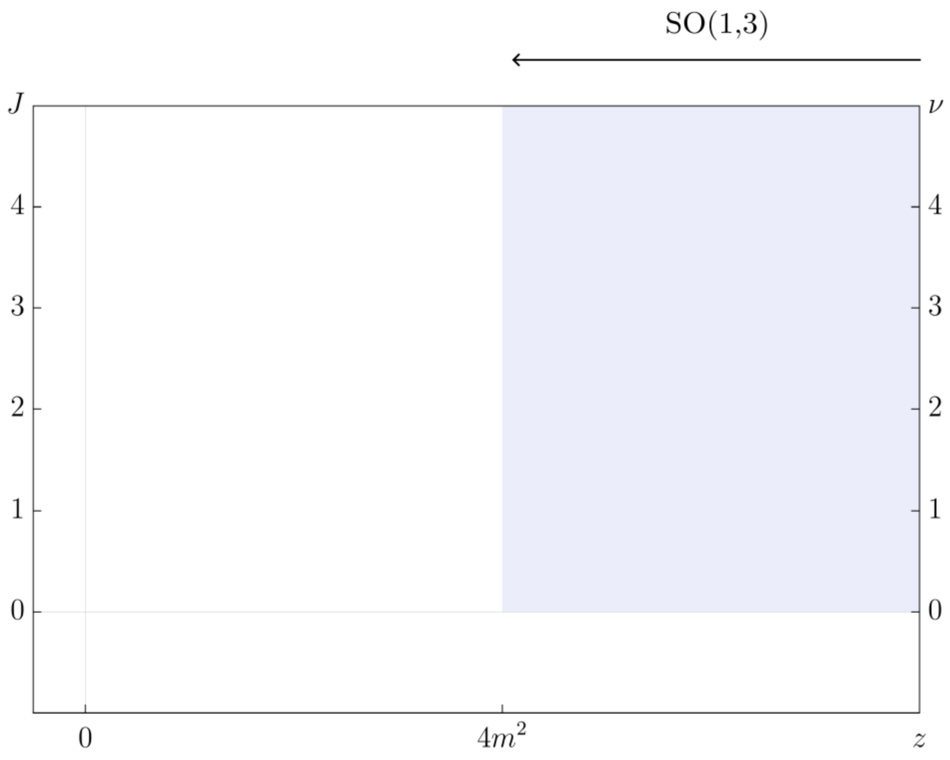
Put bounds on Wilson coefficients:

$$f(s,t) + \frac{1}{st} = \boxed{a_{0,0}} + a_{1,0}(s+t) + a_{2,0}(s^2+t^2) + a_{2,1}st + \dots \qquad \bar{a}_{n,l} = \frac{a_{n,l}}{a_{0,0}} m^{2n}_{\text{gap}}$$

one-loop exact (SUSY) [Dine, Seiberg '97]

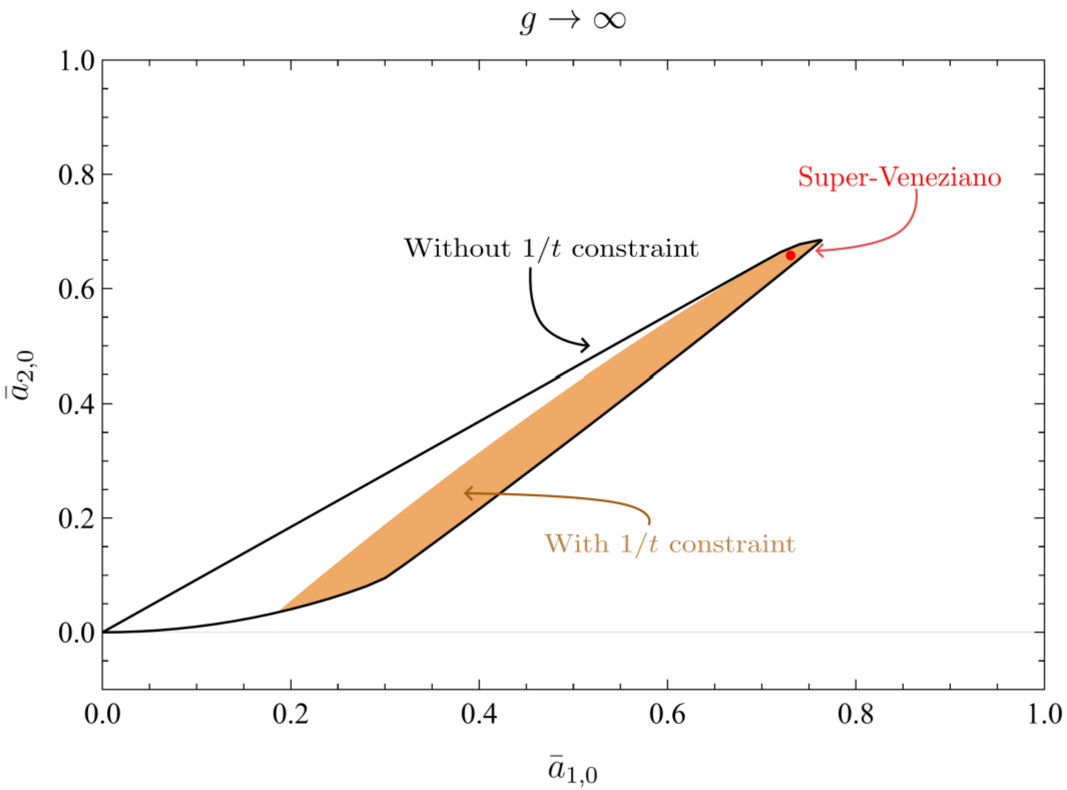
Bounds at $g \to 0$ [Bellazzini, Miró, Rattazzi, Riembau, Riva '20]

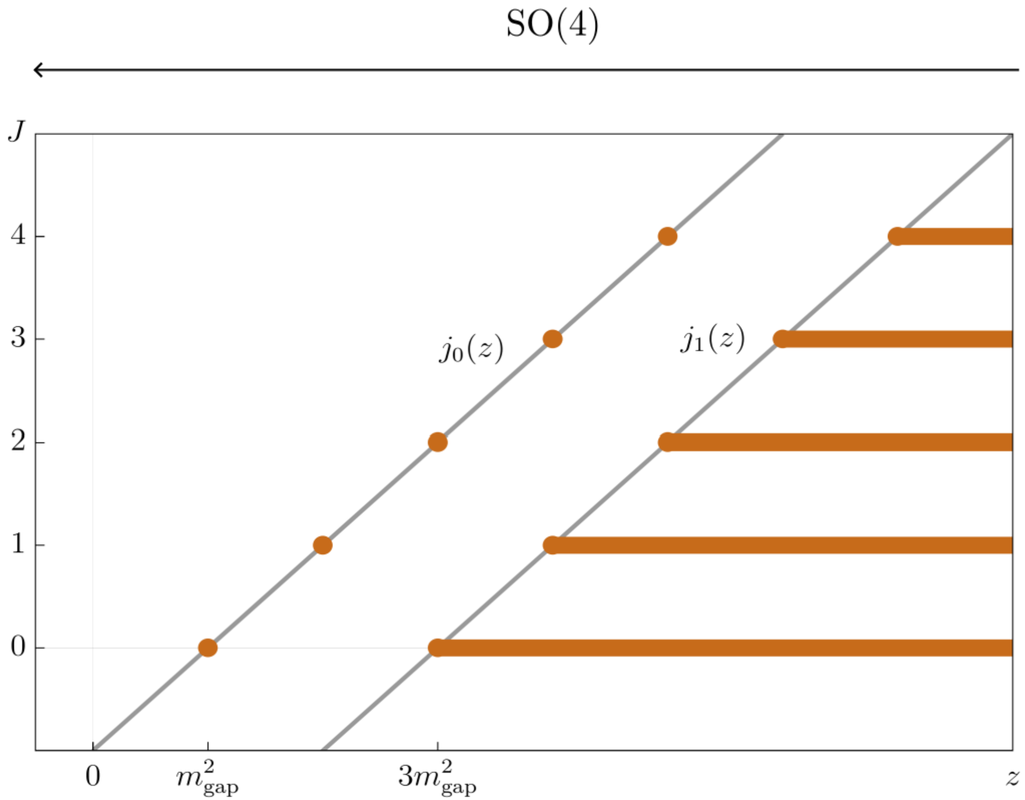




$\mathcal{N} = 4$ SYM on the Coulomb branch at one-loop is extremal!

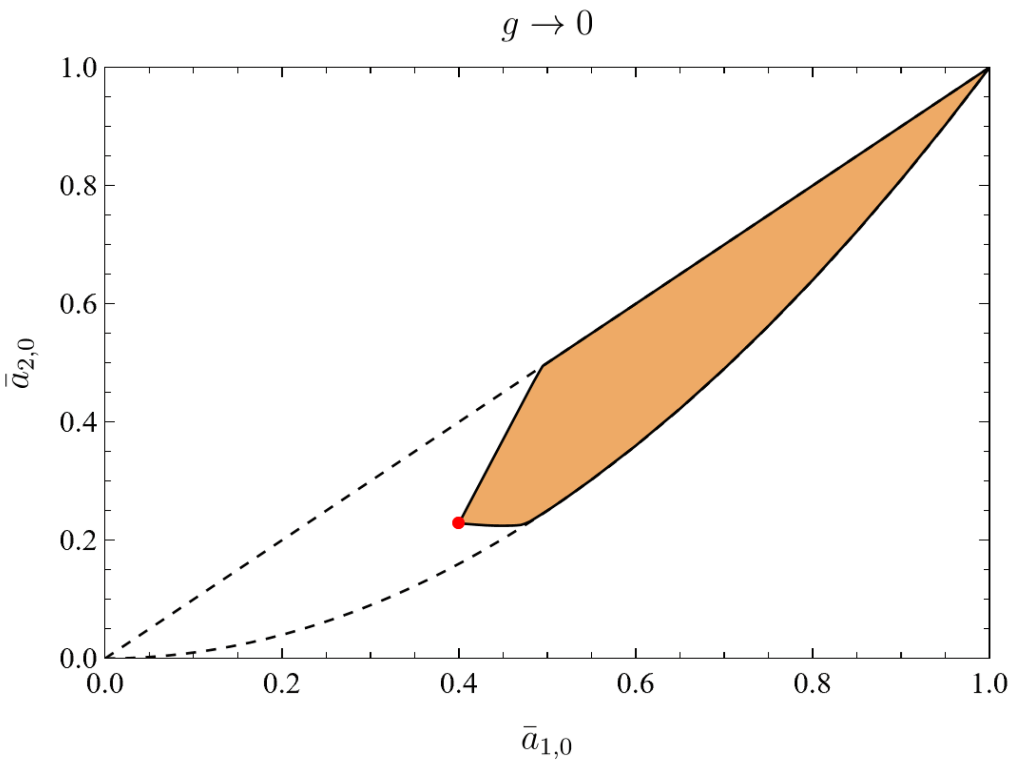
Bounds at $g \to \infty$





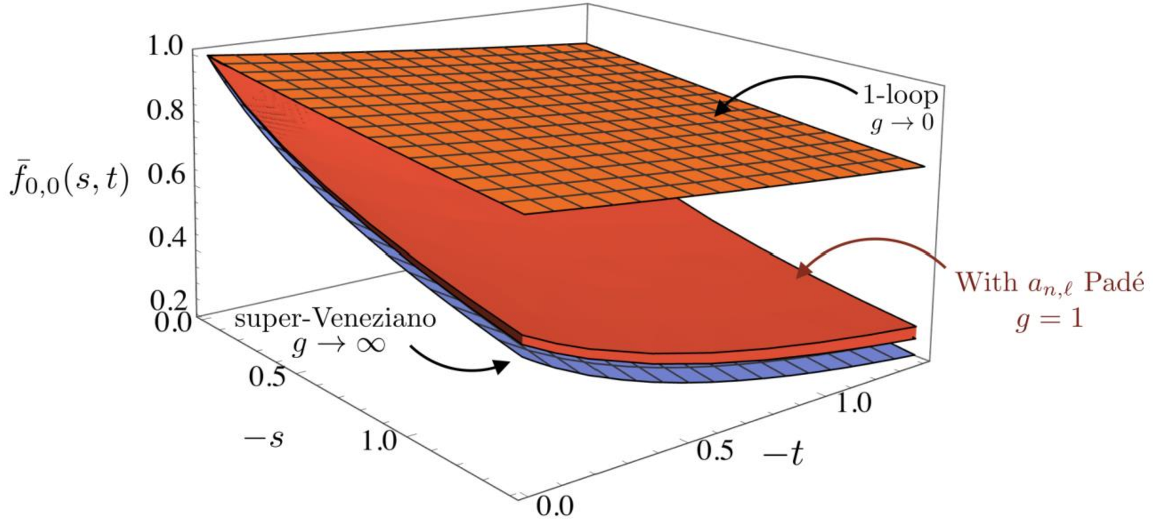
$$f(s,t) = -\frac{1}{st} + \frac{g^2}{6m^4} + \dots, \qquad -\frac{1}{t} = \frac{1}{\pi}\int_{m^2_{\text{gap}}(g)}^{\infty} ds' \, \mathrm{Im}(f(s',t))$$

Bounds at finite coupling



- Not enough to fix the amplitude at finite coupling!
- Padé island is small compared to the bounds

- Bootstrap with the Padé bands as constraints:



- Need further input from integrability (three-point couplings, ...), or SUSY localization
- Look at a simpler amplitude…

The amplituhedron

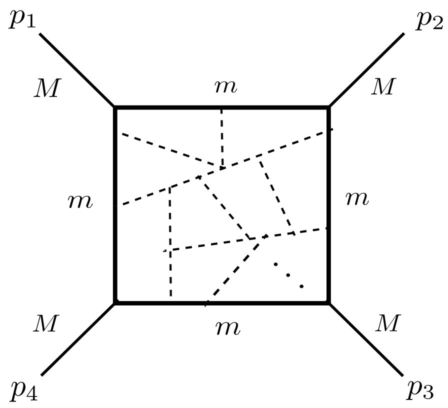
[Arkani-Hamed, Trnka '13]

Preliminary results

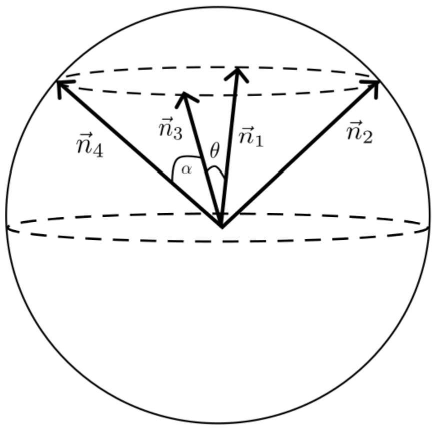
Massive amplitude on the Coulomb branch

$$p_i^2 = M^2 = 2m^2(1 - \vec{n}_i \cdot \vec{n}_{i+1}) > 0$$

Choose: $\alpha = \pi/2\,, \theta$ generic $\longrightarrow M = \sqrt{2}m$



$S^3 \subset S^5$



$$\vec{n}_i \cdot \vec{n}_{i+1} = \cos\alpha$$
$$\vec{n}_1 \cdot \vec{n}_3 = \vec{n}_2 \cdot \vec{n}_4 = \cos\theta$$

L-loop amplitude

$$\updownarrow \quad M^{(L)} = \int \Omega^{(L)}$$

deformed amplituhedron geometry

[Arkani-Hamed, Flieger, Henn, Schreiber, Trnka '23]

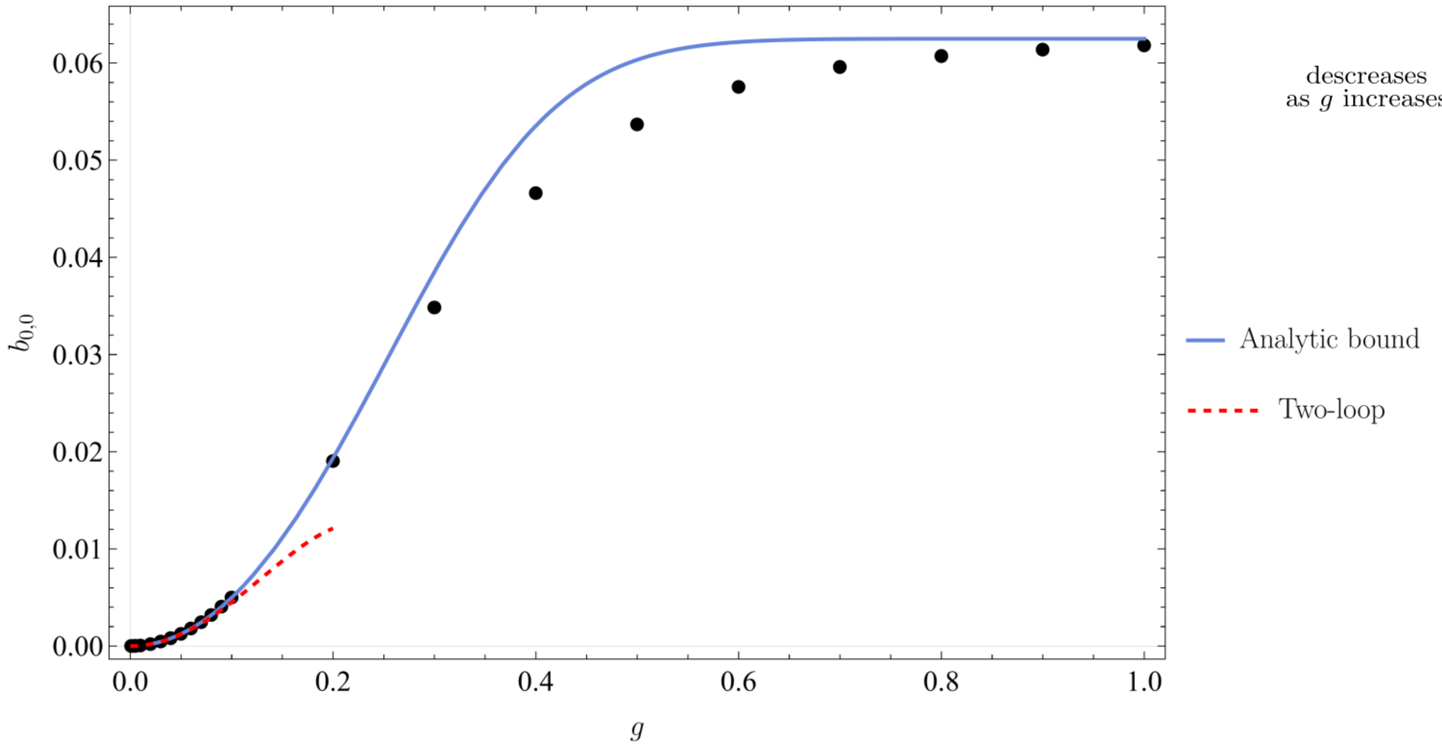
The amplituhedron

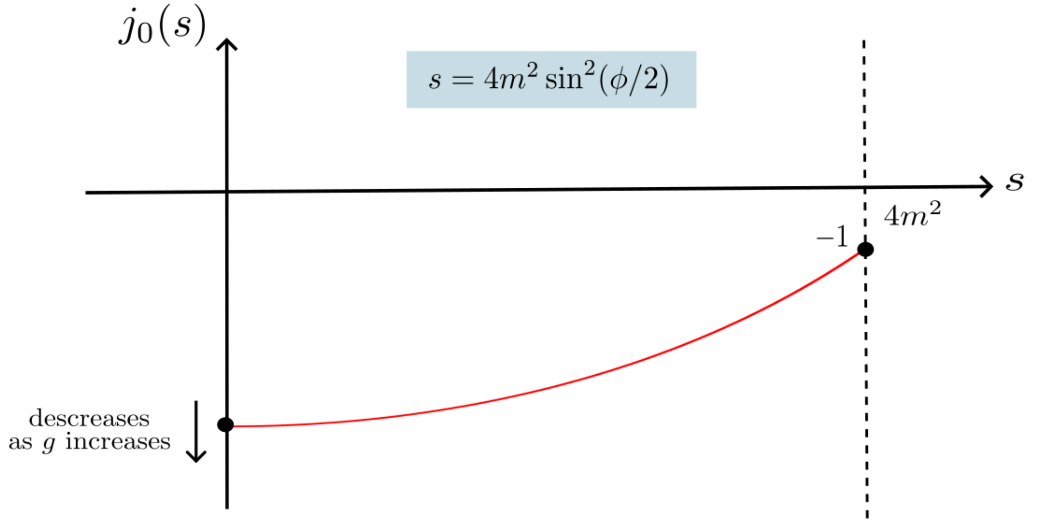
Preliminary results

Massive amplitude on the Coulomb branch

$$p_i^2 = M^2 = 2m^2(1 - \vec{n}_i \cdot \vec{n}_{i+1}) > 0$$

Choose: $\alpha = \pi/2\,, \theta = \pi \longrightarrow$ no bound states





$$j_0(s) + 1 = -\Gamma_{\text{cusp}}(\phi, \theta = \pi)$$

$$j_1(s) + 1 = -\Gamma_{\text{cusp},\Phi_\perp}(\phi, \theta = \pi)$$

[Flieger, Henn, Schreiber, Trnka '25]

Conclusions and outlook

Unique laboratory to study scattering at finite coupling

- Combined different techniques to study the amplitude at finite g:

1. Perturbative techniques and worldsheet bootstrap
2. Integrability
3. S-matrix Bootstrap

not enough to constrain the amplitude at finite g

Work in progress/open questions:

- Massive amplitudes: amplituhedron, octagon [Alday, EA, Belitsky, Häring, Zhiboedov '26], …
- Higher point amplitudes, connection to DBI
- Form factor/amplitude bootstrap [Karateev, Kuhn, Penedones '21], [Coronado, Komatsu, Zarembo '25]
- Finite N: open/closed mixed bootstrap

Thank you for your attention!

From the cusp to the octagon

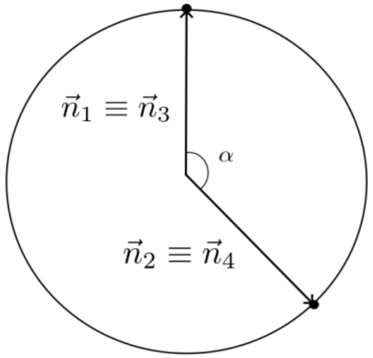
[Alday, EA, Belitsky, Häring, Zhiboedov '26]

Massive amplitude on the Coulomb branch

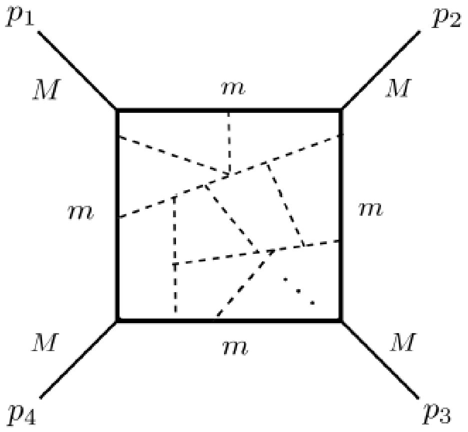
$$p_i^2 = M^2 = 2m^2(1 - \vec{n}_i \cdot \vec{n}_{i+1}) \longrightarrow p_i^2 = -M^2 < 0$$

- No anomalous-thresholds
- Interpolation between the cusp and the octagon

$S^1 \subset S^5$



$$\vec{n}_i \cdot \vec{n}_{i+1} = \cos\alpha$$
$$\vec{n}_1 \cdot \vec{n}_3 = \vec{n}_2 \cdot \vec{n}_4 = 1$$



$$\xrightarrow{(s,t)\to\infty,\,(m,M)\text{-fixed}} \quad e^{-\frac{\Gamma_{\text{cusp}}(g)}{2}\log s \log t}$$

$$\uparrow (\eta_s, \eta_t) \to 0$$

$$\xrightarrow{(s,t)\to\infty,\, s^{\eta_s}/t^{\eta_t}\text{-fixed}} \quad e^{-\Gamma_{\text{walk}}(\eta_s,\eta_t,g)\log\left(\frac{-s}{M^2}\right)\log\left(\frac{-t}{M^2}\right)} \qquad \eta_s = \frac{\log(M^2/m^2)}{\log(-s/M^2)},\ \eta_t = \frac{\log(M^2/m^2)}{\log(-t/M^2)}$$

$$\downarrow (\eta_s, \eta_t) \to 1$$

[Caron-Huot, Coronado '21]

$$\xrightarrow{(s,t)\to\infty,\, m=0} \quad e^{-\frac{\Gamma_{\text{oct}}(g)}{4}\log^2\left(\frac{st}{M^4}\right)}$$